COMPLEX METHODS IN REAL INTEGRAL GEOMETRY

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Though I take responsibility for the exposition, the work described here is joint with Toby N. Bailey and C. Robin Graham and specific attributions are made for each lecture. Conversations with A. Rod Gover, Lionel J. Mason, and Michael A. Singer have been extremely useful. Indeed, the methods are a direct continuation of joint work of Bailey, Gover, Mason, and myself [4] which itself continues from earlier work of the same authors [3]. These lectures were presented at the 16$^{\text{th}}$ Winter School on Geometry and Physics, Srní, Czech Republic, January 1996.

The aim is to present a general machine which will analyse some real integral transforms using complex methods. The machine will be illustrated by its application to the X-ray transform. An elaboration upon these aims will be postponed until the second lecture.

Lecture One

The Involutive Structure on a Totally Real Blow-Up

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Let Ω be a connected complex n-dimensional manifold and $M \subset \Omega$ a totally real, real-analytic submanifold. An example would be $\mathbb{R}^n \subset \mathbb{C}^n$ and, in general, one can choose coördinates to put $M \subset \Omega$ locally into this form. Let $\eta : \widetilde{\Omega} \to \Omega$ denote the real blow-up of Ω along M and denote by Σ the exceptional variety $\eta^{-1}(M)$. Thus,

$$\begin{array}{ccc} \Sigma & \subset & \widetilde{\Omega} \\ \big\downarrow & & \big\downarrow{\scriptstyle \eta} \\ M & \subset & \Omega \end{array}$$

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with Σ a smooth hypersurface in $\widetilde{\Omega}$ whose fibre over $m \in M$ is the real $(n-1)$-dimensional projective space of normal directions at m to M in Ω. The mapping $\eta : \widetilde{\Omega} \setminus \Sigma \to \Omega \setminus M$ is a diffeomorphism. Therefore, $\widetilde{\Omega} \setminus \Sigma$ has a complex structure.

Proposition 1 *The complex structure on $\widetilde{\Omega} \setminus \Sigma$ extends to $\widetilde{\Omega}$ as an involutive structure.*

Proof. An involutive structure (in the sense of Treves [18]) is a complex sub-bundle $T^{0,1}$ of the complexified tangent bundle $\mathbb{C}T$, closed under Lie bracket. Equivalently, by setting $\Lambda^{1,0} = (T^{0,1})^\perp$, it is a sub-bundle of the bundle of complex-valued 1-forms which generates a differentially closed ideal. Of course, the notation is selected to generalise that for complex manifolds. The involutive structure on $\widetilde{\Omega}$ is obtained by pulling back the complex structure on Ω, specifically $\Lambda^{1,0}_{\widetilde{\Omega}} = \eta^* \Lambda^{1,0}_{\Omega}$. Evidently, this will be closed provided it is indeed a sub-bundle. This is ensured by M being totally real. Rather than check this abstractly, it is convenient for later use to work in local coördinates (and here we use that M is real-analytic):

$$
\begin{array}{c}
(z, s^1, t^1, s^2, t^2, \ldots, s^{n-1}, t^{n-1}) \in \mathbb{C} \times \mathbb{R}^{2n-2} \\
\big\downarrow \eta \\
(z, w^1, w^2, \ldots, w^{n-1}) = (z, s^1 + zt^1, s^2 + zt^2 \ldots, s^{n-1} + zt^{n-1}) \in \mathbb{C}^n.
\end{array}
$$

Charts of this form cover Σ, the image of each being a double wedge in Ω with edge in M. Writing $z = x + iy$, the hypersurface Σ is defined in these coördinates by $y = 0$. The bundle $\Lambda^{1,0}_{\Omega}$ is spanned by $dz, dw^1, dw^2, \ldots, dw^{n-1}$ so $\Lambda^{1,0}_{\widetilde{\Omega}}$ is spanned by

$$
dz, ds^1 + z\,dt^1 + t^1 dz, ds^2 + z\,dt^2 + t^2 dz, \ldots, ds^{n-1} + z\,dt^{n-1} + t^{n-1} dz
$$

which are manifestly linearly independent. $\square$

Dually, the bundle $T^{0,1}$ on $\widetilde{\Omega}$ is spanned by the commuting vector fields

$$
\overline{Z} = \frac{\partial}{\partial \bar{z}} \qquad \text{and} \qquad \overline{W}_j = \frac{\partial}{\partial t^j} - z \frac{\partial}{\partial s^j} \quad \text{for } j = 1, \ldots, n-1. \tag{1}
$$

These vector fields extend the Cauchy-Riemann equations on $\widetilde{\Omega} \setminus \Sigma$ across Σ. On Σ the vector fields $\overline{W}_j$ are real and span $T^{0,1} \cap \overline{T^{0,1}}$. They generate the fibres of $\eta : \Sigma \to M$.

To any real smooth hypersurface in a smooth manifold, there is a canonically associated locally constant line bundle. Write $\widetilde{\mathcal{E}}$ for this bundle associated to $\Sigma \subset \widetilde{\Omega}$. We recall its construction. It is defined by transition functions taking the values ± 1. Choose a covering of Σ by open sets U_α in $\widetilde{\Omega}$ with Σ defined as the jump discontinuity of a chosen Heaviside function H_α (for example, $\operatorname{sign} y$ in local coördinates as above). Add $\widetilde{\Omega} \setminus \Sigma$ as a final set to give a covering of $\widetilde{\Omega}$. Now specify transition functions:

- on $U_\alpha \cap (\widetilde{\Omega} \setminus \Sigma)$ use H_α
- on $U_\alpha \cap U_\beta$ use H_α / H_β.

By definition, $\widetilde{\mathcal{E}}$ is trivial on $\widetilde{\Omega} \setminus \Sigma$ and we write H as the trivialising section (i.e. the function which is identically 1 in our chosen trivialisation). When restricted to a fibre $\eta^{-1}(m)$ for $m \in M$, the bundle $\widetilde{\mathcal{E}}$ becomes the standard bundle of twisted functions on $\mathbb{RP}_{n-1}$ whose sections correspond to odd functions on S^{n-1}.

