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The Monge-Ampère equation and its geometric applications

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Abstract

In this paper we present the basic theory of the Monge-Ampère equation together with a selection of geometric applications, mainly to affine geometry. First we introduce the Monge-Ampère measure and the resultant notion of generalized solution of Aleksandrov. Then we discuss a priori estimates and regularity, followed by the existence and uniqueness of solutions to various boundary value problems. As applications we consider the existence of smooth convex hypersurfaces of prescribed Gauss curvature, as well as various topics in affine geometry, including affine spheres, affine completeness and affine maximal hypersurfaces. In particular we describe our recent work concerning the affine Bernstein and the affine Plateau problems.

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Contents

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1 Introduction

In this paper we consider the Monge-Ampère equation

$$\det D^2 u = f(x, u, Du) \tag{1.1}$$

and its geometric applications, particularly in affine geometry. Here $\det D^2 u$ denotes the determinant of the Hessian matrix $D^2 u$, u is a function in the Euclidean space $\mathbb{R}^n$, and f is a given function. This is a fully nonlinear, second order partial differential equation. It is elliptic when the Hessian matrix $D^2 u$ is positive definite, namely when u is locally uniformly convex.

When $f = K(x)[1 + |Du|^2]^{(n+2)/2}$, equation (1.1) becomes the prescribed Gauss curvature equation, and has been extensively studied in the last century. The Monge-Ampère equation is also fundamental in affine geometry. Let $\mathcal{M}$ be a locally convex hypersurface in $\mathbb{R}^{n+1}$. One can define the affine metric $g = K^{-\frac{1}{n+2}} II$ on $\mathcal{M}$, where K is the Gauss curvature and II is the second fundamental form. There are two important topics in affine geometry which are closely related to the Monge-Ampère equation, one is affine spheres and the other is affine maximal surfaces. An affine sphere in the graph case satisfies the Monge-Ampère equation (1.1) while an affine maximal surface satisfies the fourth order equation

$$\sum_{i,j=1}^{n} U^{ij} \partial_{x_i} \partial_{x_j} [\det D^2 u]^{-\frac{n+1}{n+2}} = 0, \tag{1.2}$$

where $\{U^{ij}\}$ is the cofactor matrix of $D^2 u$.

In many other applications, such as in isometric embedding, optimal transportation, reflector shape design, there arise equations of Monge-Ampère type in the more general form,

$$\det[D^2u - A(x, u, Du)] = f(x, u, Du), \tag{1.3}$$

where $A = \{a_{ij}\}$ is a symmetric matrix. The Monge-Ampère equation also occurs in the study of special Lagrangian sub-manifolds, prescribing Weingarten curvature, and in complex geometry on toric manifolds. It also arises in meteorology and fluid mechanics where the Monge-Ampère equation is coupled with a transport equation, such as the semi-geostrophic equation. Due to this profusion of applications and beautiful theory, equations of Monge-Ampère type are the most important fully nonlinear partial differential equations.

The Monge-Ampère equation draws its name from its initial formulation in two dimensions, by the French mathematicians, Monge [Mon] and Ampère [Am], about two hundred years ago. It was subsequently studied by Minkowski, Bernstein, Schauder, Lewy and many others. See [Go] for an early treatment of the equation. In the last century the development of the Monge-Ampère equation was closely related to geometric problems, such as the Minkowski problem of finding a convex body, whose boundary has its Gauss curvature prescribed as a function of its normal [M1], the problem of local isometric embedding of Riemannian surfaces in $\mathbb{R}^3$ and the related Weyl problem [We]. The first notable result is by Minkowski [M1, M2], who proved the existence of a weak solution to the Minkowski problem by approximation by convex polyhedra with given face areas. Using convex polyhedra with given generalized curvatures at the vertices, Aleksandrov [A1, A3] also proved the existence of a weak solution to the Minkowski problem (in all dimensions), as well as the C^1 smoothness of solutions in two dimensions [A2]. Moreover, Minkowski, Aleksandrov, and Lewy [L2], also proved the uniqueness of weak solutions. Both notions of weak solutions by Minkowski and Aleksandrov have continued to be frequently used in recent years. In fact, Aleksandrov's generalized solution corresponds to the curvature measure in the theory of convex bodies while the weak solution of Minkowski is related to the area measure [Sch].

The regularity of weak solutions has been a much more difficult challenge. Lewy [L1] proved that a C^2 convex surface with analytic Gauss curvature is also analytic. Lewy's argument involves a "characteristic" theory, which was followed by Heinz [H1, H2, H3] in a series of papers and was applied to the more general Monge-Ampère type equation (1.3). In particular Heinz obtained the genuine interior $C^{2,\alpha}$ a priori estimates for the two dimensional Monge-Ampère equation (1.1), assuming the inhomogeneous term f is Hölder continuous. A main ingredient in this "characteristic" theory is the so called partial Legendre transform, which was introduced by Darboux in the 19th century and used by many authors. This transform reduces the Monge-Ampère equation to a quasilinear elliptic system. We refer the reader to [Scu] for details of Heinz' work and the partial Legendre transform.

For the existence of smooth (but not necessarily analytic) solutions to the Minkowski problem, it suffices to establish global $C^{2,\alpha}$ a priori estimate for solutions with sufficiently smooth Gauss curvature. The second derivative estimate for the Weyl problem was already proved by Weyl in 1916 [We], and by Miranda [Mir] for the Minkowski problem. By the second derivative bound, the equation becomes uniformly elliptic. So the $C^{2,\alpha}$ a priori estimate follows from Morrey's regularity result for two dimensional uniformly elliptic equation. Therefore Nirenberg [N1], based on Morrey and Miranda's estimates, developed a regularity theory for two dimensional elliptic equations and resolved the Minkowski problem by the continuity method [N2]. Meanwhile, Pogorelov [P1] established the $C^{2,\alpha}$ estimate, as well as the corresponding estimate for the Dirichlet problem in a small disc [P2, P3], which implies the regularity of weak solutions to the Minkowski problem. Note that the Weyl problem is of the same nature and can be solved by similar arguments.

In high dimensions, based on his earlier works [A1, A3], Aleksandrov [A4], and also Bakelman [B1] in dimension two, introduced a generalized solution to the Monge-Ampère equation and proved the existence and uniqueness of solutions to the Dirichlet problem. The treatment also lead to the Aleksandrov-Bakelman maximum principle which plays a fundamental role in the study of non-divergence elliptic equations [GT]. The regularity of generalized solutions in high dimensions is a much harder problem. A breakthrough was made by Calabi [Ca1]. He established the interior third derivative estimate in terms of a uniform bound for the second derivatives. His computation is intrinsic, regarding $u_{x_i x_j} dx_i dx_j$ as a metric. But in high dimensions ($n \geq 3$), there is no genuine local a priori estimate for the second derivatives. Pogorelov [P8] found a convex function which contains a line segment and satisfies the Monge-Ampère equation (1.1) with positive, analytic right hand side f. However, Pogorelov [P5] established an interior second derivative estimate for solutions satisfying affine boundary condition. A different proof was later given by Ivochkina [I1]. Therefore by Calabi and Pogorelov, the a priori estimates were established for strictly convex solutions, or for solutions which do not contain a line segment with both endpoints on boundary. In early 1980's, Evans [Ev] and Krylov [K1, K3] established the interior regularity for fully nonlinear, uniformly elliptic equations satisfying a concavity or convexity condition. Therefore Pogorelov's second derivative estimate together with Evans [Ev] and Krylov [K1, K3] also implied the smoothness of Aleksandrov's generalized solutions [GT].

But to prove the interior regularity of Aleksandrov's generalized solution, from the a priori estimates is still very tricky, and was achieved independently by Cheng and Yau [CY2] and Pogorelov [P8]. They first proved by the continuity method the existence of a unique smooth solution to the multi-dimensional Minkowski problem [CY1, P8], and then reduced the existence of regular solutions of the Dirichlet problem to the Minkowski problem, assuming the boundary function is C^2 smooth. See also [CW2] for a Gauss curvature flow approach to the Minkowski problem. The C^2 boundary condition can be relaxed to $C^{1,\alpha}$ with $\alpha > 1 - \frac{2}{n}$ [P6, U3]. The existence of a regular solution was also obtained by Cheng and Yau [CY4] and Lions [Ls1] by different arguments.

Because the Monge-Ampère equation is fully nonlinear, the estimates at the boundary required different techniques and advances were not made until the 1980's. The global second derivative estimate, for general Dirichlet boundary value, was obtained by Ivochkina [I2] (see also [CNS, GT]), and also by Nirenberg, Cheng and Yau. The third derivative estimate requires completely new ideas and was obtained independently by Caffarelli-Nirenberg-Spruck [CNS] and Krylov [K2, K3]. Therefore by the continuity method, one obtains globally smooth solutions to the Dirichlet problem, and so Aleksandrov's generalized solution is smooth up to the boundary provided all given data are smooth. Recently the authors proved the global $C^{2,\alpha}$ estimate for the Dirichlet problem under optimal conditions [TW7].

For the prescribed Gauss curvature equation, in addition to the Minkowski problem, one may also consider the Dirichlet problem and the more general Plateau type problem, as suggested by Yau [Y2]. A necessary condition for the existence of a generalized solution to the Dirichlet problem is that the total curvature is less than or equal to $\frac{1}{2}|S^n|$. In the case when the total curvature is less than $\frac{1}{2}|S^n|$, the existence and regularity were treated in [B2, TU]. When the total curvature is equal to $\frac{1}{2}|S^n|$, the gradient of solutions will necessarily tend to infinity near the boundary, and one can not prescribe the boundary value of the solutions anymore. The existence and regularity in this extreme case were studied by Urbas [U1, U5, U6], who established a very deep regularity result, namely the graph must be smooth and the boundary value must also be a smooth function. The Plateau type problem was also treated and solved for smooth manifolds by the authors [TW4], see also [GS2].

In the above discussions, one assumes that the inhomogeneous term f is positive and sufficiently smooth (except in dimension two). When f is merely continuous, Caffarelli proved, by a perturbation argument, a surprising interior $W^{2,p}$ estimate for any $p > 1$, assuming that f is continuous, which turns out to be necessary [W2]. Caffarelli also obtained an interior $C^{2,\alpha}$ estimate for Hölder continuous f. See [W7, JW] for details of the $C^{2,\alpha}$ estimate and extension to Dini continuous f. By the a priori estimates of Calabi and Pogorelov, a crucial point for the perturbation argument is the strict convexity of solutions, which in dimension two was known to Aleksandrov and Heinz [H2] long ago, and was established by Caffarelli [C1] in high dimensions. In the two dimensional case, the $W^{2,p}$ estimate has been previously obtained in [NS] and the $C^{2,\alpha}$ estimate is due to Heinz [H2], see also [Scu].

When the assumption $f > 0$ is relaxed to $f \geq 0$, equation (1.1) is degenerate elliptic and has also been studied by many authors. A significant result, obtained by C.S. Lin [Ln1] for dimension two, and Hong and Zuily [HZ] in high dimensions, is the existence of a local smooth solution. A nontrivial global second derivative estimate for the Dirichlet problem can be found in [GTW], but a satisfactory general regularity theory is still lacking.

In recent years advances have been made in the investigation of the more general Monge-Ampère equation (1.3), which arises in applications such as optimal transportation and reflector design, as prescribing the Jacobian determinant of an associated mapping. A natural boundary condition, called the second boundary condition, is to prescribe the image of the mapping. For the standard Monge-Ampère equation (1.1), the interior regularity of solutions was proved in [C4] and the global regularity in [C5]. See also in [D3] for $n = 2$ and [U7] for $n \geq 2$. For the general equation (1.3), a sufficient condition for the interior regularity (see (3.43) below), whose degenerate form turns out also to be necessary [Lo], was found in [MTW], and the global regularity was established in [TW8], under the degenerate form of the condition. We note that the global regularity and existence of solutions to the Neumann problem were obtain in [LTU], but the oblique boundary problem does not enjoy the global regularity in general [U4,W1], see §4.2 below.

