Some brief notes by Michael Eastwood on

Conformal Differential Geometry

Honours/PhD Special Topics Math3349/6209

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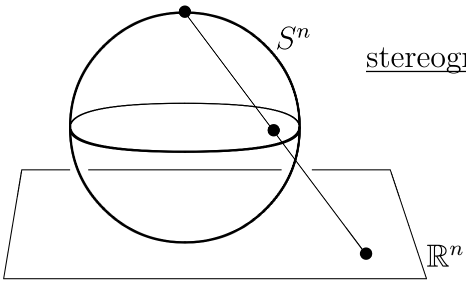
There are many 'map projections' [4]. All are distorting in some way but one can preserve:–

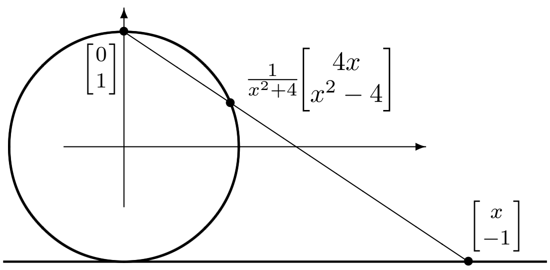
- geodesics (gnomomic (Thales circa 590 B.C.)),
- areas (Archimedes circa 240 B.C., Lambert 1772),
- angles (Mercator 1569, stereographic (Halley 1695)).

Map projections that preserves angles are called 'conformal' and we shall be concerned with



stereographic projection

One can see that this is conformal either by pure thought [6] or by calculus, as follows. Notice that



so inverse stereographic projection (in arbitrary dimension) is given by

$$\mathbb{R}^n \ni x \stackrel{\phi}{\longmapsto} \frac{1}{\|x\|^2+4}\begin{bmatrix}4x\\ \|x\|^2-4\end{bmatrix} \in S^n \subset \begin{array}{c}\mathbb{R}^n\\ \oplus\\ \mathbb{R}\end{array} = \mathbb{R}^{n+1}$$

and we may calculate the $(n+1)\times n$ matrix (the Jacobian)

$$\phi'(x) = \frac{4}{(\|x\|^2+4)^2}\begin{bmatrix}(\|x\|^2+4)\mathrm{Id}_{n\times n} - 2\,x\,x^t\\ 4x^t\end{bmatrix}. \tag{1}$$

We may use $\phi'(x)$ to map vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^{n+1}$ tangent to S^n at $\phi(x)$ according to $X \mapsto \phi'(x)X$. The inner product between two such vectors in $\mathbb{R}^{n+1}$ is $\langle \phi'(x)X, \phi'(x)Y\rangle = (\phi'(x)X)^t(\phi'(x)Y)$. However,

$$\begin{aligned}
&\begin{bmatrix} (\|x\|^2+4)\mathrm{Id}_{n\times n} - 2\,x\,x^t \\ 4x^t \end{bmatrix}^t \begin{bmatrix} (\|x\|^2+4)\mathrm{Id}_{n\times n} - 2\,x\,x^t \\ 4x^t \end{bmatrix} \\
&= \begin{bmatrix} (\|x\|^2+4)\mathrm{Id}_{n\times n} - 2\,x\,x^t, 4x \end{bmatrix} \begin{bmatrix} (\|x\|^2+4)\mathrm{Id}_{n\times n} - 2\,x\,x^t \\ 4x^t \end{bmatrix}. \\
&= (\|x\|^2+4)^2\mathrm{Id}_{n\times n} - 4(\|x\|^2+4)\,x\,x^t + 4\|x\|^2\,x\,x^t + 16\,x\,x^t \\
&= (\|x\|^2+4)^2\mathrm{Id}_{n\times n}
\end{aligned}$$

Therefore, from (1) we find

$$(\phi'(x))^t\phi'(x) = \frac{16}{(\|x\|^2+4)^2}\mathrm{Id}_{n\times n} = \Omega^2(x)\mathrm{Id}_{n\times n},$$

where

$$\Omega(x) = \frac{4}{\|x\|^2+4} = \frac{1}{1+\|x/2\|^2}.$$

It follows that

$$\langle \phi'(x)X, \phi'(x)Y\rangle_{\mathbb{R}^{n+1}} = X^t(\phi'(x))^t\phi'(x)Y = \Omega^2(x)\langle X, Y\rangle_{\mathbb{R}^n}. \tag{2}$$

Finally, observe that the formula for the angle θ between X and Y,

$$\cos\theta = \frac{\langle X, Y\rangle}{\sqrt{\langle X, X\rangle\langle Y, Y\rangle}}$$

does not see a rescaling of the form (2), no matter what is $\Omega(x)$.

The Mercator projection from the two-sphere to the cylinder may be obtained by following stereographic projection with the complex logarithm $z \mapsto \log z$ (where they are defined). All complex analytic functions are conformal in two dimensions.

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Smooth surfaces S in $\mathbb{R}^3$ may locally be viewed, according to the implicit function theorem [10], in three equivalent ways:–

- implicit $S = \{f = 0\}$, where $df|_S \neq 0$,
- explicit $S = \{z = F(x, y)\}$ for some choice of coördinates,
- parametric $S = \phi(U)$, for $\mathbb{R}^2 \supset^{\text{open}} U \xrightarrow{\phi} \mathbb{R}^3$ with $\operatorname{rank}\phi' = 2$.

Gauss initiated the study of such surfaces under 'Euclidean motions.' The group of Euclidean motions in $\mathbb{R}^n$ comprises transformations of the form

$$x \mapsto Ax + b, \text{ where } A \in \mathrm{SO}(n) \text{ and } b \in \mathbb{R}^n.$$

Here, $\mathrm{SO}(n)$ denotes the 'special orthogonal group' comprising $n \times n$ matrices A satisfying $A^t A = \mathrm{Id}$ and $\det A = 1$. Noting that

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix},$$

we may view the group of Euclidean motions as a subgroup of the group of invertible $(n+1) \times (n+1)$ matrices. In particular, there are two natural subgroups

$$\mathrm{SO}(n) \cong \left\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathbb{R}^n \cong \left\{ \begin{bmatrix} \mathrm{Id} & b \\ 0 & 1 \end{bmatrix} \right\}$$

of which the latter is normal. In other words, the group of Euclidean motions is a 'semi-direct product' [9], usually written as $\mathrm{SO}(n) \ltimes \mathbb{R}^n$. Elements from these two subgroups are referred to as 'rotations' and 'translations,' respectively.