Let V be a holomorphic vector bundle on Ω and denote by $\mathcal{O}(V)$ (respectively $\mathcal{E}(V)$) its sheaf of germs of holomorphic (respectively smooth) sections. The cohomology spaces $H^r(\Omega, \mathcal{O}(V))$ will be realised as Dolbeault cohomology. From this point of view the bundle V is equipped with a $\overline{\partial}$-operator, the initial operator in the usual complex:

$$\mathcal{E}(V) \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1}(V) \xrightarrow{\overline{\partial}} \mathcal{E}^{0,2}(V) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{0,n}(V) \to 0.$$

An involutive structure is all that is needed to define a complex of $\overline{d}$-operators

$$\mathcal{E} = \mathcal{A}^{0,0} \xrightarrow{\overline{d}} \mathcal{A}^{0,1} \xrightarrow{\overline{d}} \mathcal{A}^{0,2} \xrightarrow{\overline{d}} \cdots \tag{2}$$

where $\mathcal{A}^{0,r}$ is the sheaf of germs of smooth sections of the r^{th} exterior power of $\Lambda^{0,1} = \Lambda^1/\Lambda^{1,0}$. This is true on $\widetilde{\Omega}$ where the operators may also be coupled with $\eta^* V$ (a bundle compatible with the involutive structure in the sense of [4]). Let $\widetilde{V} = \widetilde{\mathcal{E}} \otimes \eta^* V$. Since $\widetilde{\mathcal{E}}$ is given by locally constant transition functions, this bundle is also compatible with the involutive structure and we obtain a complex

$$\Gamma(\widetilde{\Omega}, \mathcal{A}^{0,0}(\widetilde{V})) \xrightarrow{\overline{d}} \Gamma(\widetilde{\Omega}, \mathcal{A}^{0,1}(\widetilde{V})) \xrightarrow{\overline{d}} \Gamma(\widetilde{\Omega}, \mathcal{A}^{0,2}(\widetilde{V})) \xrightarrow{\overline{d}} \cdots \tag{3}$$

whose cohomology we shall denote by $H^r_{\overline{d}}(\widetilde{\Omega}, \widetilde{V})$. (Warning: there is no reason to suppose these are sheaf cohomologies.) The main purpose of this lecture is to prove the following:

Theorem 1 *There is an exact sequence*

$$0 \to \Gamma(\Omega, \mathcal{O}(V)) \to \Gamma(M, \mathcal{E}(V)) \to H^1_{\overline{d}}(\widetilde{\Omega}, \widetilde{V}) \to H^1(\Omega, \mathcal{O}(V)) \to 0$$

and isomorphisms

$$H^r_{\overline{d}}(\widetilde{\Omega}, \widetilde{V})) \cong H^r(\Omega, \mathcal{O}(V)) \quad \textit{for } r \geq 2$$

except for $r = n-1$ and $r = n$ when n is odd in which case there is an exact sequence

$$0 \to H^{n-1}(\Omega, \mathcal{O}(V)) \to H^{n-1}_{\overline{d}}(\widetilde{\Omega}, \widetilde{V}) \to \Gamma(M, \mathcal{E}(V)) \to H^n(\Omega, \mathcal{O}(V)) \to H^n_{\overline{d}}(\widetilde{\Omega}, \widetilde{V}) \to 0.$$

The proof proceeds by the following Lemmata.

Lemma 1 *The complex*

$$0 \to \Gamma(\Omega, \mathcal{E}^{0,0}(V)) \xrightarrow{\overline{\partial}} \Gamma(\Omega, \mathcal{E}^{0,1}(V)) \xrightarrow{\overline{\partial}} \Gamma(\Omega, \mathcal{E}^{0,2}(V)) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Gamma(\Omega, \mathcal{E}^{0,n}(V)) \to 0$$

is formally exact along M except in the zeroth position where the formal cohomology is $\Gamma(M, \mathcal{E}(V))$.

Proof. To say that the complex is formally exact at $\Gamma(\Omega, \mathcal{E}^{0,r}(V))$ is to say that if $\omega \in \Gamma(\Omega, \mathcal{E}^{0,r}(V))$ satisfies $d\omega \sim 0$ (meaning that $d\omega$ vanishes to infinite order along M), then there exists $\nu \in \Gamma(\Omega, \mathcal{E}^{0,r-1}(V))$ so that $d\nu \sim \omega$ (meaning that $\omega - d\nu$ vanishes to infinite order along M). To say that the zeroth formal cohomology is $\Gamma(M, \mathcal{E}(V))$ is to say that

- every smooth section f_0 of V along M may be extended to a smooth section f of V over Ω so that $\overline{\partial} f \sim 0$
- if $g \in \Gamma(\Omega, \mathcal{E}(V))$ satisfies $\overline{\partial} g \sim 0$ and $g|_M = 0$, then $g \sim 0$.

(This is the well-known manœuvre of taking an almost analytic extension.)

Let us check the zeroth cohomology first. Choose local coördinates $z^j = x^j + iy^j$ with M given by $y^1 = y^2 = \cdots = y^n = 0$. The vector fields

$$\overline{\partial}_j = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)$$

are transverse to M so, in these coördinates, we may define the extension of a function off M to infinite order by specifying all derivatives $\overline{\partial}_j \cdots \overline{\partial}_k f$ along M. These may be chosen arbitrarily (provided they are symmetric in the indices $j \cdots k$) and, in particular, we may take $f|_M = f_0$ and all higher derivatives zero. This is precisely a local version of what we want to prove in the case when V is trivial. A standard partition of unity argument gives the global result and also allows non-trivial V.

For higher cohomology first use the zeroth result to extend any partition of unity on M to Ω so that it is $\overline{\partial}$-closed to infinite order along M. This reduces to the local case where we can use the coördinates above. Thus, we are given

$$\omega_{\underbrace{kl\cdots m}_{r}} = \omega_{[kl\cdots m]} \text{ such that } \overline{\partial}_h \cdots \overline{\partial}_i \overline{\partial}_{[j} \omega_{kl\cdots m]}|_M = 0$$

and we are required to specify $\overline{\partial}_h \cdots \overline{\partial}_i \overline{\partial}_j \overline{\partial}_k \nu_{l\cdots m}$ on M, skew in the indices $l \cdots m$ so that

$$\overline{\partial}_h \cdots \overline{\partial}_i \overline{\partial}_j \overline{\partial}_{[k} \nu_{l\cdots m]}|_M = \overline{\partial}_h \cdots \overline{\partial}_i \overline{\partial}_j \omega_{kl\cdots m}|_M.$$

This is easily accomplished by setting $\nu_{l\cdots m}|_M = 0$ and

$$\underbrace{\overline{\partial}_h \cdots \overline{\partial}_i \overline{\partial}_j \overline{\partial}_k}_{d} \nu_{l\cdots m}|_M = \frac{rd}{r+d-1} \overline{\partial}_{(h} \cdots \overline{\partial}_i \overline{\partial}_j \omega_{k)l\cdots m}|_M \quad \text{for } d \geq 1.$$

□

For the next lemma, recall the local coördinates introduced on $\widetilde{\Omega}$ and the vector field $\overline{Z}$ constructed from them. Let us write Σ_0 and $\widetilde{\Omega}_0$ for the charts of Σ and $\widetilde{\Omega}$ covered by this coördinate system.