The application to affine geometry was a main motivation for the development of the Monge-Ampère equation. A classical problem in affine geometry is the classification of affine spheres. It was proved by Blaschke [Bls] in dimension two, and by Deicke [De] and Calabi [Ca2] for high dimensions, that a (complete) elliptic affine sphere must be an ellipsoid. See also [An] for a different proof by geometric flow. The classification of parabolic affine spheres is related to the study of entire solutions to the Monge-Ampère equation

$$\det D^2 u = 1 \quad \text{in} \quad \mathbb{R}^n. \tag{1.4}$$

It was proved by Jörgens [Jo] for $n = 2$, Calabi [Ca1] for $3 \le n \le 5$ [Ca1], and Pogorelov [P7] for all $n \ge 2$, that an entire solution must be a quadratic function. Calabi's restriction to the dimensions $n \le 5$ is due to that in these dimensions he was able to show that his third derivative estimate at a point depends only on that of the second derivative at the given point. Jörgens, Calabi, and Pogorelov's result was reproved by Cheng and Yau [CY3] by a different argument. The hyperbolic affine sphere problem is related to a Dirichlet problem for the Monge-Ampère equation (1.1) and was also resolved by Cheng and Yau [CY2, CY3].

Applications to affine maximal surfaces are a more recent development. Note that the affine maximal surface equation (1.2) can also be written as a system

$$U^{ij} w_{ij} = f, \tag{1.5}$$

$$\det D^2 u = w^{-\frac{n+2}{n+2}}, \tag{1.6}$$

(with $f = 0$), where (1.5) is a linearized Monge-Ampère equation and (1.6) is the standard Monge-Ampère equation. By the $C^{2,\alpha}$ estimate for the Monge-Ampère equation [C2] and the Hölder estimate for the linearized Monge-Ampère equation [CG2], the authors [TW2, TW6] established the $W^{4,p}$ and $C^{4,\alpha}$ estimates for strictly convex solution u to equation (1.6). As in the minimal surface theory, one is interested in the Bernstein and Plateau problems for the affine maximal surface equation. Chern [Ch] conjectured that a locally convex, Euclidean complete, affine maximal surface in R^3 is an elliptic paraboloid. Calabi [Ca3] gave an affirmative answer to this Bernstein type problem provided the surface is also affine complete. This conjecture was resolved by the authors [TW2]. In a separate paper [TW3] we proved that affine completeness implies Euclidean completeness for $n \ge 2$, therefore the Euclidean completeness in Chern's conjecture can be replaced by affine completeness (see also [LJ] for a direct proof).

The affine Plateau problem is considerably more complicated. A special case is the first boundary value problem, that is prescribing the solution and its gradient on the boundary. The existence of weak solutions (in any dimension) and interior regularity (in dimension 2) were obtained by the authors in [TW6]. We also proved the global regularity of solutions to the second boundary problem to the affine maximal surface equation, namely prescribing the solution and its Hessian determinant on the boundary. The second boundary problem was used in a penalty argument for the regularity of solutions of the first boundary problem and the affine Plateau problem.

We now collect some basic properties of the Monge-Ampère operator which will be used in later sections.

First, the Monge-Ampère operator $\det D^2u$ is the Jacobian determinant of the gradient mapping Du; and it is equal to the product of all eigenvalues of the Hessian matrix D^2u, namely,

$$\det D^2u = \lambda_1 \cdots \lambda_n. \tag{1.7}$$

The Monge-Ampère operator is also of divergence form,

$$\det D^2u = \frac{1}{n} U^{ij}\partial_{ij}u = \frac{1}{n}\partial_i[U^{ij}u_j], \tag{1.8}$$

where the co-factor matrix U^{ij} is divergence free,

$$\sum_i \partial_i U^{ij} = 0 \quad \forall\ j = 1, \cdots, n. \tag{1.9}$$

It is easy to verify that the operator is invariant under linear transforms, namely for any matrix A,

$$\det D^2u(Ax) = |\det A|^2 f(Ax). \tag{1.10}$$

The Monge-Ampère operator is also invariant if one subtracts a linear function φ, namely,

$$\det D^2u = \det D^2(u-\varphi), \tag{1.11}$$

and obviously it is also homogeneous,

$$\det D^2(tu) = t^n f. \tag{1.12}$$

The Legendre transform plays an important role in the study of the Monge-Ampère equation. Let Ω be a convex domain and u be a convex function in Ω. The Legendre transform of u is a convex function u^* defined in $\Omega^* = Du(\Omega)$, given by

$$u^*(y) = \sup\{\widetilde{x}\cdot y - u(\widetilde{x}) : \quad \widetilde{x} \in \Omega\}. \tag{1.13}$$

It follows that $u^{**} = u$ and if $u \in C^2(\Omega)$ is uniformly convex, then $u^*(y) = x\cdot y - u(x)$, where x is determined by $y = Du(x)$, and u^* is also C^2 and uniformly convex. By the relation $y = Du(x)$, we have $x = Du^*(y)$ and

$$\{D^2u(x)\} = \{D^2u^*(y)\}^{-1}. \tag{1.14}$$

In particular if u satisfies (1.1), its Legendre transform u^* satisfies

$$\det D^2u^* = 1/f. \tag{1.15}$$

In this survey, we discuss the existence and regularity of solutions to the Monge-Ampère equation and the prescribed Gauss curvature equation, and their applications in affine geometry. We will consider convex functions in the Euclidean space, and deal with elliptic solutions only. The hyperbolic or mixed type Monge-Ampère equation is much more difficult but we refer the interested readers to [Ef, HH, Ho2, Ln2, Zu] for works in these directions. For the arrangement of the paper, see the contents above.

To conclude this introduction, we point out that this paper is dedicated to Professor Shing-Tung Yau on the occasion of his 60th birthday. But we also wish to recognize Professor Yau for the dominant and immense role he has played in the nurturing, promotion, and eventual supremacy of geometric analysis during the last thirty to forty years.

2 The Monge-Ampère measure

2.1 Locally convex hypersurfaces

A *locally convex hypersurface* $\mathcal{M}$ in the Euclidean space $\mathbb{R}^{n+1}$ is an immersion of an n-dimensional oriented and connected manifold $\mathcal{N}$ (possibly with boundary) in $\mathbb{R}^{n+1}$, that is a mapping T from $\mathcal{N}$ to $\mathcal{M} \subset \mathbb{R}^{n+1}$, with the following property: for any point $q \in \mathcal{N}$, there exists a neighborhood $\omega_q \subset \mathcal{N}$ such that (i) T is a homeomorphism from ω_q to $T(\omega_q)$; (ii) $T(\omega_q)$ is a convex graph; (iii) the convexity of $T(\omega_q)$ agrees with the orientation.

A hyperplane P is a *local supporting plane* of $\mathcal{M}$ at p if there exists a neighborhood ω_p which lies on one side of P. We say $\mathcal{M}$ is *locally strictly convex* if $\omega_p \cap P = \{p\}$ for all $p \in \mathcal{M}$; and $\mathcal{M}$ is *locally uniformly convex* if furthermore $\mathcal{M}$ is C^2 and the principal curvatures of $\mathcal{M}$ are positive.

We say $\mathcal{M}$ is *convex* if for any point $p \in \mathcal{M}$ and any local supporting plane P at p, the whole surface $\mathcal{M}$ lies on one side of P. In this case, P is simply called a *supporting plane*. Accordingly we can define strict convexity and uniform convexity of $\mathcal{M}$.

Note that in our definition, we allow nonsmooth locally convex hypersurfaces, and condition (iii) is to rule out hypersurfaces such as $x_{n+1} = x_1 \max(|x_1| - 1, 0)$. Note also that since $\mathcal{M}$ is an immersion of a manifold $\mathcal{N}$ in $\mathbb{R}^{n+1}$, when referring to a point $p \in \mathcal{M}$ we actually mean a point $q \in \mathcal{N}$ such that $p = T(q)$. Similarly we say $\omega_p \subset \mathcal{M}$ is a neighborhood of p if it is the image of a neighborhood of $q \in \mathcal{N}$ under the mapping T, and so on.

Let u be a function defined in a domain $\Omega \subset \mathbb{R}^n$. We say that u is *convex* (*locally convex*, resp) if its graph $\mathcal{M}_u = \{(x, u(x)) : x \in \Omega\}$ is convex (locally convex). For any $x_0 \in \Omega$, a linear function $x_{n+1} = \varphi(x)$ is a (local) *supporting function* of u at x_0 if its graph is a (local) supporting plane of $\mathcal{M}_u$. Accordingly we say that u is *strictly convex* or *uniformly convex* if $\mathcal{M}_u$ is strictly convex or uniformly convex. If u is locally convex in Ω, it is Lipschitz continuous in any compact subset of Ω. By a classical theorem of Aleksandrov, u is twice differentiable almost everywhere.

Let $\mathcal{M}$ be a locally convex hypersurface. Then at any given point $p \in \mathcal{M}$, one can choose a coordinate system such that locally $\mathcal{M}$ is the graph of a convex function u,

$$x_{n+1} = u(x), \quad x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

Let η be a *mollifier*, namely η is a C^∞ smooth, nonnegative function in $\mathbb{R}^n$, with support in the unit ball $B_1(0)$, satisfying the integral condition $\int_{\mathbb{R}^n} \eta = 1$. Then the mollification

$$u_\varepsilon(x) = \varepsilon^{-n} \int_{\mathbb{R}^n} u(y)\eta(\frac{x-y}{\varepsilon}) \tag{2.1}$$

is a smooth, locally convex function. In particular, a locally convex function u can be approximated by smooth, locally convex functions.

Let u be a locally convex function in Ω. If the domain Ω is convex, then by Taylor's formula, u is convex. We also have the following lemma, which is useful in our argument below.

Lemma 2.1. *Let $\mathcal{M}$ be a locally convex hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$. Suppose that $\mathcal{M}$ is strictly convex at some point p_0 and the boundary $\partial\mathcal{M}$ lies on a hyperplane $\hat{P}$. Then $\mathcal{M}$ is convex.*

Proof. By choosing proper axes, we assume that p_0 is the origin and locally $\mathcal{M}$ is contained in $\{x_{n+1} \geq 0\}$. For $h > 0$, denote by $\mathcal{M}_h$ the component of the set $\{(x_1, \cdots, x_{n+1}) \in \mathcal{M} : \ x_{n+1} \leq h\}$ which contains p_0. Then when $h > 0$ is small, $\mathcal{M}_h$ is a convex hypersurface and its boundary $\partial\mathcal{M}_h$ is itself a closed convex hypersurface of dimension $n-1$. If for any point $p \in \partial\mathcal{M}_h$ and any supporting plane P of $\mathcal{M}$ at p, $|\gamma \cdot e_{n+1}| < 1$, where γ is the normal of P, and $e_{n+1} = (0, \cdots, 0, 1)$, one can move the plane $\{x_{n+1} = h\}$ upwards by a small distance $\delta > 0$ such that $\mathcal{M}_{h+\delta}$ is a convex hypersurface and $\partial\mathcal{M}_{h+\delta}$ is a closed convex hypersurface. Therefore we can increase h to a level $\bar{h}$ such that either $\mathcal{M}_{\bar{h}} \cap \partial\mathcal{M} \neq \emptyset$, or e_{n+1} is the normal of a supporting plane of $\mathcal{M}$ at some point in $\mathcal{M}_{\bar{h}} \cap \{x_{n+1} = \bar{h}\}$.