The naïve approach to surfaces $S \hookrightarrow \mathbb{R}^3$ under Euclidean motions is to move them into some normal form from which one can read off some useful information. Evidently, one can spend the translation freedom to move a given point on the surface to the origin and then partially spend the rotation freedom to align the unit normal (let's suppose the surface is oriented so that it has a preferred unit normal) with the z-axis. In other words, we may arrange that S is given explicitly as

$$z = x^t Q x + \ldots,$$

in some orthogonal coördinate system (x^1, x^2, z) where Q is a symmetric 2×2 matrix and the ellipsis $\ldots$ denotes cubic and higher order terms. The remaining coördinate freedom is rotation about the z-axis and may be spent in orthogonally diagonalising Q. We therefore arrive at a defining equation for S of the form

$$z = \lambda_1 (x^1)^2 + \lambda_2 (x^2)^2 + \ldots$$

where the only remaining ambiguity is to swop $\lambda_1 \leftrightarrow \lambda_2$. Therefore, we are led to the invariant quantities

- mean curvature $H \equiv \lambda_1 + \lambda_2$,
- Gauss curvature $K \equiv \lambda_1 \lambda_2$.

In fact, one can back-track a little to obtain more 'effective' expressions

$$H \equiv \operatorname{trace} Q \quad \text{and} \quad K \equiv \det Q.$$

$\boxed{\text{Gauss' Theorema Egregium[1]: the quantity } K \text{ is intrinsic to } S.}$!!

We shall now explain what this theorem means and how to prove it in a manner that introduces all sorts of useful geometric machinery.

[1]Remarkable Theorem

Firstly, a precise statement of Theorema Egregium. In the derivation above, the quantities H and K were defined at a point but now let us regard them as smooth functions on S. Let us write S parametrically by means of $\phi : U \to \mathbb{R}^3$ for some open subset $U \subseteq \mathbb{R}^3$. We obtain a function on U with values in the 2×2 symmetric matrices

$$g(x) \equiv \left(\phi'(x)\right)^t \phi'(x) \tag{3}$$

and Theorema Egregium says that the function $K(x)$ can be obtained solely from $g(x)$. The geometric interpretation of this statement, that '$K(x)$ is intrinsic to S,' emerges because the matrix $g(x)$ records exactly the inner product on the tangent space to S at $\phi(x)$:–

- Tangent space at $\phi(x) \in S \;=\; \{\phi'(x)X \text{ for } X \in \mathbb{R}^2\}$,
- $\langle \phi'(x)X, \phi'(x)Y\rangle_{\mathbb{R}^3} = X^t g(x)Y$.

In other words, the matrix $g(x)$ is what beings living on S can measure. They may need the help of 'local coördinates' $\phi(x)$ for $x \in U \subseteq \mathbb{R}^2$ to write down $g(x)$ as a matrix but the remarkable fact is that they can then compute the function $K(x)$ for themselves, without knowing how S sits inside $\mathbb{R}^3$. In particular, its value at any point on S will be independent of choice of local coördinates near that point.

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At this point it is not beyond the pale to prove Theorema Egregium by brute force calculation. Specifically, if we write

$$g(x) = g(x^1, x^2) = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

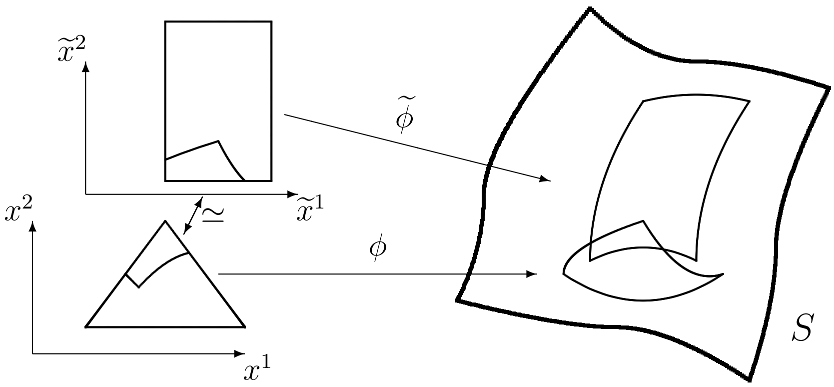
for smooth functions $E = E(x)$, $F = F(x)$, $G = G(x)$, and we write partial derivatives $\partial E/\partial x^a$ as E_a et cetera, then

$$K = \frac{\left[\begin{array}{l} 2(EG-F^2)(2F_{12}-E_{22}-G_{11}) \\ \quad + F(4F_1F_2 - 2F_1G_1 - 2F_2E_2 - E_2G_1 + E_1G_2) \\ \qquad + E({G_1}^2 - 2F_1G_2 + E_2G_2) + G({E_2}^2 - 2E_1F_2 + E_1G_1) \end{array}\right]}{16(EG-F^2)^2}.$$

This sheds absolutely no light on what's really going on here!

To discover the true picture (Riemann, Levi-Civita, Ricci,...) let's consider what happens to the matrix $g(x)$ if we change coördinates. The plan is to incorporate the result into a definition (of Riemannian manifold) and then work solely with this definition to construct some natural machinery in which freedom from any further choices (such as a system of coördinates) is manifest.