Lemma 2 *Given a smooth function f_0 on Σ_0 and a smooth function g on $\widetilde{\Omega}_0$, there is a smooth solution f on $\widetilde{\Omega}_0$ to*

$$\overline{Z} f \sim g \quad \textit{with} \quad f|_{\Sigma_0} = f_0$$

and f is formally unique along Σ. If f_0 has compact support then f can be chosen to have compact support.

Proof. This is a formal non-characteristic Cauchy problem for a complex vector field and is proved inductively by power series calculation. More generally, the Cauchy-Kovalevski theorem, usually stated in the real-analytic category, is true on the level of formal power series. □

Lemma 3 *Let $\widetilde{\mathbb{C}}$ denote the sheaf of locally constant complex-valued twisted functions on $\mathbb{RP}_{n-1}$.*

$$\text{Then } \begin{cases} H^r(\mathbb{RP}_{n-1},\widetilde{\mathbb{C}}) = 0 & \text{for all } r \text{ except} \\ H^{n-1}(\mathbb{RP}_{n-1},\widetilde{\mathbb{C}}) = \mathbb{C} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $\Gamma_{\mathrm{odd}}(S^{n-1},\Lambda^r)$ denote the smooth r-forms on S^{n-1} satisfying $\iota^*\omega = -\omega$ where ι is the antipodal mapping. Then

$$H^r(\mathbb{RP}_{n-1},\widetilde{\mathbb{C}}) = H^r(\Gamma_{\mathrm{odd}}(S^{n-1},\Lambda^\bullet))$$

and the result follows from the de Rham theorem on S^{n-1}. Notice that the mapping $H^{n-1}(\mathbb{RP}_{n-1},\widetilde{\mathbb{C}}) \to \mathbb{C}$ may be defined by integrating a twisted volume form over $\mathbb{RP}_{n-1}$ (a well-defined procedure when n is odd though $\mathbb{RP}_{n-1}$ is not orientable). □

Lemma 4 *The complex*

$$0 \to \Gamma(\widetilde{\Omega},\mathcal{A}^{0,0}(\widetilde{V})) \xrightarrow{\overline{d}} \Gamma(\widetilde{\Omega},\mathcal{A}^{0,1}(\widetilde{V})) \xrightarrow{\overline{d}} \Gamma(\widetilde{\Omega},\mathcal{A}^{0,2}(\widetilde{V})) \xrightarrow{\overline{d}} \cdots \xrightarrow{\overline{d}} \Gamma(\widetilde{\Omega},\mathcal{A}^{0,n}(\widetilde{V})) \to 0$$

is formally exact along Σ if n is even. If n is odd it is formally exact except at $\Gamma(\widetilde{\Omega},\mathcal{A}^{0,n-1}(\widetilde{V}))$ where the formal cohomology may be identified with $\Gamma(M,\mathcal{E}(V))$.

Proof. We use the vector fields (1) to write out the $\overline{d}$-operator in local coördinates. An r-form ω may be realised as a pair

$$\alpha_{\underbrace{kl\cdots m}_{r-1}} = \alpha_{[kl\cdots m]} \quad \beta_{\underbrace{jkl\cdots m}_{r}} = \beta_{[jkl\cdots m]}$$

(dual to $\overline{Z}\wedge\overline{W}_k\wedge\overline{W}_l\wedge\cdots\wedge\overline{W}_m$ and $\overline{W}_j\wedge\overline{W}_k\wedge\overline{W}_l\wedge\cdots\wedge\overline{W}_m$, respectively) and $\overline{d}\omega \sim 0$ means

$$\overline{W}_{[j}\alpha_{kl\cdots m]} \sim \overline{Z}\beta_{jkl\cdots m} \quad\text{and}\quad \overline{W}_{[i}\beta_{jkl\cdots m]} \sim 0$$

whilst $\omega \sim d\nu$ reads

$$\alpha_{kl\cdots m} \sim \overline{Z}\delta_{kl\cdots m} - \overline{W}_{[k}\gamma_{l\cdots m]} \quad\text{and}\quad \beta_{jkl\cdots m} \sim \overline{W}_{[j}\delta_{kl\cdots m]}$$

for suitable $\gamma_{l\cdots m} = \gamma_{[l\cdots m]}$ and $\delta_{kl\cdots m} = \delta_{[kl\cdots m]}$. Recall that the vector fields $\overline{W}_j$ are tangent on Σ to the fibres of η. The forms β and δ expressed above in local coördinates have a well-defined meaning on Σ as relative r- and $(r-1)$-forms respectively. On each fibre $\eta^{-1}(m) \cong \mathbb{RP}_{n-1}$ for $m \in M$, the bundle η^*V plays no rôle whilst the bundle $\widetilde{\mathcal{E}}$ is the bundle of twisted functions. From Lemma 3, we may always solve for δ on Σ unless n is odd and $r = n-1$. Having found $\delta|_\Sigma$ we may take γ to be identically zero, using Lemma 2 and a partition of unity to extend δ off Σ so that

$$\overline{Z}\delta_{kl\cdots m} \sim \alpha_{kl\cdots m}.$$

Since the vector fields commute

$$\begin{array}{rcl}\overline{Z}(\overline{W}_{[j}\delta_{kl\cdots m]}-\beta_{jkl\cdots m}) & = & \overline{W}_{[j}\overline{Z}\delta_{kl\cdots m]}-\overline{Z}\beta_{jkl\cdots m}\\ & \sim & \overline{W}_{[j}\alpha_{kl\cdots m]}-\overline{Z}\beta_{jkl\cdots m}\\ & \sim & 0\end{array}$$

so, by the formal uniqueness in Lemma 2, $\overline{W}_{[j}\delta_{kl\cdots m]}\sim\beta_{jkl\cdots m}$, as required. Finally, when n is odd and $r=n-1$, the obstruction to carrying out this reasoning is precisely the integral of β along the fibres of η. This obstruction lies in $\Gamma(M,\mathcal{E}(V))$. □