In the latter case, by delicate reasoning one concludes that $\mathcal{M}$ is a closed convex hypersurface. In the former case, one can rotate $\mathcal{M}$ step by step, and apply the above moving plane argument in each step, such that $\mathcal{M}$ is in a position such that the boundary $\partial\mathcal{M}$ is on the plane $\hat{P} = \{x_{n+1} = \hat{h}\}$ for some constant $\hat{h}$ and $\mathcal{M}_h$ is convex for all $h < \hat{h}$. For more details, see [TW4]. □

If we assume that $\mathcal{M}$ is compact with $\partial\mathcal{M}$ lying on a plane, then it is easy to see that $\mathcal{M}$ must be strictly convex at a point. The moving plane argument above also implies the following extension of the classical Hadamard theorem for smooth convex bodies.

Corollary 2.1. *Let $\mathcal{M}$ be a complete, locally convex hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$. If $\mathcal{M}$ is strictly convex near some point p_0. Then $\mathcal{M}$ is the boundary of an open convex set in $\mathbb{R}^{n+1}$. If $\mathcal{M}$ is locally strictly convex, $\mathcal{M}$ is the graph of a convex function.*

2.2 The Monge-Ampère measure

Let u be a locally convex function in a domain $\Omega \subset \mathbb{R}^n$. The *normal mapping* N_u is a set valued mapping from Ω to $\mathbb{R}^n$ such that for any $x_0 \in \Omega$,

$$N_u(x_0) = \{y : \quad y \text{ is the gradient of a local supporting function of } u \text{ at } x_0\}.$$

For any subset $\omega \subset \Omega$, we denote $N_u(\omega) = \bigcup_{x\in\omega} N_u(x)$. By our definition, N_u is a local quantity, namely $N_u(x_0)$ depends only on the behavior of u near x_0.

Recall that the *Gauss mapping* G is a mapping from $\mathcal{M}$ to the unit sphere S^n. For any point $p \in \mathcal{M}$, $G(p)$ is the set of normals of all local supporting planes of $\mathcal{M}$ at p. When $\mathcal{M}$ is the graph of u, the normal mapping and Gauss mapping are related by

$$y \in N_u(x_0) \quad \text{if and only if} \quad \gamma = \frac{(y, -1)}{\sqrt{1+|y|^2}} \in G_{\mathcal{M}}(p).$$

If u is differentiable at x_0, it has a unique local supporting function at x_0 and $N_u(x_0)$ is a single point. Otherwise $N_u(x_0)$ is a convex set.

Let u be a locally convex function in Ω. If ω is closed in Ω, so is $N_u(\omega)$. Therefore $N_u(\omega)$ is measurable for any open and closed subsets $\omega \Subset \Omega$, and so also for any Borel subset $\omega \Subset \Omega$. Let Π denote the collection of all Borel subsets of Ω. The *Monge-Ampère measure* μ_u is a function defined on the set Π, given by

$$\mu_u(\omega) = |N_u(\omega)| \quad \forall\ \omega \in \Pi, \tag{2.2}$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$.

First let us consider two examples.

(i) The function u is C^2 smooth. In this case, the normal mapping N_u coincides with the gradient mapping Du, and

$$\mu_u(\omega) = |Du(\omega)| = \int_\omega \det D^2 u. \tag{2.3}$$

Therefore we have

$$\mu_u = \det D^2 u \, dx.$$

(ii) The function u is piecewise linear. Let $\{p_i = (z_i, u(z_i)) : \ i = 1, 2, \cdots, k\}$ be the vertices of the graph $\mathcal{M}_u$. Then $|N_u(\Omega - Z)| = 0$, where $Z = \{z_1, \cdots, z_k\}$. Suppose all z_i are interior point of Ω. Then

$$\mu_u(\omega) = \sum_{z_i \in Z\cap\omega} |N_u(z_i)|. \tag{2.4}$$

Hence we have

$$\mu_u = \sum c_i \delta_{z_i},$$

where $c_i = |N_u(z_i)|$, and δ_z is the Dirac measure at z.

To see that μ_u is a measure, first note that μ_u satisfies a *monotonicity formula*. That is, if u, v are two *convex* functions in Ω, satisfying $u \le v$ in Ω and $u = v$ on $\partial\Omega$, then

$$\mu_u(\Omega) \ge \mu_v(\Omega).$$

Indeed, if φ is a supporting function of v, then $\varphi - c$ is a supporting function of u for an appropriate constant $c \ge 0$. Hence $N_v(\Omega) \subset N_u(\Omega)$ and so $\mu_u(\Omega) \ge \mu_v(\Omega)$. The following lemma shows that μ_u is a measure.

Lemma 2.2. *Let u_k be a sequence of locally convex functions which converges to u locally uniformly. Then μ_{u_k} converges weakly to μ_u.*

Proof. It suffices to prove Lemma 2.2 for smooth u_k. For any compact set $K \Subset \Omega$, by the monotonicity formula and the finite covering, $\mu_{u_k}(K)$ is uniformly bounded. Hence $\{\mu_{u_k}\}$ contains weakly convergent subsequence. Suppose μ_{u_k} sub-converges to a measure $\hat{\mu}$ weakly. To prove that $\hat{\mu} = \mu_u$, it suffices to prove that for any closed small balls $\overline{B}_r \Subset \Omega$, $\hat{\mu}(\overline{B}_r) = \mu_u(\overline{B}_r)$.

Since $\overline{B}_r$ is compact, $N_{u_k}(\overline{B}_r)$ is contained in the δ-neighborhood of $N_u(\overline{B}_r)$ with $\delta \to 0$ as $k \to \infty$. Hence by Fatou's lemma, $\mu_u(\overline{B}_r) \geq \lim_{k\to\infty} \mu_{u_k}(\overline{B}_r) = \hat{\mu}(\overline{B}_r)$.

To prove $\mu_u(\overline{B}_r) \leq \hat{\mu}(\overline{B}_r)$, first we show that if w is a smooth, convex function in $\overline{B}_{r+\varepsilon}$, then

$$\mu_{w_\delta}(B_r) \leq \mu_w(B_r) + C\delta$$

where $\varepsilon > 0$ is a small constant, $w_\delta = w + \delta|x|^2$, and the constant C depends only on n, r, and $\sup_{B_{r+\varepsilon}} |Dw|$. Indeed, extend w to $B_{2r}(0)$ such that w is smooth, convex, and w =constant on ∂B_{2r}. Then extend w to $B_{3r}(0)$ such that w is smooth, convex, and rotationally symmetric in the annulus $\frac{5}{2}r < |x| < 3r$. We have

$$\begin{aligned}\mu_{w_\delta}(B_r) - \mu_w(B_r) &\leq \mu_{w_\delta}(B_{3r}) - \mu_w(B_{3r}) \\ &= \int_{\partial B_{3r}} [W_\delta^{ij}\partial_i w_\delta - W^{ij}\partial_i w]\gamma_j \leq C\delta,\end{aligned}$$

where γ is the unit outer normal, W^{ij} is the co-factor of the matrix $D^2 w$.

For any fixed small constants $\varepsilon, \delta > 0$, it is easy to see that $N_u(\overline{B}_r) \subset N_{u_{k,\delta}}(\overline{B}_{r+\varepsilon})$, provided k is sufficiently large. Hence

$$\mu_u(\overline{B}_r) \leq \lim_{k\to\infty} \mu_{u_{k,\delta}}(\overline{B}_{r+\varepsilon}) \leq \hat{\mu}(\overline{B}_{r+\varepsilon}) + C\delta.$$

Letting $\delta, \varepsilon \to 0$, we obtain $\mu_u(\overline{B}_r) \leq \hat{\mu}(\overline{B}_r)$. $\square$

Lemma 2.2 asserts that the Monge-Ampère measure is weakly continuous with respect to the convergence of functions. By Lemma 2.2 and approximation by smooth functions, we have

$$\mu_{u+v} \geq \mu_u + \mu_v \tag{2.5}$$

for any locally convex functions u and v.

Recall that a measure has a regular part and a singular part,

$$\mu = \mu^{(r)} + \mu^{(s)}, \tag{2.6}$$

where $\mu^{(r)}$ is a measurable function and $\mu^{(s)}$ is a measure supported on a set of Lebesgue measure zero.

Lemma 2.3. *Let u be a locally convex function in Ω. Then*

$$\mu_u^{(r)} = (det\,\partial^2 u)\, dx, \tag{2.7}$$

where $\partial^2 u(x) = D^2 u(x)$ if u is twice differentiable at x, and $\partial^2 u(x) = 0$ otherwise.

Proof. Let u_h be the mollification of u. Then at any point $x \in \Omega$ where u is twice differentiable, $D^2 u_h(x) \to \partial^2 u(x)$ [Zi]. Hence for any measurable set $E \subset\subset \Omega$,

$$\int_E \det \partial^2 u \le \lim_{h \to 0} \int_E \det D^2 u_h.$$

By the weak convergence (Lemma 2.2), it follows that

$$\det \partial^2 u \le \mu_u^{(r)} \quad a.e..$$

To show the equality holds, we have, for a.e. $x_0 \in \Omega$,

$$\mu_u^{(r)}(x_0) = \lim_{\varepsilon \to 0} \frac{\mu_u^{(r)}(B_\varepsilon(x_0))}{|B_\varepsilon(x_0)|} \le \lim_{\varepsilon \to 0} \frac{\mu_u(B_\varepsilon(x_0))}{|B_\varepsilon(x_0)|}.$$

If u is twice differentiable at x_0, then for any $x \in B_\varepsilon(x_0)$,

$$|Du(x) - Du(x_0) - \partial^2 u(x_0)(x - x_0)| \le \delta |x - x_0|$$

for some constant $\delta > 0$, with $\delta \to 0$ as $\varepsilon \to 0$. Suppose for simplicity that $Du(x_0) = 0$. Let $w = \frac{1}{2} x \cdot (\partial^2 u(x_0) + \delta I) \cdot x$, where I is the unit matrix. Then for $|x - x_0|$ sufficiently small, $N_u(B_\varepsilon(x_0)) \subset N_w(B_\varepsilon(x_0))$. It follows

$$\begin{aligned} \mu_u^{(r)}(x_0) &\le \lim_{\varepsilon \to 0} \frac{|N_u(B_\varepsilon(x_0))|}{|B_\varepsilon(x_0)|} \le \lim_{\varepsilon \to 0} \frac{|N_w(B_\varepsilon(x_0))|}{|B_\varepsilon(x_0))|} \\ &= \det(\partial^2 u(x_0) + \delta I) \le \det(\partial^2 u(x_0)) + C\delta, \end{aligned}$$

where C depends on $\partial^2 u(x_0)$ but is independent of ε. Sending ε to zero, we obtain $\mu_u^{(r)} \le \det \partial^2 u$ at x_0. Hence $\mu_u^{(r)} = \det \partial^2 u$. □

2.3 Generalized solutions

Consider the equation

$$\det D^2 u = \nu \ \text{ in } \ \Omega, \tag{2.8}$$

where ν is a nonnegative measure in Ω. We say a function u is a *generalized solution* in the sense of Aleksandrov if u is a locally convex function and $\mu_u = \nu$. In (2.8) we regard $\det D^2 u$ as a measure when u is a generalized solution. But when u is smooth, we also regard $\det D^2 u$ as its density, as in the usual sense.

We say u is a *subsolution* if u is a locally convex function and $\mu_u \ge \nu$. It is easy to see that if u_1, u_2 are two subsolutions, then $u = \max(u_1, u_2)$ is also a subsolution. In particular the sup of a family of subsolutions is a subsolution.