The picture is like this



and, where it makes sense, $\widetilde{\phi}(\widetilde{x}) = \phi(x)$ (and $x = (x^1, x^2)$ is regarded as a function of $\widetilde{x} = (\widetilde{x}^1, \widetilde{x}^2)$). By the chain rule

$$\widetilde{\phi}'(\widetilde{x}) = \phi'(x)\frac{\partial x}{\partial \widetilde{x}}, \quad \text{where } \frac{\partial x}{\partial \widetilde{x}} \equiv \begin{bmatrix} \partial x^1/\partial \widetilde{x}^1 & \partial x^1/\partial \widetilde{x}^2 \\ \partial x^2/\partial \widetilde{x}^1 & \partial x^2/\partial \widetilde{x}^2 \end{bmatrix}.$$

Therefore, the matrix $g(x)$ changes as follows.

$$\widetilde{g}(\widetilde{x}) = \big(\widetilde{\phi}'(\widetilde{x})\big)^t \widetilde{\phi}'(\widetilde{x}) = \left(\phi'(x)\frac{\partial x}{\partial \widetilde{x}}\right)^t \phi'(x)\frac{\partial x}{\partial \widetilde{x}} = \left(\frac{\partial x}{\partial \widetilde{x}}\right)^t g(x)\frac{\partial x}{\partial \widetilde{x}}. \tag{4}$$

Henceforth, we shall regard a parameterisation $\mathbb{R}^2 \supseteq U \xrightarrow{\phi} S$ as giving a 'system of local coördinates' (x^1, x^2) on S. Then for each such local coördinate system, the matrix-valued function $g(x)$ defined by (3) may be written out in terms of its components $g(x) = \big(g_{ab}(x)\big)$, which are themselves simply smooth functions on S (defined only where the local coördinates are themselves defined). In terms of these components, the transformation (4) reads

$$\widetilde{g}_{ab}(\widetilde{x}) = \sum_{c=1}^{2}\sum_{d=1}^{2} \frac{\partial x^c}{\partial \widetilde{x}^a} g_{cd}(x) \frac{\partial x^d}{\partial \widetilde{x}^b}.$$

We will abbreviate expressions such as this by omitting the summation sign and instead using the 'Einstein summation convention,' namely that repeated indices implicitly require that they be summed. Also, when matrix multiplication is written out explicitly like this, one can freely reorder the expression. In summary, (4) may be written as

$$\boxed{\widetilde{g}_{ab} = g_{cd}\frac{\partial x^c}{\partial \widetilde{x}^a}\frac{\partial x^d}{\partial \widetilde{x}^b}} \tag{5}$$

where all quantities are regarded as smooth function on S. A <u>smooth manifold</u> is defined by abstractly gluing together open subsets of $\mathbb{R}^n$ by smooth coördinate changes (to obtain a Hausdorff topological space).

Using (5) as a prototype, there are many <u>tensors</u> that can similarly be defined on an arbitrary smooth manifold M either operationally or conceptually as follows (supposing some knowledge of vector bundles).

	in Local Coördinates	as Vector Bundle
vector field	$X^a \dfrac{\partial}{\partial x^a} \qquad \widetilde{X}^a = X^b \dfrac{\partial \widetilde{x}^a}{\partial x^b}$	$\begin{array}{l} TM \\ \ \downarrow \!\uparrow X \\ \ M \end{array}$
1-form	$\omega_a \, dx^a \qquad \widetilde{\omega}_a = \omega_b \dfrac{\partial x^b}{\partial \widetilde{x}^a}$	$\begin{array}{l} T^*M \\ \ \downarrow \!\uparrow \omega \\ \ M \end{array}$ or $\omega \in \Lambda^1$
2-form	$\omega_{ab} = -\omega_{ba} \quad \widetilde{\omega}_{ab} = \omega_{cd} \dfrac{\partial x^c}{\partial \widetilde{x}^a} \dfrac{\partial x^d}{\partial \widetilde{x}^b}$	$\omega \in \Lambda^2$
metric	$g_{ab} = g_{ba} \qquad \widetilde{g}_{ab} = g_{cd} \dfrac{\partial x^c}{\partial \widetilde{x}^a} \dfrac{\partial x^d}{\partial \widetilde{x}^b}$ and (g_{ab}) is positive definite	$g \in \bigodot^2 \Lambda^1$ (+ve def)

Also notice that the <u>exterior derivative</u> on functions

$$\Lambda^0 \ni f \longmapsto df \equiv \frac{\partial f}{\partial x^a} \, dx^a$$

and the action of vector fields on functions

$$Xf \equiv X^a \frac{\partial f}{\partial X^a} = X \lrcorner \, df$$

are coördinate-free and, therefore, intrinsically defined on any smooth manifold.

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On $\mathbb{R}^n$ there are some useful operations that can be defined using the standard coördinates, namely

$$X^b \mapsto \partial_a X^b \equiv \frac{\partial X^b}{\partial x^a} \quad \text{and} \quad \omega_b \mapsto \partial_a \omega_b \equiv \frac{\partial \omega_b}{\partial x^a} \tag{6}$$

on vector fields and 1-forms respectively. In fact, these are really the same operation. Specifically, if we use the standard Euclidean metric and its inverse on $\mathbb{R}^n$

$$\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \qquad \delta^{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

to identify vector fields and 1-forms according to

$$\omega_a = \delta_{ab} X^b \qquad X^a = \delta^{ab} \omega_b,$$

then the operations (6) clearly coincide.

Remarkably, the differential operators (6) on $\mathbb{R}^n$ (and the fact that they agree under the natural identification of vector fields and 1-forms) generalise to any Riemannian manifold. Indeed, this is the key to a completely intrinsic proof of Theorema Egregium. Before setting up this generalisation, however, here is an example that shows the utility of these operations on $\mathbb{R}^n$.