Proof of Theorem 1. Let $\Gamma(M\subset\Omega,\mathcal{E}^{0,\bullet}(V))$ denote the subcomplex of $\Gamma(\Omega,\mathcal{E}^{0,\bullet}(V))$ consisting of those V-valued $(0,r)$-forms on Ω which vanish to infinite order along M. Write $\Gamma_{[M]}(\Omega,\mathcal{E}^{0,\bullet}(V))$ for the quotient so that

$$0\longrightarrow\Gamma(M\subset\Omega,\mathcal{E}^{0,\bullet}(V))\longrightarrow\Gamma(\Omega,\mathcal{E}^{0,\bullet}(V))\longrightarrow\Gamma_{[M]}(\Omega,\mathcal{E}^{0,\bullet}(V))\longrightarrow 0\qquad(4)$$

is an exact sequence of complexes. Lemma 1 says that

$$H^r(\Gamma_{[M]}(\Omega,\mathcal{E}^{0,\bullet}(V)))=\begin{cases}\Gamma(M,\mathcal{E}(V)) & \text{if } r=0\\ 0 & \text{if } r\geq 1.\end{cases}$$

In a similar vein, Lemma 4 says that

$$H^r(\Gamma_{[\Sigma]}(\widetilde{\Omega},\mathcal{A}^{0,\bullet}(\widetilde{V})))=\begin{cases}0 & \text{for all } r \text{ except}\\ \Gamma(M,\mathcal{E}(V)) & \text{for } r=n-1 \text{ when } n \text{ is odd.}\end{cases}$$

The long exact sequence derived from (4) gives, by Lemma 1,

$$\begin{array}{l}0\to H^0(\Gamma(M\subset\Omega,\mathcal{E}^{0,\bullet}(V)))\to\Gamma(\Omega,\mathcal{O}(V))\to\Gamma(M,\mathcal{E}(V))\\ \qquad\qquad\swarrow\\ \qquad\qquad H^1(\Gamma(M\subset\Omega,\mathcal{E}^{0,\bullet}(V)))\to H^1(M,\mathcal{O}(V))\to 0\end{array}$$

and isomorphisms

$$H^r(\Gamma(M\subset\Omega,\mathcal{E}^{0,\bullet}(V)))\cong H^r(\Omega,\mathcal{O}(V))\quad\text{for all } r\geq 2.$$

Now $H^0(\Gamma(M\subset\Omega,\mathcal{E}^{0,\bullet}(V)))$ is precisely the holomorphic sections of V which vanish (to infinite order) along M. Since Ω is connected, these vanish identically. Therefore, we have the exact sequence

$$0\to\Gamma(\Omega,\mathcal{O}(V))\to\Gamma(M,\mathcal{E}(V))\to H^1(\Gamma(M\subset\Omega,\mathcal{E}^{0,\bullet}(V)))\to H^1(M,\mathcal{O}(V))\to 0.$$

Similarly, Lemma 4 gives isomorphisms

$$H^r(\Gamma(\Sigma\subset\widetilde{\Omega},\mathcal{A}^{0,\bullet}(\widetilde{V})))\cong H^{\frac{r}{d}}(\widetilde{\Omega},\widetilde{V})$$

for all r except for $r=n-1$ and $r=n$ when n is odd in which case there is an exact sequence

$$\begin{array}{l}0\to H^{n-1}(\Gamma(\Sigma\subset\widetilde{\Omega},\mathcal{A}^{0,\bullet}(\widetilde{V})))\to H^{\frac{n-1}{d}}(\widetilde{\Omega},\widetilde{V})\to\Gamma(M,\mathcal{E}(V))\\ \qquad\qquad\swarrow\\ \qquad\qquad H^n(\Gamma(\Sigma\subset\widetilde{\Omega},\mathcal{A}^{0,\bullet}(\widetilde{V})))\to H^{\frac{n}{d}}(\widetilde{\Omega},\widetilde{V})\to 0.\end{array}$$

We make the following observations

- A smooth function on Ω vanishing to infinite order along M is equivalent to a smooth function on $\widetilde{\Omega}$ vanishing to infinite order along Σ.
- If f is a smooth function on $\widetilde{\Omega}$ vanishing to infinite order along Σ, then Hf is a smooth twisted function on $\widetilde{\Omega}$ vanishing to infinite order along Σ and vice versa.

Here, recall that $H \in \Gamma(\widetilde{\Omega}, \widetilde{\mathcal{E}})$ is the canonically defined trivialising section. These observations apply equally well to sections of a vector bundle on Ω and its pull back to $\widetilde{\Omega}$. They combine to give an isomorphism of complexes

$$\Gamma(M \subset \Omega, \mathcal{E}^{0,\bullet}(V)) \xrightarrow{\simeq\ \times H} \Gamma(\Sigma \subset \widetilde{\Omega}, \mathcal{A}^{0,\bullet}(\widetilde{V}))$$

and, hence, induced isomorphisms on cohomology. Substituting into the various exact sequences above completes the proof. □

Corollary 1 *The spaces $H^r_{\bar{d}}(\widetilde{\Omega}, \widetilde{V})$ are finite-dimensional except for $r = 1$ when n is arbitrary and for $r = n - 1$ when n is odd.*

It is interesting to compare this with the Hodge theory discussed by Hanges and Jacobowitz [12]. The involutive structure on $\widetilde{\Omega}$ is elliptic except on Σ but the Levi form is degenerate unless $n = 2$. In this case $\widetilde{\Omega}$ is Example V_2 on page 501 of [12] save for a change of coördinates and the Levi form has one positive and one negative eigenvalue. Then [12, Theorem on p. 501] implies that the Laplacian is hypoelliptic with one loss of derivative on 0-forms and 2-forms and standard arguments [12, pp. 509–510] imply that the cohomology is finite-dimensional in degree 0 and 2.

Lecture Two

Geometry on the Correspondence Space

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The prototype for our discussions will be the X-ray transform introduced by John [13] in 1938. A compactified version of this transform starts with a smooth function f on the three-sphere S^3 and for each plane P through the origin in $\mathbb{R}^4$, integrates f over the geodesic $\gamma = P \cap S^3$:

$$\phi(P) = \frac{1}{2\pi} \oint_\gamma f.$$

The smooth function ϕ is defined on the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$ of two-planes in $\mathbb{R}^4$ and satisfies a second order linear differential equation, the ultrahyperbolic wave equation. In a closely related case, by arguing that both spaces are irreducible under an action of $\mathrm{SL}(4, \mathbb{R})$, Guillemin and Sternberg [11] conclude that the X-ray transform is an isomorphism:

$$\left\{ \begin{array}{l} \text{Smooth even} \\ \text{functions on } S^3 \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{l} \text{Smooth even solutions of the} \\ \text{ultrahyperbolic wave equation on } \mathrm{Gr}(\mathbb{R}^4) \end{array} \right\}. \quad (5)$$

Woodhouse [19] reaches the same conclusion by expanding both side in (generalised) spherical harmonics. John's original article [13] proves a corresponding result for sufficiently smooth functions on $\mathbb{R}^3$ subject to decay conditions at infinity. In these lectures we intend to prove (5) by complex methods.