Lemma 2.4. *(The comparison principle) Let $u, v \in C(\overline{\Omega})$ be two locally convex functions. Suppose that $\mu_u \ge \mu_v$ and $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.*

Proof. If $\{v < u\} \ne \emptyset$, let $u_{\varepsilon,\delta} = u + \varepsilon(|x|^2 - R^2) - \delta$, where we choose R large and $\varepsilon, \delta > 0$ such that $u_{\varepsilon,\delta} < u$ on $\partial\Omega$ and the set $\omega_{\varepsilon,\delta} = \{v < u_{\varepsilon,\delta}\} \ne \emptyset$ is compactly contained in Ω. By choosing ε, δ properly, we can assume the diameter of $\omega_{\varepsilon,\delta}$ is small, so that both u and v are convex in it. Observe that $N_{u_{\varepsilon,\delta}}(\omega_{\varepsilon,\delta}) \subset N_v(\omega_{\varepsilon,\delta})$. We have $\mu_{u_{\varepsilon,\delta}}(\omega_{\varepsilon,\delta}) \le \mu_v(\omega_{\varepsilon,\delta})$. But by (2.5) we also have $\mu_{u_{\varepsilon,\delta}}(\omega_{\varepsilon,\delta}) \ge \mu_u(\omega_{\varepsilon,\delta}) + \varepsilon^n |\omega_{\varepsilon,\delta}|$, which is a contradiction. □

Lemma 2.5. *Let Ω be a Lipschitz domain and let φ be a continuous function on $\partial\Omega$. Let*

$$\overline{u} = \sup\{a(x) : \ a \text{ is a linear function and } a \le \varphi \text{ on } \partial\Omega\}.$$

Then $\mu_{\overline{u}} = 0$ in Ω.

Proof. Note that $\overline{u}$ is the sup of linear functions and so it is convex. Recall that by our definition, the Monge-Ampère measure is a local quantity. Hence we may restrict $\overline{u}$ to an arbitrary ball $B \Subset \Omega$. Choose a sequence of convex, piecewise linear function u_k such that $u_k \nearrow \overline{u}$ on ∂B. Let

$$\overline{u}_k = \sup\{a(x) : \ a \text{ linear and } a \le u_k \text{ on } \partial B\}.$$

Then $\overline{u}_k$ is convex, piecewise linear. One easily verifies that $\mu_{\overline{u}_k} = 0$ and $\overline{u}_k \to \overline{u}$ in B. Hence $\mu_{\overline{u}} = 0$ by Lemma 2.2. □

Theorem 2.1. *Let Ω be a bounded, convex domain, ν a finite, nonnegative measure, and φ a convex function in $\overline{\Omega}$. Then there is a unique generalized solution u to (2.8) satisfying the Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$.*

Proof. First we consider the case when $\nu = \sum_{i=1}^N c_i \delta_{z_i}$ is a discrete measure, where z_i are interior points. For any $b = (b_1, b_2, \cdots, b_N)$, where $b_i (i = 1, \cdots, N)$ are constants, let

$$u_b = \sup\{a(x) : \ a \text{ is linear, } a(z_i) \le b_i \text{ and } a \le \varphi \text{ on } \partial\Omega\},$$

Then $u_b = \varphi$ on $\partial\Omega$, $\mu_{u_b} = 0$ in $\Omega - \bigcup_{i=1}^N \{z_i\}$. For any fixed i $(1 \le i \le N)$, $\mu_{u_b}(z_i)$ is decreasing in b_i and increasing in b_j for $j \ne i$. The monotonicity implies that if there exists a subsolution, then there is a solution.

To show that there is a subsolution, note that for any $z_0 \in \Omega$, there is a unique convex function u vanishing on $\partial\Omega$, whose graph is a convex cone, such that $\mu_u = \delta_{z_0}$. Let u_i be a convex function vanishing on $\partial\Omega$, such that $\mu_{u_i} = c_i \delta_{z_i}$. Then $\mu_{\Sigma u_i} \ge \nu$ in Ω and $\sum u_i = 0$ on $\partial\Omega$. Hence $u_0 = \varphi + \sum u_i$ is a subsolution.

Next we consider the case when ν is a general finite measure. Let $\nu_k = \sum_{i=1}^{N_k} c_{i,k} \delta_{x_{i,k}}$ be a sequence of discrete measures converging to ν weakly. Let u_k be the corresponding generalized solution. By the Hölder estimate in Lemma 3.3 below, u_k is uniformly bounded and sub-converges to a convex function u_0, which is a generalized solution by Lemma 2.2. The uniqueness follows by the comparison principle. □

Corollary 2.2. *Assume ν is a finite, nonnegative measure in Ω and φ is a locally convex function in $\overline{\Omega}$ satisfying $\mu_\varphi \geq \nu$. Then there exists a locally convex function u satisfying $\mu_u = \nu$ in Ω and $u = \varphi$ on $\partial\Omega$.*

Proof. Let u be the sup of all subsolutions. For any small ball $B \subset \Omega$, restricting u to the ball and applying Theorem 2.1, we conclude that $\mu_u = \nu$ in B. □

The proof of Lemma 2.2 is from [TW1] and that of Lemma 2.3 from [TW6]. We refer the reader to [CY2, Gut, P8, RT, U2] for more discussions on the Monge-Ampère measure and generalized solution.

3 A priori estimates

3.1 Minimum ellipsoid

The following lemma, due to John [J], plays an important role in the study of the Monge-Ampère equation.

Lemma 3.1. *Let Ω be a bounded, convex domain in $\mathbb{R}^n$. Then among all ellipsoids containing Ω, there is a unique ellipsoid E of smallest volume such that*

$$\frac{1}{n}E \subset \Omega \subset E. \tag{3.1}$$

In this paper we denote by $\alpha\Omega$ the α-dilation of Ω with respect to the center of its minimum ellipsoid. Lemma 3.1 is now a well known result. There is also a unique ellipsoid contained in Ω with maximal volume. We call E the *minimum ellipsoid* of Ω. By a rotation of the coordinates, we may assume that E is given by

$$E = \left\{ \sum\nolimits_{i=1}^{n} \left(\frac{x_i - x_{0,i}}{r_i} \right)^2 < 1 \right\},$$

where $x_0 = (x_{0,1}, \cdots, x_{0,n})$. By the unimodular linear transform $T: \ x \to y$,

$$y_i = \frac{r}{r_i}(x_i - x_{0,i}) + x_{0,i}, \quad i = 1, \cdots, n, \tag{3.2}$$

where $r = (r_1 \cdots r_n)^{1/n}$, E becomes the ball $B_r(x_0)$ with

$$B_{r/n}(x_0) \subset T(\Omega) \subset B_r(x_0).$$

We say Ω is *normalized* if its minimum ellipsoid is a ball (namely when T is the identity mapping).

To prove Lemma 3.1, let $V_0 = \inf\{|E| : \ E \in \Phi\}$, where Φ is the set of ellipsoids containing Ω. Let E_k be a sequence of ellipsoids in Φ with $|E_k| \to V_0$. Since E_k contains Ω, it must be uniformly bounded and converges in Hausdorff distance to an minimum ellipsoid E.

To show that E satisfies (3.1), we assume by a linear transform that E is the unit sphere with center at the origin. If (3.1) is not true, let $x_0 \in \partial\Omega$ such that $|x_0| = \inf_{x\in\partial\Omega} |x|$. By a rotation of axes, we assume $x_0 = (0, \cdots, 0, -t)$ with $t \le \frac{1}{n} - \varepsilon$ for some $\varepsilon > 0$, such that the plane $\{x_n = -t\}$ is a tangent plane of $\partial\Omega$ at x_0. Then we have $\Omega \subset G =: B_1(0) \cap \{x_n > -t\}$. It suffices to prove that the unit ball is not the minimum ellipsoid of G.

The proof is very elementary. Let $y_i = x_i/(1+\delta)$ for $i = 1, \cdots, n-1$, and $y_n = x_n(1+\delta)^{n-1}$, where $\delta = \varepsilon^2$. In the new coordinates, G is *strictly* contained in the unit sphere with center at $(0, \cdots, 0, h)$ provided ε is sufficiently small, where $h = (1+\delta)^{n-1} - 1 + \delta^2$. We reach a contradiction as E is a minimum ellipsoid.

For the uniqueness we refer the reader to [J]. We remark that the uniqueness of minimum ellipsoids is not needed in our treatment below.

3.2 Uniform and Hölder estimates

Consider the Dirichlet problem for the Monge-Ampère equation,

$$\det D^2 u = \nu \quad \text{in} \ \ \Omega, \tag{3.3}$$

where Ω is a bounded, convex domain in $\mathbb{R}^n$, ν is a finite measure.

Lemma 3.2. *Let u be a bounded, convex solution to (3.3). Suppose $u = 0$ on $\partial\Omega$ and $\nu(\Omega) \le b\nu(\frac{1}{2}\Omega)$ for some positive constant b. Then*

$$C^{-1}\big\{|\Omega|\nu(\Omega)\big\}^{1/n} \le \sup|u| \le C\big\{|\Omega|\nu(\Omega)\big\}^{1/n}, \tag{3.4}$$

where C is a constant depending only on n and b. In particular if $\nu = f\,dx$ and $c_0 \le f \le c_1$ for positive constants c_0, c_1, then

$$C^{-1}|\Omega|^{2/n} \le \sup|u| \le C|\Omega|^{2/n}, \tag{3.5}$$

where C depends only on n, c_0, c_1.

Proof. The Monge-Ampère equation is affine invariant and homogeneous, hence we may assume that $B_{1/n}(0) \subset \Omega \subset B_1(0)$ and $\nu(\Omega) = 1$. To prove (3.4), by convexity it suffices to prove that $C^{-1} \le |u(0)| \le C$.

To prove $|u(0)| \le C$, let w be a convex function which vanishes on $\partial\Omega$, such that its graph is a convex cone with vertex at $(0, u(0))$. Then one easily verifies that $N_w(\Omega) \subset N_u(\Omega)$ and $B_r(0) \subset N_w(\Omega)$ for $r = |u(0)|$ as $\Omega \subset B_1(0)$. Hence $|u(0)|^n \le C|N_w(\Omega)| \le C$.

To prove $|u(0)| \ge C$, let w be the solution of $\det D^2 w = \hat{\nu}$ in Ω, $w = 0$ on $\partial\Omega$, where $\hat{\nu} = \nu$ in $\frac{1}{2}\Omega$ and $\hat{\nu} = 0$ elsewhere. Then $w \ge u$ in Ω. Since $\mu_w = 0$ outside $\frac{1}{2}\Omega$, we have for any $x \in \partial\Omega$, $|Dw(x)| \le \sup|w|/\text{dist}\{x, \partial(\frac{1}{2}\Omega)\} \le 2n\sup|u|$. Hence $\nu(\frac{1}{2}\Omega) = |N_w(\Omega)| \le C\sup|u|^n$. □

Lemma 3.3. *Let u be a generalized solution to (3.3). Suppose $\Omega \subset B_1$ for some unit ball B_1, $u = \varphi$ on $\partial\Omega$ for some convex function $\varphi \in C^\alpha(\overline{\Omega})$, and $\nu(\Omega) \le c_1$ for a constant c_1. Then*

$$|u(x) - u(y)| \le C|x-y|^{\bar\alpha} \quad \forall\ x, y \in \Omega, \tag{3.6}$$

where C depends on n, c_1, and $\|\varphi\|_{C^{\bar\alpha}(\overline{\Omega})}$, and $\bar\alpha = \min(\frac{1}{n}, \alpha)$.