The operations (6) certainly depend on choice of coördinates. Let's see this explicitly on 1-forms. Suppose $\widetilde{x}^a$ is another choice of local coördinates and recall that $\widetilde{\omega}_a = \omega_b(\partial x^b/\partial\widetilde{x}^a)$. We compute

$$\begin{aligned}\widetilde{\partial}_a\widetilde{\omega}_b &= \frac{\partial}{\partial\widetilde{x}^a}\left(\omega_c\frac{\partial x^c}{\partial\widetilde{x}^b}\right) = \frac{\partial\omega_d}{\partial\widetilde{x}^a}\frac{\partial x^d}{\partial\widetilde{x}^b} + \omega_c\,\frac{\partial^2 x^c}{\partial\widetilde{x}^a\partial\widetilde{x}^b}\\ &= \frac{\partial\omega_d}{\partial x^c}\frac{\partial x^c}{\partial\widetilde{x}^a}\frac{\partial x^d}{\partial\widetilde{x}^b} + \omega_c\,\frac{\partial^2 x^c}{\partial\widetilde{x}^a\partial\widetilde{x}^b} = \partial_c\omega_d\frac{\partial x^c}{\partial\widetilde{x}^a}\frac{\partial x^d}{\partial\widetilde{x}^b} + \omega_c\,\frac{\partial^2 x^c}{\partial\widetilde{x}^a\partial\widetilde{x}^b}.\end{aligned}$$

In particular, notice that $\partial^2x^c/\partial\widetilde{x}^a\partial\widetilde{x}^b$ is symmetric in a and b. Hence,

$$\widetilde{\partial}_{[a}\widetilde{\omega}_{b]} = \partial_{[c}\omega_{d]}\frac{\partial x^c}{\partial\widetilde{x}^a}\frac{\partial x^d}{\partial\widetilde{x}^b}$$

where $\partial_{[a}\omega_{b]} \equiv \frac{1}{2}(\partial_a\omega_b - \partial_b\omega_a)$ denotes the skew part of $\partial_a\omega_b$. But this is exactly the operational definition of a 2-form and we have constructed the exterior derivative, an intrinsic differential operator $d : \Lambda^1 \to \Lambda^2$ on any smooth manifold. It continues with the de Rham complex

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{n-1} \xrightarrow{d} \Lambda^n \tag{7}$$

intrinsically defined on any smooth n-manifold where the operators d in local coördinates are given by

$$\Lambda^p \ni \omega_{b\cdots d} \mapsto \partial_{[a}\omega_{b\cdots c]} \in \Lambda^{p+1}$$

where, again, square brackets $[\cdots]$ mean to take the skew part over the indices they enclose. We have generalised the familiar operations of grad, curl, and div on $\mathbb{R}^3$ to coördinate-free operators on an arbitrary smooth manifold. The sequence (7) is a complex meaning that the composition of any two consecutive operators is zero.

Just as $n \times n$ matrices naturally split into symmetric and skew parts

$$A = \tfrac{1}{2}(A + A^t) + \tfrac{1}{2}(A - A^t) \tag{8}$$

so it is natural to consider the symmetric part of $\partial_a\omega_b$, namely

$$\partial_{(a}\omega_{b)} \equiv \tfrac{1}{2}(\partial_a\omega_b + \partial_b\omega_a),$$

having seen above that this operator is very much tied to $\mathbb{R}^n$ with its standard coördinates and Euclidean metric. It is the Killing operator on $\mathbb{R}^n$ and its solutions enjoy a geometric interpretation, namely that the corresponding vector field be an infinitesimal Euclidean isometry. We shall come back to this geometric interpretation shortly, but it is already illuminating to solve $\partial_{(a}\omega_{b)} = 0$ explicitly as follows. It is clear that

$$\partial_{(a}\omega_{b)} = 0 \iff F_{ab} \equiv \partial_a\omega_b \text{ is skew.}$$

Observe that

$$K_{abc} \equiv \partial_a F_{bc} = \partial_a\partial_b\omega_c$$

is skew in bc since F_{bc} is and symmetric in ab by the equality of mixed partial derivatives. The following algebraic observation is well-known.

Lemma 1. *Suppose K_{abc} is symmetric in ab and skew in bc. Then it vanishes.*

Proof. We compute

$$K_{abc} = K_{bac} = -K_{bca} = -K_{cba} = K_{cab} = -K_{acb} = -K_{abc}$$

so $K_{abc} = 0$, as advertised. □

Thus, we have shown that the Killing equation is equivalent to the following system

$$\begin{array}{rcl} \partial_a\omega_b &=& F_{ab} \\ \partial_a F_{bc} &=& 0 \end{array} \qquad \text{for } F_{ab} = F_{[ab]}, \tag{9}$$

which is closed in the sense that all the partial derivatives of all of the unknown functions ω_b and F_{bc} are determined in terms of the functions themselves. It is straightforward to solve, firstly for F_{ab}, which must be constant, and then for ω_a to conclude that

$$\omega_a = s_a + m_{ab}x^b \text{ for constant tensors } s_a \text{ and } m_{ab} = -m_{ba}.$$

We shall see later how these solutions correspond to Euclidean motions.

There is a more refined splitting of $n \times n$ matrices

$$A = \left(\tfrac{1}{2}(A + A^t) - \tfrac{1}{n}(\text{trace}A)\text{Id}\right) + \tfrac{1}{2}(A - A^t) + \tfrac{1}{n}(\text{trace}A)\text{Id}$$

into a symmetric trace-free part, a skew part, and a pure-trace part. Correspondingly, a weaker version of the Killing equation known as the conformal Killing equation is obtained by requiring only that the trace-free symmetric part of $\partial_a\omega_b$ vanish. We shall write this equation as $\partial_{(a}\omega_{b)}{}^\circ = 0$. Immediately, it may be rewritten as

$$\partial_a\omega_b = F_{ab} + \Lambda\delta_{ab}, \text{ where } F_{ab} \text{ is skew.}$$

Like the Killing equation, this equation has a geometric interpretation. As the name suggests, it is that the corresponding vector field should be an infinitesimal conformal symmetry. Pending a full explanation of this interpretation, let us try to solve it as we did for the Killing equation (the general approach being known as 'prolongation'). As above, we consider

$$K_{abc} \equiv \partial_a \partial_b \omega_c = \partial_a F_{bc} + (\partial_a \Lambda)\delta_{bc},$$

noting that this tensor is symmetric in ab and has the property that $K_{a(bc)} = L_a \delta_{bc}$ for some L_a. Here is the counterpart of Lemma 1.