The X-ray transform may be viewed in terms of the real correspondence

$$\begin{array}{ccc} & \mathbb{RF} = \text{The real flag manifold of lines in planes in } \mathbb{R}^4 & \\ \swarrow & & \searrow \\ \mathbb{RP}_3 & & \mathrm{Gr}_2(\mathbb{R}^4). \end{array}$$

Properties of the Penrose transform may be deduced (as, for example, in [8]) by analysing the complex correspondence

$$\begin{array}{ccc} & \mathbb{CF} = \text{The complex flag manifold of lines in planes in } \mathbb{C}^4 & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \mathbb{CP}_3 & & \mathrm{Gr}_2(\mathbb{C}^4). \end{array}$$

To analyse the X-ray transform we shall use a halfway house

$$\begin{array}{ccc} & F & \\ {}^{\eta}\swarrow & & \searrow^{\tau} \\ \mathbb{RP}_3 \subset \mathbb{CP}_3 & & \mathrm{Gr}_2(\mathbb{R}^4) \end{array} \tag{6}$$

where $F \equiv \nu^{-1}(\mathrm{Gr}_2(\mathbb{R}^4))$, $\tau \equiv \nu|_F$, and $\eta \equiv \mu|_F$. The fibres of ν are $\mathbb{CP}_1$'s so the same is true of τ. However, η is no longer a fibration:

Proposition 2 *The mapping η is the real blow-up of $\mathbb{CP}_3$ along $\mathbb{RP}_3$.*

Proof. Let us view the complex correspondence in local coördinates on $\mathbb{CF}$. A generic line inside a generic plane in $\mathbb{C}^4$ may be written as

$$\mathrm{span}\begin{pmatrix} 1 \\ z \\ s^1 + zt^1 \\ s^2 + zt^2 \end{pmatrix} \subset \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ s^1 \\ s^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ t^1 \\ t^2 \end{pmatrix}\right\}$$

and $(z, s^1, t^1, s^2, t^2) \in \mathbb{C}^5$ give standard affine coördinates on $\mathbb{CF}$. The mapping μ is then given by

$$(z, s^1, t^1, s^2, t^2) \longmapsto (z, s^1 + zt^1, s^2 + zt^2)$$

using standard affine coördinates on $\mathbb{CP}_3$. The submanifold F is where s^1, t^1, s^2, t^2 are real. This is the coördinate system used in Lecture One for a real blow-up. $\square$

Since the fibres of τ are complex, we may consider the bundle $\Lambda_\tau^{0,1}$ of relative $(0,1)$-forms. The complex structure on each fibre is induced from its image in $\mathbb{CP}_3$ under η. Therefore, the composition $\eta^*\Lambda_{\mathbb{CP}_3}^{1,0} \to \Lambda_F^1 \to \Lambda_\tau^{0,1}$ is zero and we obtain a well-defined homomorphism of vector bundles

$$\Lambda_F^{0,1} \longrightarrow \Lambda_\tau^{0,1}$$

where $\Lambda_F^{0,1}$ defines the involutive structure of Lecture One. We shall verify soon that this is a surjection. In other words, the complex structure on the fibres of τ may be obtained by restricting the involutive structure on F.

Let ι denote the inclusion $\iota : F \hookrightarrow \mathbb{CF}$. Then $\eta = \mu \circ \iota$ so $\eta^*\Lambda_{\mathbb{CP}_3}^{1,0} = \iota^*(\mu^*\Lambda_{\mathbb{CP}_3}^{1,0})$. Thus, the composition

$$\iota^*\Lambda_{\mathbb{CF}}^{1,0} \longrightarrow \Lambda_F^1 \longrightarrow \Lambda_F^{0,1}$$

kills $\mu^*\Lambda_{\mathbb{CP}_3}^{1,0}|_F$. Hence, there is a well-defined homomorphism

$$\Lambda_\mu^{1,0}|_F \longrightarrow \Lambda_F^{0,1}$$

where $\Lambda_\mu^{1,0}$ denotes the bundle $\Lambda_{\mathbb{CF}}^{1,0}/\mu^*\Lambda_{\mathbb{CP}_3}^{1,0}$ of $(1,0)$-forms along the fibres of μ. Let us write $\Lambda_\eta^{1,0}$ for $\Lambda_\mu^{1,0}|_F$. We have constructed the following homomorphisms

$$0 \to \Lambda_\eta^{1,0} \longrightarrow \Lambda_F^{0,1} \longrightarrow \Lambda_\tau^{0,1} \to 0. \tag{7}$$

Proposition 3 *The sequence* (7) *is exact.*

Proof. In our usual local coördinates on F,

$$\begin{aligned}
\Lambda_\eta^{1,0} &= \frac{\operatorname{span}\{dz, ds^1, dt^1, ds^2, dt^2\}}{\operatorname{span}\{dz, ds^1 + z\,dt^1, ds^2 + z\,dt^2\}} \\
&\cong \operatorname{span}\{dt^1, dt^2\} \\
\Lambda_F^{0,1} &= \frac{\operatorname{span}\{dz, d\bar{z}, ds^1, dt^1, ds^2, dt^2\}}{\operatorname{span}\{dz, ds^1 + z\,dt^1, ds^2 + z\,dt^2\}} \\
&\cong \operatorname{span}\{d\bar{z}, dt^1, dt^2\} \\
\Lambda_\tau^{0,1} &= \frac{\operatorname{span}\{d\bar{z}, ds^1, dt^1, ds^2, dt^2\}}{\operatorname{span}\{ds^1, dt^1, ds^2, dt^2\}} \\
&\cong \operatorname{span}\{d\bar{z}\}
\end{aligned}$$

with the obvious mappings. The result follows. □

Let us see how this fits with the general real blow-up $\eta : \widetilde{\Omega} \to \Omega$ along M as in Lecture One. The involutive structure on $\widetilde{\Omega}$ is sufficient to define holomorphic curves in $\widetilde{\Omega}$. We shall suppose that we are given a foliation of $\widetilde{\Omega}$ by such curves. Then we may simply define $\Lambda_\eta^{1,0}$ so that (7) is exact. In order to proceed further we shall also suppose that the space of leaves of the foliation is a smooth manifold X. In other words, there is a submersion $\tau : \widetilde{\Omega} \to X$ whose fibres are the leaves of the foliation.