Proof. First consider the case $u \equiv 0$ on $\partial\Omega$. Since u is convex, it suffices to estimate $\sup|u(x_0) - u(y_0)| \le C|y_0 - x_0|^{1/n}$ for $x_0 \in \Omega, y_0 \in \partial\Omega$. For any point $x_0 \in \Omega$, by choosing proper coordinates, we assume that $x_0 = de_n$ and $\Omega \subset \{x_n > 0\}$, where $d = \text{dist}(x_0, \partial\Omega)$ and $e_n = (0, \cdots, 0, 1)$. Then $\Omega \subset \hat\Omega = \{x \in \mathbb{R}^n :\ |x'| < 2, 0 < x_n < 4\}$. Let v and w be convex functions such that their graphs are convex cones, with vertex at $(x_0, u(x_0))$ and bases $\partial\Omega$ and $\partial\hat\Omega$, respectively. Then $N_u(\Omega) \supset N_v(\Omega) = N_v(x_0) \supset N_w(x_0)$. Since w is a convex cone, one easily verifies that $|N_w(x_0)| \ge \frac{C}{d}|u(x_0)|^n$, namely $|u(x_0)| \le C[d\nu(\Omega)]^{1/n}$.

When $u = \varphi$ on $\partial\Omega$ for a convex function $\varphi \in C^\alpha(\overline{\Omega})$, we let u_0 be a solution of $\det D^2 u = \nu$ in Ω which vanishes on $\partial\Omega$. Then $u_0 + \varphi$ is a sub-barrier and we also obtain (3.6). □

Corollary 3.1. *Let u be a generalized solution to (3.3) which vanishes on $\partial\Omega$. Suppose $\nu(\Omega) \le b\nu(\frac{1}{2}\Omega)$ for some constant $b > 0$. Let ℓ be a line segment in Ω with two endpoints $z', z'' \in \partial\Omega$. Let z be a point on ℓ such that $u(z) \le \frac{1}{2}\inf_\Omega u$. Then $|z' - z| \ge C|z'' - z'|$ for some $C > 0$ depending only on n and b.*

Note that the ratio $\frac{|z'-z|}{|z''-z'|}$ is invariant under linear transforms. Hence by making a linear transform we may assume that $|\Omega| = 1$ and Ω is normalized. By Lemma 3.2 and the assumption $\nu(\Omega) \le b\nu(\frac{1}{2}\Omega)$, we may assume furthermore that $\inf u = -1$ and $\nu(\Omega) \le C$. Hence when $u(z) < -\frac{1}{2}$, by Lemma 3.3 we have $\text{dist}(z, \partial\Omega) \ge C_0 > 0$, and so Corollary 3.1 follows.

3.3 Strict convexity and $C^{1,\alpha}$ regularity

We say a measure ν satisfies the *doubling condition* if there exists a constant $b > 0$ such that for any convex set $\omega \subset \Omega$,

$$\nu(\omega) \le b\nu(\frac{1}{2}\omega). \tag{3.7}$$

This condition is invariant under linear transforms. First we consider the strict convexity of solutions.

Lemma 3.4. *Let u be a generalized solution to (3.3). Assume $u = 0$ on $\partial\Omega$, $B_{1/n} \subset \Omega \subset B_1$ is normalized, and ν satisfies the doubling condition (3.7). Then there exists $\beta > 0$, depending on n and b, such that for any $x_0 \in \Omega_\delta$,*

$$u(x) \ge C|x - x_0|^{1+\beta} + \ell_{x_0}(x), \tag{3.8}$$

where ℓ_{x_0} is a support function of u at x_0, $\Omega_\delta = \{x \in \Omega :\ dist(x, \partial\Omega) > \delta\}$, and C is a constant depending on n, b, δ, $\nu(\Omega)$.

Proof. Let $x_0 = 0$ be a point in Ω_δ. By subtracting a linear function, we assume $u(0) = 0$, $u \geq 0$ in Ω, and $u = \varphi$ on $\partial\Omega$, where φ is a linear function with $|D\varphi| \leq C_1$.

We claim that $\inf_{\partial\Omega} \varphi \geq C_0$ for a positive constant C_0 depending only on n, b, δ, $\nu(\Omega)$. If the claim is not true, by the weak continuity of the Monge-Ampère measure (Lemma 2.2) and the Hölder continuity (Lemma 3.3), there is a sequence of generalized solutions u_k which converges in $C^{1/2n}(\overline{\Omega})$ to a generalized solution u of (3.3) which satisfies $u(0) = 0$, $u \geq 0$ in Ω, and $u = \varphi$ on $\partial\Omega$ for a linear function φ with $|D\varphi| \leq C_1$, such that $\inf_{\partial\Omega} u = 0$.

By a rotation of coordinates we assume that $\varphi(x) = a_0 x_n + a_1$ for some $a_0 > 0$. Then the set $\{x \in \partial\Omega : \ u(x) = 0\} \subset \{x_n = -a_1/a_0\}$. Since $0 \in \Omega_\delta$, we have $a_1/a_0 \geq \delta$. Let $z'' = (z''_1, \cdots, z''_n) \in \partial\Omega$ be a boundary point such that $u(z'') = 0$. Choose a point $z = (z_1, \cdots, z_n) \in \{x \in \Omega : \ u(x) = 0\}$ such that $\{x \in \Omega : \ u(x) = 0\} \subset \{x_n \leq z_n\}$. Then $z_n - z''_n \geq -z''_n = a_1/a_0 \geq \delta$.

Let $G_\varepsilon = \{x \in \Omega : u(x) < \varepsilon(x_n - z''_n)\}$. Let $z' \in \partial G_\varepsilon$ such that the three points z, z', and z'' lie on a straight line. Observe that G_ε shrinks to the set $\{u = 0\}$ as $\varepsilon \to 0$, namely $z' \to z$ as $\varepsilon \to 0$. We reach a contradiction by Corollary 3.1. The claim is proved.

Denote $S^0_h = \{x \in \Omega : \ u(x) < h\}$. Make a linear transform $y = Tx$ and $v(y) = u(x)/h$ such that $B_{1/n} \subset T(S^0_h) \subset B_1$. When $h \leq C_0$, $v = 1$ on $T(\partial S^0_h)$. The doubling condition (3.7) and the uniform estimate (Lemma 3.2) implies that $\nu[T(S^0_h)] \leq C_2$. Hence by the Hölder continuity of v (Lemma 3.3), we have $\text{dist}(T(S^0_{h/2}), T(\partial S^0_h)) \geq C_3$, where C_2, C_3 depends only on n, b. Changing back we obtain

$$u(\theta x) \geq \frac{1}{2} u(x) \tag{3.9}$$

for any $x \in \partial S^0_h$, where $\theta = 1 - \frac{1}{2}C_3$. As h is any small constant, it follows that for any x near the origin,

$$u(x) \geq 2^{-k} u(\theta^{-k} x)$$

provided $\theta^{-k} x \in \Omega$. Hence we obtain (3.8) with β given by $\theta^{1+\beta} = 1/2$. □

Remark 3.1. If $\nu = f\,dx$ and $c_0 \leq f \leq c_1$ for positive constants c_0, c_1, the condition $u = 0$ on $\partial\Omega$ in Lemma 3.4 can be relaxed to $u = \varphi$ for some $\varphi \in C^{1,\alpha}$, $\alpha > 1 - \frac{2}{n}$. Indeed, if u is not strictly convex, there is a line segment connecting two boundary points, such that (after subtracting a linear function), $u \geq 0$ in Ω and $u = 0$ on the segment. One easily verifies that

$$|\{x \in \Omega : \ u(x) < h\}| \geq K h^{\frac{n-1}{1+\alpha}} \delta, \tag{3.10}$$

where $|\cdot|$ denotes the Lebesgue measure, K is a constant with $K \to \infty$ as $h \to 0$. When $\alpha > 1 - \frac{2}{n}$, by the uniform estimate (Lemma 3.2), we must have $\inf u < 0$, which is a contradiction.

Lemma 3.5. *Let u, Ω and ν be the same as in Lemma 3.4. Then there exists $\alpha \in (0, 1]$, depending on n, b such that for any $x_0 \in \Omega_\delta$,*

$$u(x) \leq C|x - x_0|^{1+\alpha} + \ell_{x_0}(x), \tag{3.11}$$

where ℓ_{x_0} is a support function of u at x_0, C is a constant depending on n, b, δ, $\nu(\Omega)$.

Proof. Let $x_0 = 0$ be a point in Ω_δ. By subtracting a linear function, we assume $u(x_0) = 0$, $u \ge 0$ in Ω. By the strict convexity of u (Lemma 3.4), the set $S_h^0 = \{u < h\} \Subset \Omega$ if $h > 0$ is small. Suppose there exists $\sigma > 0$ depending only on n and b such that for any small $h > 0$ and any $x \in \partial S_h^0$,

$$u(\frac{1}{2}x) < \frac{1-\sigma}{2}u(x). \tag{3.12}$$

Define α by $1 - \sigma = 2^{-\alpha}$. Then for any $x \in \partial\Omega$ and any $t \in (\frac{1}{2^{k+1}}, \frac{1}{2^k})$,

$$u(tx) \le 2^{-k}(1-\sigma)^k u(x) = [2^{-k}]^{1+\alpha}u(x) \le 2t^{1+\alpha}u(x).$$

Hence $u \in C^{1,\alpha}$.

Inequality (3.12) follows from (3.9) with $\sigma = \frac{1-\theta}{5}$. Indeed, consider the convex function $g(t) = u(tx)$, $t \in [-1, 1]$. Replacing g by $g/g(1)$, we may assume that $g(1) = 1$. Let $\psi(t) = g(t + \frac{1}{2}) - g'(\frac{1}{2})t - g(\frac{1}{2})$. Then $\psi(0) = 0$, $\psi \ge 0$. If $g(\frac{1}{2}) > \frac{1-\varepsilon}{2}$, by convexity we have $1 + \varepsilon \ge g'(\frac{1}{2}) \ge 1 - \varepsilon$ and $\psi(-\frac{1}{2}) \le \varepsilon$. Applying (3.9) to ψ, we have $\psi(-\frac{1}{2}\theta^{-1}) \le 2\psi(-\frac{1}{2}) \le 2\varepsilon$. Hence $g(-\frac{1}{2}\theta^{-1} + \frac{1}{2}) < 0$ when $\varepsilon < \frac{1-\theta}{5}$, we reach a contradiction as $u \ge 0$. □

The strict convexity and $C^{1,\alpha}$ estimate are due to Caffarelli [C1, C3]. Estimate (3.8) can also be found in [CW1].

Remark 3.2. In dimension two, by Aleksandrov and Heinz, a generalized solution to

$$\det D^2 u \ge c_0 \quad \text{in} \quad B_1(0) \tag{3.13}$$

must be strictly convex. We give an elementary proof here. By subtracting a linear function we may suppose that $u(1, 0) = u(-1, 0) = 0$, and $\inf_{x\in\partial B_1}[u(x) + u(-x)]$ is attained at $x = (1, 0)$. We need only to show that $u(0) \le -C$.

By convexity we have $0 \ge u(x_1, 0) \ge -2|u(0)|$ for $x_1 \in (-1, 1)$ and for any $x \in B_{3/4}$,

$$u(x_1, x_2) \ge u(x_1, 0) - C|x_2| \ge -2|u(0)| - C|x_2|,$$
$$u(x_1, x_2) \le u(x_1, 0) + C|x_2| \le C|x_2|.$$

It follows that for any $x \in (-\frac{1}{4}, \frac{1}{4}) \times (-\frac{1}{4}, \frac{1}{4})$,

$$\partial_{x_1}u(x) \le 8(u(\frac{1}{2}, x_2) - u(x_1, x_2)) \le 16(|u(0)| + C|x_2|).$$

Similarly we have $\partial_{x_1}u(x) \ge -16(|u(0)| + C|x_2|)$. Hence by Lemma 2.3, or by approximation by smooth functions, we have $u_{11}u_{22} \ge c_0$ almost everywhere. Hence

$$\int_{-1/4}^{1/4} \frac{1}{u_{22}(x)}dx_1 \le c_0^{-1}\int_{-1/4}^{1/4} u_{11}dx_1 \le \frac{16}{c_0}(|u(0)| + C|x_2|).$$

We obtain

$$\int_{-1/4}^{1/4} u_{22}(x)dx_1 \geq \frac{1}{4}\Big(\int_{-1/4}^{1/4} \frac{1}{u_{22}(x)}dx_1\Big)^{-1} \geq \frac{c_0}{64}(|u(0)|+C|x_2|)^{-1}.$$

It follows that

$$\begin{aligned} C &\geq \int_{-1/4}^{1/4}\Big[\int_0^{1/4} u_{22}(x)dx_2\Big]dx_1 = \int_0^{1/4}\Big[\int_{-1/4}^{1/4} u_{22}(x)dx_1\Big]dx_2 \\ &\geq \frac{c_0}{64}\int_0^{1/4}(|u(0)|+C|x_2|)^{-1}dx_2. \end{aligned}$$

Hence $u(0) \leq -\varepsilon_0$ for some $\varepsilon_0 > 0$ depending only on c_0 and the gradient of u in B_1.