Lemma 2. *Suppose K_{abc} satisfies*

$$K_{[ab]c} = 0 \quad \textit{and} \quad K_{a(bc)} = L_a \delta_{bc} \ \textit{for some } L_a.$$

Then $K_{abc} = L_a \delta_{bc} + L_b \delta_{ac} - L_c \delta_{ab}$.

Proof. Consider

$$\widehat{K}_{abc} = K_{abc} - (L_a \delta_{bc} + L_b \delta_{ac} - L_c \delta_{ab}),$$

noting that $\widehat{K}_{abc}$ satisfies the hypotheses of Lemma 1. □

Lemma 2 shows that

$$\partial_a F_{bc} = (\partial_b \Lambda)\delta_{ac} - (\partial_c \Lambda)\delta_{ab},$$

which we may rewrite as

$$\partial_a F_{bc} = \delta_{ab} Q_c - \delta_{ac} Q_b, \quad \text{where } \partial_a \Lambda = -Q_a.$$

Now observe that

$$K_{abcd} \equiv \partial_a \partial_b \partial_c \omega_d = \partial_a \partial_b F_{cd} + (\partial_a \partial_b \Lambda)\delta_{cd}$$

is symmetric in abc and has the property that $K_{ab(cd)} = L_{ab}\delta_{cd}$ for some tensor L_{ab}, namely $L_{ab} = \partial_a \partial_b \Lambda = -\partial_a Q_b$.

Lemma 3. *Suppose K_{abcd} is symmetric in abc and satisfies*

$$K_{ab(cd)} = L_{ab}\delta_{cd} \ \textit{for some } L_{ab} \ \textit{(necessarily symmetric).}$$

If $n \geq 3$, then $K_{abcd} = 0$.

Proof. Lemma 2 implies that $K_{abcd} = L_{ab}\delta_{cd} + L_{ac}\delta_{bd} - L_{ad}\delta_{bc}$ and because K_{abcd} is symmetric in ab we conclude that

$$0 = 2K_{[ab]cd} = L_{ac}\delta_{bd} - L_{bc}\delta_{ad} - L_{ad}\delta_{bc} + L_{bd}\delta_{ac}.$$

Tracing this expression over ac gives

$$0 = L\delta_{bd} + (n-2)L_{bd}, \ \text{ where } L \equiv \delta^{ab} L_{ab}.$$

Tracing over bd now implies $L = 0$ and if $n \geq 3$, then substituting back implies that $L_{ab} = 0$, as required. □

This lemma is just what we need to conclude that $\partial_a Q_b = 0$ and we have closure. Specifically, the conformal Killing equation $\partial_{(a}\omega_{b)}{}^\circ = 0$ is equivalent to the following closed system

$$(10)\qquad \begin{array}{c}\partial_a\omega_b = F_{ab} + \Lambda\delta_{ab}\\ \partial_a F_{bc} = \delta_{ab}Q_c - \delta_{ac}Q_b \qquad \partial_a\Lambda = -Q_a\\ \partial_a Q_b = 0\end{array} \qquad \text{for } F_{ab} = F_{[ab]}.$$

It follows immediately that the dimension of the solution space is

$$n + \frac{n(n-1)}{2} + 1 + n = \frac{(n+1)(n+2)}{2}.$$

Indeed, the general solution is

$$\omega_a = s_a + m_{ab}x^b + \lambda x_a + r^b x_b x_a - \tfrac{1}{2} r_a x^b x_b,$$

where $s_a, m_{ab}, \lambda, r_b$ are constant tensors with $m_{ab} = -m_{ba}$. We shall see later how these solutions correspond to conformal motions. In particular, we shall see how the vector field

$$\left(r^b x_b x^a - \tfrac{1}{2} r^a x^b x_b\right)\frac{\partial}{\partial x^a}$$

corresponds to the family of conformal transformations

$$x^a \longmapsto \frac{x^a - \frac{1}{2} t r^a \|x\|^2}{1 - t r_b x^b + \frac{1}{4} t^2 \|r\|^2 \|x\|^2}, \quad \text{for } t \in \mathbb{R}.$$

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In this lecture we shall be concerned with the theory of connections on vector bundles [7]. Connections in this generality will turn out to be considerably more useful than our initial aim, which is to extend the differential operator (6) to an arbitrary Riemannian manifold.

A connection on a smooth vector bundle E is a linear differential operator $\nabla : E \to \Lambda^1 \otimes E$ satisfying the Leibniz rule:–

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma \quad \text{for } f \in \Gamma(\Lambda^0) \text{ and } \sigma \in \Gamma(E).$$

In particular, the exterior derivative itself $d : \Lambda^0 \to \Lambda^1$ is a connection on the trivial bundle and the general connection on the trivial bundle is given by

$$\nabla f = df + \gamma f, \quad \text{for any fixed } \gamma \in \Gamma(\Lambda^1).$$

Locally, a connection on any smooth vector bundle may be constructed by trivialising the vector bundle $E|_U \cong U \times \mathbb{R}^N$. Then $\sigma \in \Gamma(U, E)$ can be identified with an array $(\sigma_1, \sigma_2, \cdots, \sigma_N)$ of smooth functions and $\nabla : E|_U \to \Lambda^1 \otimes E|_U$ defined by

$$\nabla(\sigma_1, \sigma_2, \cdots, \sigma_N) = (d\sigma_1, d\sigma_2, \cdots, d\sigma_N).$$

If ∇ and $\widetilde{\nabla}$ are connections on E, then so is $h\nabla + (1-h)\widetilde{\nabla}$ for any smooth function h. It follows that local connections may be patched together by a partition of unity and, therefore, that any smooth vector bundle will admit a connection. This deals with existence. As far as freedom is concerned, suppose that ∇ and $\widetilde{\nabla}$ are two connections on E and consider $\Phi \equiv \widetilde{\nabla} - \nabla : E \to \Lambda^1 \otimes E$. It is a homomorphism of vector bundles, equivalently a differential operator of degree zero. To see this, note that

$$\begin{aligned} \Phi(f\sigma) &= \widetilde{\nabla}(f\sigma) - \nabla(f\sigma) \\ &= df \otimes \sigma + f\widetilde{\nabla}\sigma - df \otimes \sigma - f\nabla\sigma \\ &= f\widetilde{\nabla}\sigma - f\nabla\sigma \quad = \quad f\Phi(\sigma). \end{aligned}$$

One says that 'Φ is linear over the functions' and it follows easily that Φ is a homomorphism. In summary, if $\nabla : E \to \Lambda^1 \otimes E$ is a connection then the general connection on E is $\nabla + \Phi$, where $\Phi \in \Gamma(\Lambda^1 \otimes \mathrm{End}(E))$ is arbitrary.