Finally, let us consider how (7) fits with the involutive $\overline{d}$-complex (2). It induces a filtering of this complex. Abusing notation somewhat, and allowing the fibres of τ an arbitrary dimension,

$$
\begin{array}{ccccccccc}
\mathcal{A}^{0,0} & = & \mathcal{E} & & & & & & \\
\downarrow{\scriptstyle \overline{d}} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & & & & & \\
\mathcal{A}^{0,1} & = & \mathcal{E}^{0,1}_\tau & + & \mathcal{E}(\Lambda^{1,0}_\eta) & & & & \\
\downarrow{\scriptstyle \overline{d}} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & & & \\
\mathcal{A}^{0,2} & = & \mathcal{E}^{0,2}_\tau & + & \mathcal{E}^{0,1}_\tau(\Lambda^{1,0}_\eta) & + & \mathcal{E}(\Lambda^{2,0}_\eta) & & \\
\downarrow{\scriptstyle \overline{d}} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & \\
\mathcal{A}^{0,3} & = & \mathcal{E}^{0,3}_\tau & + & \mathcal{E}^{0,2}_\tau(\Lambda^{1,0}_\eta) & + & \mathcal{E}^{0,1}_\tau(\Lambda^{2,0}_\eta) & + & \mathcal{E}(\Lambda^{3,0}_\eta) \\
\downarrow{\scriptstyle \overline{d}} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & \downarrow{\scriptstyle \overline{\partial}_\tau} & & \downarrow{\scriptstyle \overline{\partial}_\tau} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array} \tag{8}
$$

where $\Lambda^{p,0}_\eta = \Lambda^p(\Lambda^{1,0}_\eta)$, $\Lambda^{0,q}_\tau = \Lambda^q(\Lambda^{0,1}_\tau)$, and $\mathcal{E}^{0,q}_\tau = \mathcal{E}(\Lambda^{0,q}_\tau)$, the sheaf of germs of smooth sections of $\Lambda^{0,q}_\tau$. The right hand sides in this diagram merely indicate the composition factors defined by the filtering. There are induced differential operators

$$\overline{\partial}_\tau : \mathcal{E}^{0,q}_\tau(\Lambda^{p,0}_\eta) \longrightarrow \mathcal{E}^{0,q+1}_\tau(\Lambda^{p,0}_\eta).$$

For $p = 0$, this is the Dolbeault resolution along the fibres of τ. We shall write Œ for the sheaf it resolves. It is the sheaf of germs of smooth functions which are holomorphic along the fibres of τ and we shall refer to such functions as partially holomorphic. Similarly, we shall write $Œ^p_\eta$ for the sheaf of germs of partially holomorphic sections of $\Lambda^{p,0}_\eta$. We may view the partially holomorphic structure on $\Lambda^{p,0}_\eta$ as arising from the diagram (8). Alternatively, in the X-ray case, $\Lambda^{p,0}_\eta$ is obtained by restricting the holomorphic bundle $\Lambda^{1,0}_\mu$ on $\mathbb{CF}$ to F. Of course, any holomorphic vector bundle on $\mathbb{CF}$ will be partially holomorphic when restricted to F.

Lecture Three

Pushing Down

T.N. Bailey M.G. Eastwood

Lecture Two ended with

$$\begin{array}{ccc} & \widetilde{\Omega} & \\ {\scriptstyle\eta}\swarrow & & \searrow{\scriptstyle\tau} \\ M\subset\Omega & & X \end{array}$$

where

- Ω is a complex manifold.
- M is a totally real, real-analytic submanifold.
- $\widetilde{\Omega}$ is the real blow-up of Ω along M endowed with the involutive structure of Lecture One.
- X is a smooth manifold.
- $\tau:\widetilde{\Omega}\to X$ is a submersion with complex fibres of complex dimension one.

From now on we shall assume that Ω, and hence $\widetilde{\Omega}$, X, and the fibres of τ, are compact. The prototype is (6).

Suppose V is a holomorphic vector bundle on Ω and recall the complex (3). Combined with the short exact sequence (7), we obtain a spectral sequence

$$E_0^{p,q}=\Gamma(\widetilde{\Omega},\mathcal{E}_\tau^{0,q}(\Lambda_\eta^{p,0}\otimes\widetilde{V}))\Longrightarrow H_{\overline{d}}^{p+q}(\widetilde{\Omega},\widetilde{V}).$$

This is the E_0-level of the spectral sequence associated with the filtered complex (8). It will enable us to interpret the involutive cohomology spaces $H_{\overline{d}}^{r}(\widetilde{\Omega},\widetilde{V})$ down on X. This interpretation follows exactly the general method explained by Bailey [1] in relation to the Penrose transform. The same methods were used by Schmid [16] in 1967 in relating the discrete series representations realised as Dolbeault cohomology on G/T to a realisation as solutions of his $\mathcal{D}$-operator on G/K. More recently, similar techniques appear in the theses of Singer [17] and Wong [20].

It is easy to pass to the E_1-level. The differentials at the E_0-level are the $\overline{\partial}_\tau$-operators. Thus,

Theorem 2 *There is a spectral sequence*

$$E_1^{p,q}=\Gamma(X,\tau_*^q\text{Œ}_\eta^p(\widetilde{V}))\Longrightarrow H_{\overline{d}}^{p+q}(\widetilde{\Omega},\widetilde{V}).$$

(Those familiar with the Penrose transform will recognise this spectral sequence from its counterpart there (for example, [5, p. 308], [6, Theorem 7.3.1], or [8, Theorem 4.1]).) Notice that since the fibres of τ are compact, Dolbeault cohomology spaces along these fibres will be finite-dimensional. Thus, provided this dimension is constant, the direct images $\tau^q_*\text{Œ}^p_\eta(\widetilde{V})$ define smooth complex vector bundles on X and the differentials of the spectral sequence define differential operators between them.

A general machine for analysing the X-ray transform and various other real integral transforms arises by combining Theorems 1 and 2. The only remaining task is to identify the direct images $\tau^q_*\text{Œ}^p_\eta(\widetilde{V})$ and the induced differential operators. The remainder of this lecture will be concerned with carrying out this task for the X-ray transform.