Note that by the Legendre transform, the strict convexity of solutions implies the C^1 smoothness of solutions to the two dimensional Monge-Ampère inequality

$$\det D^2u \leq c_1. \tag{3.14}$$

See also Lemma 6.3 below for the C^1 regularity.

3.4 Second derivative estimate

From now on, we consider smooth solutions to the Monge-Ampère equation

$$\det D^2u = f(x) \quad \text{in} \;\; \Omega, \tag{3.15}$$

where Ω is a bounded, uniformly convex domain in $\mathbb{R}^n$ with $C^{3,1}$ boundary. Assume that $f \in C^{1,1}(\overline{\Omega})$ and f satisfies

$$c_0 \leq f \leq c_1 \tag{3.16}$$

for some positive constants c_0, c_1. Write equation (3.15) in the form

$$\log \det D^2u = \log f$$

Differentiating the equation we get

$$\begin{aligned} &u^{ij}u_{ij\xi} = (\log f)_\xi, \\ &u^{ij}u_{ij\xi\xi} - u^{ik}u^{jl}u_{ij\xi}u_{kl\xi} = (\log f)_{\xi\xi}, \end{aligned} \tag{3.17}$$

where ξ is a unit vector, $\{u^{ij}\}$ is the inverse of $\{u_{ij}\}$. In the above we have used the formulae

$$\begin{aligned} &\partial_{u_{ij}} \log \det D^2u = u^{ij}, \\ &\partial_{u_{ij}}\partial_{u_{kl}} \log \det D^2u = -u^{ik}u^{jl}. \end{aligned}$$

Lemma 3.6. *Let $u \in C^4(\Omega)$ be a convex solution of (3.15). Suppose $u = 0$ on $\partial\Omega$. Then*

$$[-u(x)]D^2u(x) \le C(1+M), \tag{3.18}$$

where C depends on n, $\sup|u|$, and $\|\log f\|_{C^2}$, but is independent of $M = \sup|Du|^2$.

Proof. The proof, due to Pogorelov, is now standard and can be found in [P8, GT]. Here we include the proof for completeness. Let $w = \rho(x)\eta(\frac{1}{2}|Du|^2)u_{\xi\xi}$, where $\rho = -u$, $\eta(t) = (1 - \frac{t}{2M})^{-1/8}$. Assume that $\sup_{x\in\Omega,|\xi|=1} w$ is attained at x_0 and $\xi = e_1$. We may assume that D^2u is diagonal at x_0. Hence

$$\begin{aligned} 0 &\ge \sum_{i=1}^n u^{ii}(\log w)_{ii} \\ &\ge \sum_{i=1}^n u^{ii}\big[(\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}) + (\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}) + (\frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2})\big]. \end{aligned}$$

From $(\log w)_i = 0$ at x_0, we have $\frac{\rho_i}{\rho} = -(\frac{\eta_i}{\eta} + \frac{u_{11i}}{u_{11}})$. Noting that $u^{ii}\eta_i = \eta' u_i$, we have

$$u^{ii}\frac{\eta_i u_{11i}}{\eta u_{11}} = u_i\frac{\eta' u_{11i}}{\eta u_{11}} = -u_i\frac{\eta'}{\eta}(\frac{\rho_i}{\rho} + \frac{\eta_i}{\eta}) \le \frac{\eta'}{\eta}\frac{|Du|^2}{|u|} \le \frac{C}{|u|}.$$

Recalling that $\eta = -u$, we get

$$\begin{aligned} \sum_{i=1}^n u^{ii}(\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}) &= -\frac{n}{\rho} - \frac{u_1^2}{u^2u_{11}} - \sum_{i=2}^n u^{ii}\Big[\frac{\eta_i}{\eta} + \frac{u_{11i}}{u_{11}}\Big]^2 \\ &\ge -\frac{C}{|u|} - \sum_{i=2}^n u^{ii}\Big[\frac{\eta_i^2}{\eta^2} + \frac{u_{11i}^2}{u_{11}^2}\Big], \end{aligned}$$

where we assume that $(-u)u_{11} \ge u_1^2$ at x_0, otherwise we are through. Next by $(\log w)_i = 0$, we have $\frac{u_{111}}{u_{11}} = -(\frac{\rho_1}{\rho} + \frac{\eta_1}{\eta})$ at x_0. Hence we obtain

$$0 \ge u^{ii}(\frac{\eta_{ii}}{\eta} - 3\frac{\eta_i^2}{\eta^2}) - \frac{C}{|u|} + u^{ii}\frac{u_{11ii}}{u_{11}} - 2\sum_{i=2}^n u^{ii}\frac{u_{11i}^2}{u_{11}^2}.$$

Observe that

$$u^{ii}(\frac{\eta_{ii}}{\eta} - 3\frac{\eta_i^2}{\eta^2}) \ge \frac{C}{M}u_{11} - \frac{C}{M},$$

where $C > 0$ is independent of M. From (3.17),

$$u^{ii}\frac{u_{11ii}}{u_{11}} - 2\sum_{i=2}^n u^{ii}\frac{u_{11i}^2}{u_{11}^2} \ge -\frac{(\log f)_{11}}{u_{11}} \ge -C.$$

We obtain $\rho u_{11}(x_0) \le C(1+M)$. Hence $w(x) \le w(x_0) \le C(1+M)$ for any $x \in \Omega$. □

The second derivative estimate and the condition (3.16) implies that the Monge-Ampère operator is uniformly elliptic. Therefore by Calabi's interior third derivative estimate and the Schauder estimate for linear elliptic equations [GT], one obtains higher order derivative estimates for the Monge-Ampère equation, provided f is sufficiently smooth. In the 1980's Evans [Ev] and Krylov [K1, K3] independently established the fundamental interior $C^{2,\alpha}$ estimate for convex (or concave), uniformly elliptic equations, provided $f \in C^{1,1}$. The Evans-Krylov regularity is now well known [GT]. Therefore we have the following regularity theorem.

Theorem 3.1. *Let u be a strictly convex solution of (3.15). Suppose $f > 0$ and $f \in C^{1,1}(\Omega)$. Then $u \in C^{3,\alpha}(\Omega)$ for any $\alpha \in (0,1)$. If furthermore $f \in C^{k,\alpha}(\Omega)$ for some $k \geq 2$ and $\alpha \in (0,1)$, then $u \in C^{k+2,\alpha}(\Omega)$.*

Obviously the zero boundary condition in Lemma 3.6 can be replaced by any affine boundary condition. Pogorelov [P6] indicated that the zero boundary condition can be relaxed to $u \in C^{1,\alpha}(\partial\Omega)$ for some $\alpha > 1 - \frac{2}{n}$. That is if $u \in C^{1,\alpha}(\partial\Omega)$, then u is strictly convex, as was shown in Remark 3.1. See also [U3], where Urbas also proved that if $u \in W^{2,p}(\Omega)$ for $p > n(n-1)/2$, then u is strictly convex and so smooth.

The exponent $\alpha > 1 - \frac{2}{n}$ cannot be improved anymore. Indeed, Pogorelov [P8] found that the function

$$u(x) = (1 + x_1^2)\big[\sum\nolimits_{k>1} x_k^2\big]^{1-\frac{1}{n}} \tag{3.19}$$

satisfies equation (3.15) with

$$f = (4 - \frac{4}{n})^{n-1}(1 + x_1^2)^{n-2}(1 - \frac{2}{n} - (3 - \frac{2}{n})x_1^2),$$

which is a positive, analytic function near the origin.

Much less is known in the degenerate case, namely when (3.16) is relaxed to $0 \leq f \leq c_1$. The second derivative estimates were obtained by many authors under various different conditions, see [GTW] and the references therein. In the degenerate case, a solution is not C^∞ smooth in general, as is easily seen by considering radial solutions to the equation $\det D^2 u = |x|^2$. Here we would like to ask whether the solution of (3.15) is $C^{2,\alpha}$ smooth if u is strictly convex, $f^{1/(n-1)} \in C^{1,1}(\Omega)$ and f satisfies the doubling condition (3.7). Another interesting question is whether the eigenfunction to the Monge-Ampère equation

$$\begin{aligned} \det D^2 u = |\lambda u|^n \;\text{ in }\; \Omega, \\ u = 0 \;\text{ on }\; \partial\Omega. \end{aligned} \tag{3.20}$$

is smooth at the boundary; in particular whether $u \in C^\infty(\overline{\Omega})$ when Ω is uniformly convex with $\partial\Omega \in C^\infty$. It is readily shown by explicit calculation that this is true for balls.

3.5 $C^{2,\alpha}$ estimate

In this section we consider the $C^{2,\alpha}$ estimate for the Monge-Ampère equation (3.15), assuming that f is Hölder continuous. The $C^{2,\alpha}$ estimate is due to Caffarelli, but the proof below is from [JW]. Let u be a convex function in Ω. The *modulus of convexity*, $m = m[u]$, is defined by

$$m(t) = \inf\{u(x) - \ell_z(x) : \quad |x - z| > t\}, \tag{3.21}$$

where $t > 0$, ℓ_z is the supporting function of u at z. Obviously m is a nonnegative function of t. When u is strictly convex, it is a positive function. For any function f, we denote by

$$\omega_f(r) = \sup\{|f(x) - f(y)| : \quad |x - y| < r\}$$

the oscillation of f. The function f is *Dini continuous* if

$$\int_0^1 \frac{\omega_f(r)}{r} dr < \infty. \tag{3.22}$$

We say a convex domain Ω has a *good shape* if

$$\max r_i / \min r_i \le c^* \tag{3.23}$$

for a constant c^* under control, where $r_1, \cdots, r_n$ are the radii of the minimum ellipsoid E, as in (3.2). For any $y \in \Omega, h > 0$, we denote

$$S^0_{h,u}(y) = \{x \in \Omega : \quad u(x) < \ell_y(x) + h\}$$

the corresponding section (sub-level set) of u and denote $S_{h,u}(y) = \partial S^0_{h,u}(y)$ its boundary, where ℓ_y is the tangent plane of u at y. When no confusion arises we will drop the subscript u, and when y is the minimum point of u, we will simply write the section as S^0_h.