A connection $\nabla : E \to \Lambda^1 \otimes E$ induces linear differential operators

$$\nabla : \Lambda^k \otimes E \to \Lambda^{k+1} \otimes E$$

(the E-coupled exterior derivative) characterised by a version of the Leibniz rule, namely

$$\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^k \omega \wedge \nabla\sigma. \tag{11}$$

Notice that this is consistent

$$\begin{aligned} \nabla(f\omega \otimes \sigma) &= d(f\omega) \otimes \sigma + (-1)^k f\omega \wedge \nabla\sigma \\ &= df \wedge \omega \otimes \sigma + f d\omega \otimes \sigma + (-1)^k f\omega \wedge \nabla\sigma \\ &= d\omega \otimes f\sigma + (-1)^k \omega \wedge df \otimes \sigma + (-1)^k \omega \wedge f\nabla\sigma \\ &= d\omega \otimes f\sigma + (-1)^k \omega \wedge (df \otimes \sigma + f\nabla\sigma) \\ &= d\omega \otimes f\sigma + (-1)^k \omega \wedge \nabla(f\sigma) \quad = \quad \nabla(\omega \otimes f\sigma) \end{aligned}$$

and also that $\Lambda^k \otimes E \xrightarrow{\nabla} \Lambda^{k+1} \otimes E$ satisfies a Leibniz rule:–

$$\nabla(f\tau) = df \wedge \tau + f\nabla\tau \quad \text{for } f \in \Gamma(\Lambda^0) \text{ and } \sigma \in \Gamma(\Lambda^k \otimes E).$$

The composition $E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\nabla} \Lambda^2 \otimes E$ is actually a homomorphism of vector bundles: one checks it is linear over the functions:–

$$\begin{aligned} \nabla^2(f\sigma) &= \nabla(df \otimes \sigma + f\nabla\sigma) \\ &= d^2 f \otimes \sigma - df \wedge \nabla\sigma + df \wedge \nabla\sigma + f\nabla^2\sigma \quad = \quad f\nabla^2\sigma. \end{aligned}$$

We shall write $\nabla^2 : E \to \Lambda^2 \otimes E$ as $\kappa \in \Gamma(\Lambda^2 \otimes \mathrm{End}(E))$ and call it the <u>curvature</u> of ∇.

If two vector bundles E and F are equipped with connections then the Leibniz rule automatically equips various bundles induced from E and F with connections, including

$$E^* \qquad E\otimes E \qquad \Lambda^2 E \qquad \bigodot^2 E \qquad \mathrm{End}(E)=E^*\otimes E$$
$$E\otimes F \qquad \mathrm{Hom}(E,F)=E^*\otimes F.$$

For example, if we denote the canonical pairing between a vector bundle and its dual by $E^*\otimes E\ni \rho\otimes\sigma\mapsto \rho\,\lrcorner\,\sigma\in\Lambda^0$, then

$$d(\rho\,\lrcorner\,\sigma)=(\nabla\rho)\,\lrcorner\,\sigma+\rho\,\lrcorner\,\nabla\sigma \tag{12}$$

(another version of the Leibniz rule) characterises $\nabla: E^*\to\Lambda^1\otimes E^*$.

Lemma 4. *For the induced connection on* $\mathrm{End}(E)$ *and the coupled exterior derivative* $\nabla:\Lambda^2\otimes\mathrm{End}(E)\to\Lambda^3\otimes\mathrm{End}(E)$ *we have*

$$\nabla\kappa=0 \quad \textit{(the Bianchi indentity).}$$

Proof. Let us consider the composition

$$\Lambda^k\otimes E\xrightarrow{\nabla}\Lambda^{k+1}\otimes E\xrightarrow{\nabla}\Lambda^{k+2}\otimes E. \tag{13}$$

We find that

$$\nabla^2(\omega\otimes\sigma)=\nabla(d\omega+(-1)^k\omega\wedge\nabla\sigma)=\cdots=\omega\wedge\nabla^2\sigma.$$

In other words (13) is nothing other than $\mathrm{Id}\wedge\kappa:\Lambda^k\otimes E\to\Lambda^{k+1}\otimes E$. Now, let us consider the composition

$$E\xrightarrow{\nabla}\Lambda^1\otimes E\xrightarrow{\nabla}\Lambda^2\otimes E\xrightarrow{\nabla}\Lambda^3\otimes E.$$

We can group it in two different ways to conclude that

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & \Lambda^1\otimes E \\ \kappa\big\downarrow & & \big\downarrow\mathrm{Id}\wedge\kappa \\ \Lambda^2\otimes E & \xrightarrow{\nabla} & \Lambda^3\otimes E \end{array}$$

commutes. Untangling this conclusion implies $\nabla\kappa=0$. □

Now let us view the operation $X^b\mapsto\partial_a X^b$ from (6) as a connection on the tangent bundle of $\mathbb{R}^n$ and see what is its curvature. Writing T for the tangent bundle, we see that $\partial: T\to\Lambda^1\otimes T$ satisfies the Leibniz rule so certainly it is a connection. From (11), the induced operator $\partial:\Lambda^1\otimes T\to\Lambda^2\otimes T$ is characterised by

$$\omega_b X^c\mapsto\partial_{[a}\omega_{b]}-\omega_{[a}\partial_{b]}X^c=\partial_{[a}\omega_{b]}+\omega_{[b}\partial_{a]}X^c=\partial_{[a}(\omega_{b]}X^c)$$

and it follows that, in general, ${X_b}^c\mapsto\partial_{[a}{X_{b]}}^c$. The curvature is the composition $X^c\mapsto\partial_b X^c\mapsto\partial_{[a}\partial_{b]}X^c$, which vanishes.