Proposition 4 *Let* $\widetilde{\text{Œ}}$ *denote the sheaf of germs of partially holomorphic twisted functions on* F. *Then* $\tau_*\widetilde{\text{Œ}}$ *is the bundle of twisted functions on* $\mathrm{Gr}_2(\mathbb{R}^4)$ *whose sections correspond to odd functions on* $\mathrm{Gr}_2^+(\mathbb{R}^4)$, *the Grassmannian of oriented two-planes in* $\mathbb{R}^4$. *All higher direct images vanish.*

Proof. Each fibre of τ is a Riemann sphere and intersects Σ in a circle. We obtain two hemispheres and choosing one of them is equivalent to choosing a Heaviside function with jump discontinuity across the circle. By definition, this trivialises $\widetilde{\text{Œ}}$. A point in $\mathrm{Gr}_2^+(\mathbb{R}^4)$ orients the circle or, equivalently, specifies a preferred hemisphere. $\square$

The upshot of this proposition is that the direct images differ from the usual Penrose transform only in being twisted. As usual, we shall denote this extra twist by adding a tilde to the standard notation on $\mathrm{Gr}_2(\mathbb{R}^4)$. The induced differential operators are exactly as in the Penrose transform (being well-defined with the twisting accomplished by locally constant transition functions).

Example 1 Let V be the line-bundle $\mathcal{O}(-2)$ on $\mathbb{CP}_3$. The relevant direct images are (see, for example, [5, Table 2] or [6, p. 99])

$$\tau^1_*\widetilde{\text{Œ}}(-2) = \widetilde{\mathcal{E}}[-1] \qquad \tau_*\widetilde{\text{Œ}}^2_\eta(-2) = \widetilde{\mathcal{E}}[-3]$$

and all others vanish. Therefore, the E_1-level of the spectral sequence is

$$\begin{array}{ccccc}
| & & & & \\
\Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-1]) & & 0 & & 0 \\
| & & & & \\
0 & \text{———————} & 0 & \text{————} & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-3]) \ —
\end{array}$$

and, following one of the standard arguments for the Penrose transform (for example, [8, Theorem 6.1]), the ultrahyperbolic wave operator $\square$ emerges as the only differential at the E_2-level. Theorem 2 gives

$$\begin{aligned}
H^1_{\bar d}(F,\widetilde{\mathcal{O}}(-2)) &= \ker\square : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-1]) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-3]) \\
H^2_{\bar d}(F,\widetilde{\mathcal{O}}(-2)) &= \operatorname{coker}\square : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-1]) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-3])
\end{aligned}$$

and all other cohomology vanishes. On the other hand, $H^r(\mathbb{CP}_3,\mathcal{O}(-2)) = 0$ for all r (for example, by the Bott-Borel-Weil Theorem). Thus, by Theorem 1,

$$\begin{aligned}
\Gamma(\mathbb{RP}_3,\mathcal{E}(-2)) &\xrightarrow{\cong} \ker\square : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-1]) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-3]) \\
\Gamma(\mathbb{RP}_3,\mathcal{E}(-2)) &\cong \operatorname{coker}\square : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-1]) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-3]).
\end{aligned}$$

The first isomorphism is (5) proved by complex methods. The second is a compactified version of a result of Grinberg [10, Theorem 3] indentifying the range of $\Box$ as those elements of $\Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}[-3])$ whose integrals over all α-planes vanish.

Example 2 Let V be the trivial line-bundle. Then

$$\tau_*\widetilde{\text{Œ}} = \widetilde{\mathcal{E}} \qquad \tau_*\widetilde{\text{Œ}}^1_\eta = \widetilde{\mathcal{E}}^1 \qquad \tau_*\widetilde{\text{Œ}}^2_\eta = \widetilde{\mathcal{E}}^2_+$$

where $\widetilde{\mathcal{E}}^2_+$ is the sheaf of germs of twisted self-dual 2-forms on $\mathrm{Gr}_2(\mathbb{R}^4)$. All higher direct images vanish. Therefore, the E_1-level of the spectral sequence is

$$\begin{array}{ccccc}
| & & & & \\
0 & & 0 & & 0 \\
| & & & & \\
\Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}) & \longrightarrow & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}^1) & \longrightarrow & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}^2_+) \longrightarrow
\end{array}$$

the differentials being exterior derivatives. This time, there is a contribution from Theorem 1 in that $\Gamma(\mathbb{CP}_3, \mathcal{O}) = \mathbb{C}$. We conclude that there is a surjective X-ray transform

$$\Gamma(\mathbb{RP}_3, \mathcal{E}) \longrightarrow \frac{\ker d_+ : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}^1) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}^2_+)}{\operatorname{im} d : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}^1)} \tag{9}$$

whose kernel is the constant functions. The right hand side may be re-intepreted as follows. In view of the well-known diffeomorphism $\mathrm{Gr}_2^+(\mathbb{R}^4) \simeq \mathrm{S}^2 \times \mathrm{S}^2$, the twisted de Rham cohomology of $\mathrm{Gr}_2(\mathbb{R}^4)$ is easily computed (compare Lemma 3):

$$H^2(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$$

and all others vanish. A closed twisted anti-self-dual 2-form on $\mathrm{Gr}_2(\mathbb{R}^4)$ is zero when restricted to any α-plane and its integral over a β-plane is independent of choice of this plane. Let us call this its charge. If the charge is zero then we can find a potential. In other words, the right hand side of (9) may be re-interpreted as the space of twisted anti-self-dual Maxwell fields with zero charge. The second involutive cohomology gives an isomorphism

$$\Gamma(\mathbb{RP}_3, \mathcal{E}) \cong \operatorname{coker} d_+ : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}^1) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}^2_+),$$

parallel to Grinberg's result in Example 1.

Example 3 Let V be the line-bundle $\mathcal{O}(-4)$ on $\mathbb{CP}_3$. It may be regarded as the bundle of holomorphic 3-forms on $\mathbb{CP}_3$ with its restriction to $\mathbb{RP}_3$ being the bundle of smooth 3-forms. The relevant direct images are

$$\tau^1_*\widetilde{\text{Œ}}(-4) = \widetilde{\mathcal{E}}^2_+ \qquad \tau^1_*\widetilde{\text{Œ}}^1_\eta(-4) = \widetilde{\mathcal{E}}^3 \qquad \tau^1_*\widetilde{\text{Œ}}^2_\eta(-4) = \widetilde{\mathcal{E}}^4$$

and all others vanish. Theorem 2 gives

$$\begin{array}{c|ccc} & & & \\ & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^2_+) & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^3) & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^4) \\ & & & \\ \hline & 0 & 0 & 0 \end{array}$$

at the E_1-level. The immediate conclusion is

$$\Gamma(\mathbb{RP}_3,\mathcal{E}(-4)) \xrightarrow{\cong} \ker d : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^2_+) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^3).$$