Theorem 3.2. *Let $u \in C^2$ be a strictly convex solution of (3.15) in the unit ball $B_1(0)$. Assume that f is Dini continuous and satisfies (3.16). Then $\forall\, x, y \in B_{1/2}(0)$, we have the estimate*

$$|D^2u(x) - D^2u(y)| \le C\big[d + \int_0^d \frac{\omega_f(r)}{r} + d\int_d^1 \frac{\omega_f(r)}{r^2}\big], \tag{3.24}$$

where $d = |x - y|$, $C > 0$ depends only on n, the modulus of convexity $m[u]$, and the constants c_0, c_1 in (3.16). It follows that
(i) If f is Dini continuous, then $u \in C^2(B_{1/2})$, and the modulus of convexity of D^2u can be estimated by (3.24).
(ii) If $f \in C^\alpha(B_1)$ and $\alpha \in (0,1)$, then

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C\big[1 + \frac{\|f\|_{C^\alpha(B_1)}}{\alpha(1-\alpha)}\big]. \tag{3.25}$$

(iii) If $f \in C^{0,1}(B_1)$, *then*

$$|D^2u(x) - D^2u(y)| \le Cd\left[1 + \|f\|_{C^{0,1}}|\log d|\right]. \tag{3.26}$$

In two dimensions, the $C^{2,\alpha}$ estimate (3.25) was obtained by Heinz for the more general equation (1.3), see [Scu] for details. When $n \ge 2$, the $C^{2,\alpha}$ estimate was found by Caffarelli. Part (i) was first obtained by the second author in 1992; the theorem in the above form was proved in [JW], see also [W7]. We also note that, by approximation, the C^2 smoothness assumption for u is not needed.

Proof. By subtracting a linear function we suppose $u(0) = 0$ and $Du(0) = 0$. Let T_h be the unimodular linear transform which normalizes S_h^0 (see (3.2)). By making the change $x \to T_h x/\sqrt{h}$ and $u \to u/h$, we may suppose $h = 1$, S_1^0 is normalized, and

$$\int_0^1 \frac{\omega(r)}{r} \le \varepsilon, \tag{3.27}$$

where $\omega(r) = \omega_f(r)$, ε can be as small as we want, provided h is sufficiently small.

Let u_k $(k = 0, 1, \cdots)$ be the solution of

$$\begin{aligned} \det D^2 u_k = f(0) \quad &\text{in} \quad S^0_{4^{-k},u}, \\ u_k = 4^{-k} \quad &\text{on} \quad \partial S^0_{4^{-k},u}. \end{aligned}$$

Denote

$$\nu(t) = \sup_{z\in B_1} \{|f(x) - f(y)| : \quad x, y \in S^0_{t^2,u}(z)\}$$

and $\nu_k = \nu(2^{-k})$. By the comparison principle, we have $|u_1 - u_0| \le C\nu_0$. Since $S^0_{1,u}$ is normalized, $S^0_{4^{-1},u}$ has a good shape. It follows $\|u_0\|_{C^4(S^0_{3/4,u})} \le C$, $\|u_1\|_{C^4(S^0_{3/16,u_1})} \le C$. Observe that

$$\begin{aligned} \det D^2u_1 - \det D^2 u_0 &= \int_0^1 \frac{d}{dt}\det[D^2u_0 + t(D^2u_1 - D^2u_0)]dt \\ &= a_{ij}(x)\partial_i\partial_j(u_1 - u_0) = 0 \end{aligned}$$

and the operator $L = a_{ij}(x)\partial_i\partial_j$ is linear, uniformly elliptic, with C^2 coefficients. By the Schauder estimates for linear elliptic equations, we obtain

$$|D^m u_0(x) - D^m u_1(x)| \le C\nu_0 \quad \forall \;\; x \in S^0_{4^{-2},u_1}.$$

where $1 \le m \le 3$. The estimate also implies that $S^0_{4^{-2},u_1}$ has a good shape.

By induction we assume that $S^0_{4^{-k-1},u}$ has good shape, with the constant c^* (see (3.23)) independent of k, so that $\nu_k \le \omega(C2^{-k})$ for some $C > 0$ independent of k (depending on c^*). By scaling, we then obtain

$$|D^m u_k(x) - D^m u_{k+1}(x)| \le C2^{(m-2)k}\nu_k \quad \forall \;\; x \in S^0_{4^{-k-2},u_{k+1}}. \tag{3.28}$$

Hence

$$|D^2u_0(x) - D^2u_{k+1}(x)| \le C\sum_{i=0}^{k}\nu_i \le C\int_{2^{-k}}^{1}\frac{\omega(r)}{r}dr,$$

where $C > 0$ is independent of k. The above estimate implies that $S^0_{4^{-k-2},u_{k+1}}$ has a good shape.

For any given point z near the origin,

$$\begin{aligned}|D^2u(z) - D^2u(0)| &\le I_1 + I_2 + I_3 =:\\ |D^2u_k(z) - D^2u_k(0)| &+ |D^2u_k(0) - D^2u(0)| + |D^2u(z) - D^2u_k(z)|.\end{aligned}$$

Let $k \ge 1$ such that $4^{-k-4} \le u(z) \le 4^{-k-3}$. By (3.28) and recalling that $\nu(t) \le \omega(Ct)$, we have

$$I_2 \le C\sum_{j=k}^{\infty}\nu_j \le C\int_0^{|z|}\frac{\omega(r)}{r}.$$

To estimate I_1, denote $h_j = u_j - u_{j-1}$. By (3.28) with $m = 3$,

$$|D^2h_j(z) - D^2h_j(0)| \le C2^j\nu_j|z|.$$

Hence

$$\begin{aligned}I_1 &\le |D^2u_{k-1}(z) - D^2u_{k-1}(0)| + |D^2h_k(z) - D^2h_k(0)|\\ &\le |D^2u_0(z) - D^2u_0(0)| + \sum\nolimits_{j=1}^{k}|D^2h_j(z) - D^2h_j(0)|\\ &\le C|z|\big(1 + \sum\nolimits_{j=1}^{k}2^j\nu_j\big)\\ &\le C|z|\big(1 + \int_{|z|}^{1}\frac{\omega(r)}{r^2}\big).\end{aligned}$$

Similarly one can estimate I_3. Hence we obtain (3.24). □

Note that estimate (3.28) (with $m = 2$) implies the Monge-Ampère equation (3.15) is uniformly elliptic. Hence the estimate (3.24)-(3.26) also follows from [W7] immediately.

3.6 $W^{2,p}$ estimate

In [C2], Caffarelli proved the following $W^{2,p}$ estimate.

Theorem 3.3. *Let u be a strictly convex solution of (3.15) in $B_1(0)$. Suppose that $f \in C^0$ and satisfies (3.16). Then for any $p > 1$,*

$$\|u\|_{W^{2,p}(B_{1/2}(0))} \le C, \tag{3.29}$$

where C depends on n, p, $m[u]$, c_0, c_1, and the modulus of continuity ω_f.

The proof is extremely involved. The basic observation is that when f is continuous, the section S_h^0 is a small perturbation of a ball (after normalization), and the solution is a small perturbation of a quadratic function. It implies that in a sufficiently dense set, the second derivative is close to that of the quadratic function. But to prove the $W^{2,p}$ estimate, one also needs to prove a Calderon-Zygmund type decomposition, using sections instead of standard cubes; and estimate the ratio of major radius and minor radius of the minimum ellipsoid of the sections.

In contrast to the Laplace equation $\Delta u = f$, the continuity of f is necessary for the $W^{2,p}$ estimate for the Monge-Ampère equation [W2]. Caffarelli's proof also implies that for any $p > 1$, there is an $\varepsilon > 0$ such that if $\sup_{B_1} |f - f_0| < \varepsilon$ for positive, continuous function f_0, then $u \in W^{2,p}(B_{1/2})$. In two dimensions, the $W^{2,p}$ estimate was obtained in [NS].

By a similar perturbation argument, one also has the following gradient estimate [JW]

$$|Du(x) - Du(y)| \leq Cd[1 + e^{-2\theta\psi(d)}] \tag{3.30}$$

for any $x, y \in B_{1/2}(0)$, where $d = |x - y|$, $C = C(n, m, C_1, C_2, \omega_f)$, $\theta < \frac{1}{2}$ is a positive constant, and

$$\psi(d) = -\int_d^1 \frac{\omega_{\log f}(r)}{r} dr. \tag{3.31}$$

It follows that if

$$\omega_{\log f}(r) \leq \frac{1}{|\log r|}, \tag{3.32}$$

one has the log-Lipschitz estimate

$$|Du(x) - Du(y)| \leq Cd[1 + |\log d|). \tag{3.33}$$

As pointed out in [JW], the assumption (3.32) should be optimal.

But if (3.32) is strengthened to

$$\omega_{\log f}(r) = o(\frac{1}{|\log r|}),$$

Huang [Hu] proved that the second derivatives are in VMO, namely

$$\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} D^2 u \to D^2 u(x_0)$$

as $r \to 0$.

3.7 Hölder estimate for the linearized Monge-Ampère equation

We consider now the linear operator

$$Lu = L_\varphi u = \sum_{i,j=1}^n \Phi^{ij} u_{ij}, \tag{3.34}$$

where φ is a locally uniformly convex function and Φ^{ij} is the cofactor of the Hessian $D^2\varphi$. The operator may also be written in divergence (and double divergence) form, by virtue of (1.9), namely

$$Lu = \partial_i(\Phi^{ij}\partial_j u) = \partial_i\partial_j(\Phi^{ij}u). \tag{3.35}$$

Hölder estimates and Harnack inequalities were proved by Caffarelli and Gutierrez [CG2] for solutions of the homogeneous equation $Lu = 0$, in terms of the pinching of the Hessian determinant

$$C_0 \le \det D^2\varphi \le C_1 \tag{3.36}$$

for positive constants C_0, C_1. In fact their condition is weaker than (3.36) but still stronger than the doubling condition (3.7) for the Monge-Ampère measure μ associated with φ. However, condition (3.36) is appropriate for our applications to affine geometry.

Their approach is based on that of Krylov and Safonov for Hölder estimates for linear elliptic equations in general form, with sections replacing balls and the Monge-Ampère measure μ replacing Lebesgue measure. Accordingly they obtain the following fundamental oscillation estimate, which we formulate here for the inhomogeneous equation

$$Lu = f\det D^2\varphi. \tag{3.37}$$

Theorem 3.4. *Let φ be a locally strictly convex function satisfying (3.36) and $u \in C^2(\Omega)$ be a locally convex solution of (3.37) in some domain $\Omega \subset \mathbb{R}^n$. Then for any section $S^0_h(x_0) \subset \Omega$, we have the estimate*

$$osc_{S^0_h(x_0)} u \le C\left(\frac{h}{h_0}\right)^{\alpha}\left\{osc_{S^0_{h_0}(x_0)} u + h^{1/2}\left(\int_{S^0_{h_0}(x)} |f|^n d\mu\right)^{1/n}\right\}, \tag{3.38}$$

where C and α are positive constants depending only on n, C_0, C_1.

In the form (3.38), the estimate is not a regular Hölder estimate. However by applying the strict convexity estimate of Section 3.3, which holds under the doubling condition (see also [CG2]), we obtain the following Hölder estimate

Corollary 3.2. *Let $u \in C^2(B_1(0))$ be a solution of (3.37). Then there exist positive constants α and C, depending on n, C_0, C_1 such that*

$$\|w\|_{C^\alpha(B_{1/2}(0))} \le C\left\{\|u\|_{L^\infty(B)} + \|f\|_{L^n_\mu(B)}\right\}. \tag{3.39}$$

We remark that it is enough to assume that $u \in W^{2,n}$ or even a viscosity solution of (3.37). The inhomogeneous forms of the estimates (3.38) or (3.39) are readily obtained from the case $f \equiv 0$, proved in [CG2]. Fixing a section $S = S^0_h(x_0)$, we solve the linear Dirichlet problem

$$\begin{aligned} Lu_0 &= f\det D^2\varphi \ \text{ in } \ S, \\ u_0 &= 0 \ \text{ on } \ \partial S \end{aligned} \tag{3.40}$$

and apply the Aleksandrov-Bakelman maximum principle [GT] to obtain

$$|u_0| \le C(n)|S|^{1/n}\left(\int_S [f^n \mathrm{det} D^2\varphi]\right)^{1/n}. \tag{3.41}$$

By virtue of Lemma 3.2 and (3.36), we have $|S| \le Ch^{n/2}$, and (3.38) then follows from the homogeneous case, as $L(u - u_0) = 0$ in S.