To summarise connections: <u>Leibniz Rules, OK?</u>

12th August 2011

We shall now specialise our discussion to connections on the tangent bundle, equivalently the co-tangent bundle. We shall refer to them as affine connections. Some authors [1] write linear connections instead.

Immediately, a problem arises. We already have a natural linear differential operator $\Lambda^1 \to \Lambda^2$, namely the exterior derivative. However, a connection $\nabla : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$ gives rise to another one, namely the composition

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 \otimes \Lambda^1 \xrightarrow{\wedge} \Lambda^2.$$

There is no need for these to agree. However, if we write τ for their difference, then we see that τ is linear over the functions:–

$$\begin{aligned}
\tau(f\omega) &\equiv (\wedge \circ \nabla - d)(f\omega) \\
&= \wedge(df \otimes \omega + f\nabla\omega) - (df \wedge \omega + f d\omega) \\
&= df \wedge \omega + f \wedge \circ \nabla\omega - df \wedge \omega - f d\omega \\
&= f \wedge \circ \nabla\omega - f d\omega \;\; = \;\; f\tau(\omega).
\end{aligned}$$

Hence $\tau : \Lambda^1 \to \Lambda^2$ is a homomorphism of vector bundles. It is called the torsion of the affine connection ∇. It is convenient to regard it as a homomorphism $\Lambda^1 \to \Lambda^1 \otimes \Lambda^1$ that happens to take values in $\Lambda^1 \wedge \Lambda^1$. Then $\nabla - \tau : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$ is a torsion-free connection canonically constructed from ∇. Henceforth we shall suppose our affine connections to be torsion-free. Notice that our previous discussion shows that on any smooth manifold, torsion-free affine connections exist. It is easy to check that, by sticking with torsion-free connections, other awkward problems are avoided. For example, $\nabla : \Lambda^1 \otimes T \to \Lambda^2 \otimes T$ may be unambiguously defined either by (11) or as the composition

$$\Lambda^1 \otimes T \xrightarrow{\nabla} \Lambda^1 \otimes \Lambda^1 \otimes T \xrightarrow{\wedge \otimes \mathrm{Id}} \Lambda^2 \otimes T, \tag{14}$$

where ∇ is the induced connection on $\Lambda^1 \otimes T$.

From now on, we shall usually adorn connections and tensors with indices, consistently using upper indices for contravariant tensors such as vector fields and lower indices for covariant tensors such as k-forms or metrics. On the one hand, this reflects simply writing everything in local coördinates. But the consistent distinction between upper and lower indices means that, instead, we may view the indices as markers depicting the type of the tensor and incorporate the natural operations such as skewing, symmetrising, or contracting into the notation by altering the order of the indices and by viewing the Einstein summation convention as the natural pairing between vectors and co-vectors. This is Penrose's abstract index notation [8].

By definition, for a torsion-free affine connection ∇_a, the exterior derivative may be written as $\omega_b \mapsto \nabla_{[a}\omega_{b]}$. If ∇_a is torsion-free, the general torsion-free connection is

$$\widetilde{\nabla}_a\omega_b = \nabla_a\omega_b - \Gamma_{ab}{}^c\omega_c, \quad \text{where } \Gamma_{ab}{}^c = \Gamma_{(ab)}{}^c.$$

The corresponding connection on vector fields is

$$\widetilde{\nabla}_a X^c = \nabla_a X^c + \Gamma_{ab}{}^c X^b$$

for then (12) holds for $\widetilde{\nabla}_a$, given that it holds for ∇_a:–

$$\begin{aligned}(\widetilde{\nabla}_a X^c)\omega_c + X^b(\widetilde{\nabla}_a\omega_b) &= (\nabla_a X^c + \Gamma_{ab}{}^c X^b)\omega_c + X^b(\nabla_a\omega_b - \Gamma_{ab}{}^c\omega_c) \\ &= (\nabla_a X^c)\omega_c + X^b(\nabla_a\omega_b) = \nabla_a(X^b\omega_b).\end{aligned}$$

The corresponding formula for the induced connection on $\bigotimes^2\Lambda^1$ is

$$\widetilde{\nabla}_a\theta_{bc} = \nabla_a\theta_{bc} - \Gamma_{ab}{}^d\theta_{dc} - \Gamma_{ac}{}^d\theta_{bd} \tag{15}$$

(because the Leibniz rule allows one readily to verify this formula for simple tensors $\theta_{bc} = \phi_b\psi_c$ whence the general case follows by linearity).

The curvature of an affine connection is a tensor $R \in \Gamma(\Lambda^2 \otimes \mathrm{End}(T))$ and for a torsion-free connection may be defined by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)X^c = R_{ab}{}^c{}_d X^d \tag{16}$$

(tradition dictating that it differ by a factor of 2 from κ defined above). Notice that we are using that ∇_a is torsion-free implicitly on the left hand side of (16) since then $\nabla : \Lambda^1 \otimes T \to \Lambda^2 \otimes T$ may be realised as the composition (14). From (12) it is easily checked that the curvature on the co-tangent bundle is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_d = -R_{ab}{}^c{}_d\omega_c.$$

Theorem 1. *On a Riemannian manifold there is a unique torsion-free affine connection 'preserving the metric' in the sense that $\nabla_a g_{bc} = 0$ for the induced connection on $\bigodot^2\Lambda^1$.*

Proof. Fixing an arbitrary torsion-free affine connection $\widetilde{\nabla}_a$, the general torsion-free affine connection is given by

$$\nabla_a\phi_b = \widetilde{\nabla}_a\phi_b - \Gamma_{ab}{}^c\phi_c \quad \text{for } \Gamma_{ab}{}^c = \Gamma_{(ab)}{}^c, \text{ equivalently } \Gamma_{[ab]}{}^c = 0.$$

