Conformal circles in flat space

Useful quantities and formulæ

- η_{ab} = flat metric
- ∂_a = flat connection
- γ = curve in flat space
- $t : \gamma \to \mathbb{R}$ a smooth parameterisation
- U^a a vector field along γ such that $U^a\partial_a t \equiv 1$
- $\partial \equiv U^a\partial_a$
- $\hat{\eta}_{ab} = \Omega^2\eta_{ab}$ also supposed flat with connection $\hat{\partial}_a$ and $\therefore$ $\boxed{\partial_a\Upsilon_b = \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon.\Upsilon\eta_{ab}}$ where $\Upsilon_a \equiv \Omega^{-1}\partial_a\Omega$.
- $\hat{\partial}_a V^b = \partial_a V^b + \Upsilon_a V^b + V.\Upsilon\delta_a{}^b - V_a\Upsilon^b$ for any vector field V^b
- $\hat{\partial} V^b = \partial V^b + U.\Upsilon V^b + V.\Upsilon U^b - U.V\Upsilon^b$ for any vector field V^b
- $A^b \equiv \partial U^b$
- $\hat{A}^b = A^b + 2U.\Upsilon U^b - U.U\Upsilon^b$
- $C^b \equiv 2U.U\partial A^b - 6U.AA^b + 3A.AU^b$

We compute

$$U.\hat{A} = U.A + U.UU.\Upsilon$$

and

$$\begin{aligned}\hat{A}.\hat{A} &= A.A + 4U.U(U.\Upsilon)^2 + (U.U)^2\Upsilon.\Upsilon + 4U.\Upsilon U.A - 2U.UA.\Upsilon - 4U.U(U.\Upsilon)^2 \\ &= A.A + (U.U)^2\Upsilon.\Upsilon + 4U.\Upsilon U.A - 2U.UA.\Upsilon\end{aligned}$$

and

$$\begin{aligned}\hat{\partial}\hat{A}^b &= \partial\hat{A}^b + U.\Upsilon\hat{A}^b + \hat{A}.\Upsilon U^b - U.\hat{A}\Upsilon^b \\ &= \partial(A^b + 2U.\Upsilon U^b - U.U\Upsilon^b) \\ &\quad + U.\Upsilon(A^b + 2U.\Upsilon U^b - U.U\Upsilon^b) \\ &\quad\quad + (A^a + 2U.\Upsilon U^a - U.U\Upsilon^a)\Upsilon_a U^b \\ &\quad\quad\quad - U_a(A^a + 2U.\Upsilon U^a - U.U\Upsilon^a)\Upsilon^b \\ &= \partial A^b + 2A.\Upsilon U^b + 2U.\partial\Upsilon U^b + 2U.\Upsilon A^b - 2A.U\Upsilon^b - U.U\partial\Upsilon^b \\ &\quad + U.\Upsilon(A^b + 2U.\Upsilon U^b - U.U\Upsilon^b) \\ &\quad\quad + (A.\Upsilon + 2U.\Upsilon U.\Upsilon - U.U\Upsilon.\Upsilon)U^b \\ &\quad\quad\quad - (A.U + U.UU.\Upsilon)\Upsilon^b \\ &= \partial A^b \\ &\quad + 3U.\Upsilon A^b + (3A.\Upsilon + 2U.\partial\Upsilon + 4(U.\Upsilon)^2 - U.U\Upsilon.\Upsilon)U^b \\ &\quad\quad - (3A.U + 2U.UU.\Upsilon)\Upsilon^b - U.U\partial\Upsilon^b\end{aligned}$$

and

$$\begin{aligned}U.\hat{A}\hat{A}^b &= (U.A + U.UU.\Upsilon)(A^b + 2U.\Upsilon U^b - U.U\Upsilon^b) \\ &= U.AA^b \\ &\quad + U.UU.\Upsilon A^b + 2(U.AU.\Upsilon + U.U(U.\Upsilon)^2)U^b - U.U(U.A + U.UU.\Upsilon)\Upsilon^b.\end{aligned}$$

Therefore,

$$\begin{aligned} U.U\hat{\partial}\hat{A}^b - 3U.\hat{A}\hat{A}^b &= U.U\partial A^b - 3U.AA^b \\ &\quad + (U.U(3A.\Upsilon + 2U.\partial\Upsilon - 2(U.\Upsilon)^2 - U.U\Upsilon.\Upsilon) - 6(U.AU.\Upsilon))U^b \\ &\quad\quad + (U.U)^2 U.\Upsilon\Upsilon^b - (U.U)^2\partial\Upsilon^b \end{aligned}$$

whereas

$$\hat{A}.\hat{A} = A.A + (U.U)^2\Upsilon.\Upsilon + 4U.\Upsilon U.A - 2U.UA.\Upsilon.$$

Therefore,

$$\begin{aligned} 2U.U\hat{\partial}\hat{A}^b - 6U.\hat{A}\hat{A}^b + 3\hat{A}.\hat{A}U^b &= 2U.U\partial A^b - 6U.AA^b + 3A.AU^b \\ &\quad + U.U(4U.\partial\Upsilon - 4(U.\Upsilon)^2 + U.U\Upsilon.\Upsilon)U^b \\ &\quad\quad + 2(U.U)^2U.\Upsilon\Upsilon^b - 2(U.U)^2\partial\Upsilon^b. \end{aligned}$$

In other words

$$\begin{aligned} \hat{C}^b &= C^b + U.U(4U.\partial\Upsilon - 4(U.\Upsilon)^2 + U.U\Upsilon.\Upsilon)U^b + 2(U.U)^2U.\Upsilon\Upsilon^b - 2(U.U)^2\partial\Upsilon^b \\ &= C^b + 2U.U(2U.\partial\Upsilon - 2(U.\Upsilon)^2 + U.U\Upsilon.\Upsilon)U^b - (U.U)^2(2\partial\Upsilon^b - 2U.\Upsilon\Upsilon^b + \Upsilon.\Upsilon U^b) \\ &= C^b + 2U.UU^a(2\partial\Upsilon_a - 2(U.\Upsilon)\Upsilon_a + \Upsilon.\Upsilon U_a)U^b - (U.U)^2(2\partial\Upsilon^b - 2U.\Upsilon\Upsilon^b + \Upsilon.\Upsilon U^b) \\ &= C^b + U.U(2\partial\Upsilon_a - 2(U.\Upsilon)\Upsilon_a + \Upsilon.\Upsilon U_a)(2U^aU^b - U.U\eta^{ab}) \end{aligned}$$

but recall that $\partial_a\Upsilon_b = \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon.\Upsilon\eta_{ab}$ so

$$2\partial\Upsilon_a - 2(U.\Upsilon)\Upsilon_a + \Upsilon.\Upsilon U_a = 0 \quad \text{whence } \boxed{\hat{C}^b = C^b}.$$

In particular, the equation $C^b = 0$ is invariant under flat-to-flat conformal transformations.

MikE, 23rd December 2010