In other words, the space of smooth 3-forms on $\mathbb{RP}_3$ is isomorphic under the X-ray transform to the space of twisted self-dual Maxwell fields on $\mathrm{Gr}_2(\mathbb{R}^4)$ (as in [11]). One can easily check that the integral of the 3-form over $\mathbb{RP}_3$ gives the charge of the corresponding field. The Bott-Borel-Weil Theorem identifies $H^3(\mathbb{CP}_3,\mathcal{O}(-4)) = \mathbb{C}$ and the remaining conclusion of Theorem 1 is an exact sequence

$$0 \to H^2_{\bar{d}}(F,\widetilde{\mathcal{O}}(-4)) \longrightarrow \Gamma(\mathbb{RP}_3,\mathcal{E}(-4)) \xrightarrow{\int} \mathbb{C} \longrightarrow H^3_{\bar{d}}(F,\widetilde{\mathcal{O}}(-4)) \to 0.$$

The mapping $\int$ is the integral over $\mathbb{RP}_3$ and is surjective. The consequent vanishing of $H^3_{\bar{d}}(F,\widetilde{\mathcal{O}}(-4))$ is consistent with the surjectivity of $\Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^3) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^4)$, namely the vanishing of the twisted de Rham cohomology $H^4(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathbb{C}})$. The final conclusion is an isomorphism of the smooth 3-forms on $\mathbb{RP}_3$ whose total integral is zero with the space

$$\frac{\ker d : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^3) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^4)}{\operatorname{im} d : \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^2_+) \to \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^3)}.$$

Example 4 Let V be Λ^1, the cotangent bundle on $\mathbb{CP}_3$. From the Bott-Borel-Weil Theorem, $H^1(\mathbb{CP}_3,\mathcal{O}(\Lambda^1)) = \mathbb{C}$ and all other cohomology vanishes. Thus, Theorem 1 gives an exact sequence

$$0 \to \Gamma(\mathbb{RP}_3,\mathcal{E}(\Lambda^1)) \longrightarrow H^1_{\bar{d}}(\widetilde{\Omega},\widetilde{\Lambda}^1) \longrightarrow \mathbb{C} \to 0.$$

The non-zero direct images are

$$\tau^1_*\text{Œ}(\widetilde{\Lambda}^1) = \widetilde{\mathcal{E}} \qquad \tau_*\text{Œ}^1_\eta(\widetilde{\Lambda}^1) = \widetilde{\mathcal{E}}^2_- \oplus \widetilde{\mathcal{E}}[-2] \qquad \tau_*\text{Œ}^2_\eta(\widetilde{\Lambda}^1) = \widetilde{\mathcal{E}}[-2] \oplus \widetilde{\mathcal{E}}^3$$

and the spectral sequence of Theorem 2 is (cf. [5, p. 309] or [6, p. 104])

$$\begin{array}{c|ccc} & & & \\ & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}) & 0 & 0 \\ & & & \\ & & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^2_-) & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-2]) \\ \hline & 0 & \oplus & \oplus \\ & & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}[-2]) & \Gamma(\mathrm{Gr}_2(\mathbb{R}^4),\widetilde{\mathcal{E}}^3) \end{array}$$

Consequently, there is an X-ray transform

$$\mathcal{X} : \Gamma(\mathbb{RP}_3, \mathcal{E}(\Lambda^1)) \longrightarrow \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}})$$

whose range is

$$\{f \in \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}) \text{ s.t. } \mathcal{D}\mathrm{df} = \mathrm{d}\rho \text{ for some } \rho \in \Gamma(\mathrm{Gr}_2(\mathbb{R}^4), \widetilde{\mathcal{E}}^2_-)\}$$

where $\mathcal{D} : \widetilde{\mathcal{E}}^1 \to \widetilde{\mathcal{E}}^3$ is the operator of [9, Proposition 3.3]. (To establish this carefully requires a good deal of further argument, cf. [14].) The kernel of $\mathcal{X}$ is identified with the twisted anti-self-dual Maxwell fields of zero charge. Alternatively, in view of Example 2, the kernel consists of the exact 1-forms. The transform $\mathcal{X}$ is geometrically extremely natural. A point of $\mathrm{Gr}_2^+(\mathbb{R}^4)$ gives an oriented geodesic on $\mathbb{RP}_3$ over which a smooth 1-form may be integrated. Changing the orientation changes the sign of this integral so there results a twisted smooth function on $\mathrm{Gr}_2(\mathbb{R}^4)$. From this interpretation it is clear that $\mathcal{X}$ annihilates the exact forms. The converse is a result of Michel [15]. It generalises considerably [2]. That one can identify the range of $\mathcal{X}$ in terms of differential equations on $\mathrm{Gr}_2(\mathbb{R}^4)$ is, perhaps, more subtle.

Example 5 An X-ray transform can be constructed and analysed starting with any irreducible homogeneous vector bundle on $\mathbb{CP}_3$. The corresponding exercise for the Penrose transform is carried out in [7] and a complete analysis in the X-ray case would follow similar lines. Certainly, the necessary direct images are computed in [7]. The other ingredient, needed to compute the analytic cohomology in $\mathbb{CP}_3$ which appears in Theorem 1, is the Bott-Borel-Weil Theorem—see, for example, [6].

Concluding Remarks

The X-ray transform considered in this section could, in principle, be analysed by decomposing the spaces in question into harmonics under the action of SO(4). On the other hand, the main analytical ingredient involved in applying Theorems 1 and 2 is the Bott-Borel-Weil Theorem, either on $\mathbb{CP}_3$ or on the fibres of τ, each isomorphic to $\mathbb{CP}_1$. The Bott-Borel-Weil Theorem can itself be proved using (generalised) spherical harmonics. Thus, one possible point of view on this approach to the X-ray transform is that the geometry of (6) is automatically organising the spherical harmonics to effect the analysis. Presumably, a similar point of view can be adopted in any situation with a sufficient degree of symmetry. Of course, the Bott-Borel-Weil Theorem is an extremely efficient method of organising these harmonics. It would be interesting to find some less symmetrical examples where the analytic cohomology of Theorem 1 is beyond the Bott-Borel-Weil Theorem. Certainly, the Penrose transform goes well beyond this.

There are two limitations inherent in this development. The first comes about by using a real blow-up. Here one replaces a totally real submanifold M of a complex manifold with its space of normal directions. This is the right thing to do for a real integral transform defined by integrals over one-dimensional real cycles. For cycles of dimension k one would have to modify the blow-up procedure so as to replace M with its space of normal k-planes. Presumably, this is not a serious impediment. A tighter restriction seems to be the requirement of M having a compact complexification with the real cycles in M inducing a foliation of the blow-up by complex submanifolds. At the moment it is far from clear how stringent a requirement this is. Many more examples are needed to illuminate this issue.

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