Let us use (15) to compute

$$\nabla_a g_{bc} = \widetilde{\nabla}_a g_{bc} - \Gamma_{ab}{}^d g_{dc} - \Gamma_{ac}{}^d g_{bd} = \widetilde{\nabla}_a g_{bc} - \Gamma_{abc} - \Gamma_{acb}, \tag{17}$$

where we are using g_{ab} to identify covariant and contravariant tensors according to $X_a \equiv g_{ab}X^b$ and, conversely, $\omega^a \equiv g^{ab}\omega_b$. Here, we are denoting by g^{ab} the 'inverse' of g_{ab}, namely $g_{ab}g^{bc} = \delta_a{}^c$ where $\delta_a{}^c$ is the Kronecker delta, i.e. the canonical pairing between tangent and co-tangent vectors. (In the vernacular, one speaks of 'raising and lowering indices' and often does this without comment). From (17) we see that existence and uniqueness of our desired connection boils down precisely to existence and uniqueness for the following algebraic equations

$$\Gamma_{[ab]c} = 0 \quad \text{and} \quad \Gamma_{a(bc)} = \tfrac{1}{2}\widetilde{\nabla}_a g_{bc}, \tag{18}$$

which, in turn, boils down to Lemma 1. To see this, it is convenient to rephrase Lemma 1 as the statement that

$$\Lambda^1 \otimes \Lambda^2 \ni K_{abc} \longmapsto K_{[ab]c} \in \Lambda^2 \otimes \Lambda^1$$

is an isomorphism (precisely, Lemma 1 says that this homomorphism is injective but these vector bundles clearly have the same rank). To solve (18) we note that $\Gamma_{abc} = \frac{1}{2}\widetilde{\nabla}_a g_{bc} - K_{abc}$, where $K_{abc} = K_{a[bc]}$, provides the general solution of the second equation and then Lemma 1 is just what we need to solve $K_{[ab]c} = \frac{1}{2}\widetilde{\nabla}_{[a} g_{b]c}$ and hence the first equation. More specifically, the unique solution of (18) is easily seen to be

$$\Gamma_{abc} = \tfrac{1}{2}\big(\widetilde{\nabla}_a g_{bc} + \widetilde{\nabla}_b g_{ac} - \widetilde{\nabla}_c g_{ab}\big)$$

(cf. the statement and proof of Lemma 2). $\square$

The connection ∇_a from Theorem 1 is called the metric connection or the Levi-Civita connection associated with g_{ab}. It is the basic object in Riemannian differential geometry [1, 5]. As an aside, we remark that its construction only depends on the non-degeneracy of g_{ab} (meaning that $X^b \mapsto g_{ab}X^b$ is an isomorphism $T \xrightarrow{\simeq} \Lambda^1$) and hence applies equally well in the Lorentzian setting.

18th August 2011

- Lie derivative by formula, e.g. $\mathcal{L}_X Y^b = X^a \nabla_a Y^b + (\nabla_b X^a)\omega_a$.
- Geometric interpretation of Lie derivative.
- Derivation of Killing equation on vector fields.
- Formal definition of conformal structure via $\widehat{g}_{ab} = \Omega^2 g_{ab}$.
- Derivation of conformal Killing equation.

19th August 2011

- Conformal change in Levi-Civita connection:–

$$\widehat{\nabla}_a \omega_b = \nabla_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a + \Upsilon^c \omega_c g_{ab}, \quad \text{where } \Upsilon_a = (\nabla_a \Omega)/\Omega.$$

- Conformal change in Riemann curvature tensor:–

$$\widehat{R}_{abcd} = \Omega^2\big(R_{abcd} - (\Xi_{ac} g_{bd} - \Xi_{bc} g_{ad} - \Xi_{ad} g_{bc} + \Xi_{bd} g_{ac})\big),$$
$$\text{where } \Xi_{ab} = \nabla_a \Upsilon_b - \Upsilon_a \Upsilon_b + \tfrac{1}{2}\Upsilon_c \Upsilon^c g_{ab}.$$

- Decomposition of Riemann curvature tensor:–

$$\widehat{R}_{abcd} = W_{abcd} + \mathrm{P}_{ac}g_{bd} - \mathrm{P}_{bc}g_{ad} - \mathrm{P}_{ad}g_{bc} + \mathrm{P}_{bd}g_{ac},$$

where P_{ab} is symmetric (the <u>Rho-tensor</u> or <u>Schouten tensor</u>) and W_{abcd} is totally trace-free (the Weyl tensor).

- Discussion with Bianchi symmetry...
- The Weyl tensor is conformally invariant: $\widehat{W_{ab}{}^c{}_d} = W_{ab}{}^c{}_d$.
- $\widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon_c\Upsilon^c g_{ab}$.
- R_{abcd} obstructs Riemannian flatness for $n \geq 2$.
- W_{abcd} obstructs conformal flatness for $n \geq 4$.
- $R_{abcd} = 4K(g_{ac}g_{bd} - g_{bc}g_{cd})$ in case $n = 2$.
- The unit 2-sphere has $K = 1/4$.

<u>25th August 2011</u>

- By stereographic projection, the unit n-sphere has

$$R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad}.$$

- Liouville's Theorem and formula [3]

$$x^a \overset{p}{\longmapsto} \frac{m^a{}_b x^b - \frac{1}{2}s^b\|x\|^2}{1 + r_c x^c - \frac{1}{2}t\|x\|^2}, \quad \text{where} \quad \begin{array}{cc} \lambda > 0 & m^a{}_b \in \mathrm{SO}(n) \\ t = -\frac{1}{2\lambda}\|r\|^2 & s^a = -\frac{1}{\lambda}m^{ab}r_b. \end{array}$$

- Proof by conformal Killing equation.
- Proof by solving $\partial_a\Upsilon_b = \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon_c\Upsilon^c\delta_{ab}$.
- Proof by <u>conformal circles</u> [2]

$$2U.U\partial A^b - 6U.AA^b + 3A.AU^b = 0, \quad \text{where } \partial \equiv U^a\partial_a \text{ and } A^b \equiv \partial U^b.$$

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