3.8 Monge-Ampère equations of general form

In this section we consider the equation

$$\det[D^2u - A(x, u, Du)] = f(x, u, Du) \tag{3.42}$$

where $A = \{a_{ij}\}$ is a symmetric matrix, f is a positive, smooth function. Equations of this form arise in many applications such as isometric embedding [HH], optimal transportation [MTW], reflector design [W5, KW], and conformal geometry [T6]. Note also that the form (3.42) is invariant under coordinate transformations.

If A is linear in, or independent of, the gradient Du [HH], one has very similar a priori estimates as the standard Monge-Ampère equation (1.1). When A is nonlinear in Du, the situation is very different. Here we assume that A satisfies

$$a_{ij,p_kp_l}\xi_i\xi_j\eta_k\eta_l \ge c_0|\xi|^2|\eta|^2 \tag{3.43}$$

for any vectors $\xi, \eta \in \mathbb{R}^n$, $\xi \perp \eta$, where c_0 is a positive constant, and $a_{ij,p_kp_l} = \frac{\partial^2}{\partial p_k \partial p_l} a_{ij}$.

Equation (3.42) with A satisfying (3.43), in two dimensions, is called strongly elliptic and the interior second derivative estimate established in [P3]. But in [P3] the assumption (3.43) was assumed for all vectors ξ, η. The interior a priori estimate for equations of the form (3.42), arising in reflector design, was established in [W5] for dimension two and in [GW] for all dimensions. The following estimate was derived in [MTW].

Lemma 3.7. *Let $u \in C^4$ be a solution to (3.42) in $B_r(0)$ such that the matrix $\mathcal{W} = D^2u - A(x, u, Du)$ is positive definite. Suppose $f \in C^{1,1}$, $f \ge f_0$ for some constant $f_0 > 0$, and A satisfies (3.43). Then we have the estimate*

$$|D^2u|(x) \le C \quad \forall\, x \in B_{r/2}(x_0), \tag{3.44}$$

for some C independent of u.

Proof. Writing the above equation in the form $F[\mathcal{W}] = \bar{f}$ and differentiating it twice, we get, by the concavity of F,

$$F^{ij}W_{ij,kk} \ge D_k^2 \bar{f},$$

where $F[\mathcal{W}] = \log \det \mathcal{W}$, $\mathcal{W} = \{W_{ij}\}$, $\bar{f} = \log f$, $W_{ij,k} = \partial_{x_k} W_{ij}$, and $F^{ij} = \frac{\partial}{\partial W_{ij}} F[\mathcal{W}]$. Let $z(x, \xi) = \rho^2 \xi_i \xi_j W_{ij}$, where ρ is a cut-off function. Suppose $\sup\{z(x, \xi) : \quad x \in$

$B_r(0), |\xi| = 1\}$ is attained at $\bar{x}$ and $\xi = (1, 0, \cdots, 0)$. By a rotation of axes we assume that $\{W_{ij}\}$ and $\{F^{ij}\}$ are diagonal at $\bar{x}$, and $W_{11} \geq \cdots \geq W_{nn}$. At $\bar{x}$, we have, by direct computation,

$$\begin{aligned} F^{ii} W_{11,ii} &= F^{ii} W_{ii,11} + F^{ii} A_{ii,p_1p_1} u_{11}^2 + O(\frac{1}{\rho}(1 + \mathcal{F}\mathcal{T})) \\ &\geq F^{ii} A_{ii,p_1p_1} u_{11}^2 + O(\frac{1}{\rho}(1 + \mathcal{F}\mathcal{T})), \end{aligned}$$

where $\mathcal{F} = \sum F^{ii}$ and $\mathcal{T}$ is the trace of the matrix $\mathcal{W}$. By (3.43), $A_{ii,p_1p_1} \geq c_0 > 0$. Hence

$$F^{ii} A_{ii,p_1p_1} \geq c_0 \sum_{i>1} F^{ii} = \frac{1}{2} c_0 \mathcal{F} > 1$$

provided W_{11} is large enough. We obtain

$$\begin{aligned} 0 \geq \sum_i F^{ii} (\log z)_{ii} &\geq -\frac{C}{\rho^2} \mathcal{F} + F^{ii} \frac{W_{11,ii}}{W_{11}} \\ &\geq -\frac{C}{\rho^2} \mathcal{F} + c_0 W_{11} \mathcal{F} + O(\frac{1}{\rho}(1 + \mathcal{F})) \end{aligned}$$

We obtain $\rho^2 W_{11}(x) \leq \rho^2 W_{11}(\bar{x}) \leq C$ for any $x \in B_r(x_0)$. □

Condition (3.43) was introduced in [MTW], where we studied the regularity of potential functions in optimal transportation, and formulated the condition, called A3, in terms of the cost function. This condition is equivalent to (3.43) above, see Remark 4.1 there. We remark that for equations arising in optimal transportation, the condition (3.43) may be satisfied only when $\xi \perp \eta$ but not for general vectors ξ and η, see examples in [MTW, TW8].

Once the second derivative estimate is established, the least eigenvalue of the matrix $\mathcal{W}$ has a positive lower bound and so equation becomes uniformly elliptic. By Evans-Krylov's regularity for fully nonlinear uniformly elliptic equations, we obtain higher order derivative estimates.

The following example by Lewy, see also [Scu], shows that there is no C^2 a priori estimate if A does not satisfies (3.43).

Example. The function

$$u(x, y) = \frac{1}{4}(3x)^{4/3} + \frac{1}{2} y^2$$

is a solution to the equation

$$\begin{vmatrix} u_{xx}, & u_{xy} \\ u_{xy}, & u_{yy} + u_x^2 - 1 \end{vmatrix} = 1$$

In a recent paper [Lo], Loeper showed that for equations arising in optimal transportation, the degenerate form of condition (3.43) is also necessary. More precisely, Loeper proved that if there exist vector ξ and η, $\xi \perp \eta$, such that the left hand side of

(3.43) is negative, there exists smooth, positive f such that solution in $B_1(0)$ is not C^1 near the center but is smooth near the boundary.

The corresponding $C^{2,\alpha}$ estimates as in Theorem 3.2 is not available for the general equation (3.42). But for the equation arising in optimal transportation, the estimate has recently been established by Jiakun Liu and the authors [LTW], under appropriate assumptions on the cost function.

4 Existence and uniqueness of solutions

4.1 The Dirichlet problem

The existence and uniqueness of generalized solutions to the Dirichlet problem

$$\det D^2 u = f(x) \quad \text{in} \quad \Omega, \tag{4.1}$$

$$u = \varphi \quad \text{on} \quad \partial\Omega. \tag{4.2}$$

was proved by Aleksandrov [A4] and Bakelman [B1], see Theorem 2.1 above. Interior a priori estimates were established by Nirenberg [N2] and Pogorelov [P1] in two dimensions and by Calabi [Ca1] and Pogorelov [P5] for higher dimensions. The existence of solutions with interior regularity was obtained by Cheng and Yau [CY2] and Pogorelov [P8]. The regularity near the boundary of solutions was proved by Caffarelli-Nirenberg-Spruck [CNS] and Krylov [K2]. The following theorem, with optimal conditions on f, φ and $\partial\Omega$, was proved in [TW7].

Theorem 4.1. *Assume that Ω is a uniformly convex domain with C^3 boundary, $\varphi \in C^3(\overline{\Omega})$, $f \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, and satisfies*

$$c_0 \le f \le c_1 \tag{4.3}$$

for some positive constants c_0, c_1. Then there is a unique convex solution $u \in C^{2,\alpha}(\overline{\Omega})$ to (4.1).

By Lemma 3.4, the solution is strictly convex, so $u \in C^{2,\alpha}(\Omega)$ by Theorem 3.2. Therefore to prove Theorem 4.1, it suffices to establish the $C^{2,\alpha}$ estimate near the boundary.

An upper bound for the tangential second derivative $u_{\xi\xi}$ follows directly from the boundary condition. Under the assumptions $\partial\Omega, \varphi \in C^3$, there is also a positive lower bound for $u_{\xi\xi}$, and moreover the assumptions can not be relaxed to $\partial\Omega, \varphi \in C^{2,1}$ [W4]. The double normal derivative estimate follows from the equation immediately provided we have the mixed tangential-normal derivative estimate at the boundary.

The proof for the mixed derivative at the boundary is the most involved part in [TW7]. If $f \in C^{0,1}(\overline{\Omega})$, the mixed derivative estimate can be obtained by constructing proper barriers [CNS, W4]. When $f \in C^\alpha$ for some $\alpha \in (0,1)$, the proof involves a delicate iteration.

Once the second derivative estimate is established, all sections (level sets) of u have a good shape, namely the ratio of circum-radius and the in-radius of the minimum ellipsoid is uniformly bounded. Hence the $C^{2,\alpha}$ estimate near the boundary can be proved in the same way as the interior one, see the proof of Theorem 3.2. With the a priori estimate, the existence of solutions then follows from the continuity method.

Remarks.
(i) As mentioned above, the assumptions in Theorem 4.1 is optimal. If either $\partial\Omega$ or φ is $C^{2,1}$, the solution may not be C^2 smooth near the boundary [W4].
(ii) When $\partial\Omega \in C^{3,1}$ and $\varphi \in C^{3,1}$, the global second derivative estimate can be found in [I2]. See also [CNS, GT].
(iii) Pogorelov [P6] indicated, and Urbas [U3] proved, that if $\varphi \in C^{1,\alpha}$ for some $\alpha > 1 - \frac{2}{n}$, then the solution is strictly convex and smooth in Ω. Pogorelov's example (3.19) implies that the exponent $\alpha > 1 - \frac{2}{n}$ cannot be improved.
(vi) For general (nonconvex) domains, if there is a subsolution to the Dirichlet problem, then there is a globally smooth, locally convex solution [HRS, Gu]. The result was extended to the radial graph in [GS1]. For general locally convex hypersurface, a corresponding result was a conjecture by Spruck [Sp] and was obtained in [TW4, GS2].

4.2 Other boundary value problems

We will consider three different second boundary value problems, namely the Neumann problem, the oblique derivative problem, and the problem of prescribing the gradient mapping image.

4.2.1 Neumann problem. The Neumann boundary problem of the Monge-Ampère equation (4.1) was studied by Lions, Trudinger and Urbas [LTU]. They established the following existence and uniqueness result.

Theorem 4.2. *Consider equation (4.1) subject to the Neumann boundary condition*

$$\partial_\gamma u = \varphi(x, u) \quad on \quad \partial\Omega, \tag{4.4}$$

where γ is the unit inner normal. Assume that Ω is a uniformly convex domain with $C^{3,1}$ boundary. Assume $\varphi \in C^{1,1}$, $\varphi_u \geq c_2 > 0$, $f \in C^{1,1}(\overline{\Omega})$, and f satisfies (4.3). Then there is a unique convex solution $u \in C^{3,\alpha}(\overline{\Omega})$ to (4.4).

To prove Theorem 4.2, one first establishes the global gradient estimate. Then by (3.17) and the maximum principle, one finds that the function $u_{\xi\xi} + A|x|^2$ does not attain its maximum at an interior point if A is chosen large enough. Hence we have

$$\sup_\Omega |D^2 u| \leq \sup_{\partial\Omega} |D^2 u| + C. \tag{4.5}$$

For the second derivative estimate at the boundary, by constructing proper barriers one obtains the double normal and mixed second derivative estimates

$$|D_\gamma D u| \leq C \text{ on } \partial\Omega. \tag{4.6}$$

To get the tangential second derivative estimate, one introduces the auxiliary function

$$v(x,\xi) = u_{\xi\xi} - 2\eta(\xi - \eta\gamma)\cdot(D\varphi - DuD\gamma) + K|x|^2$$

where ξ is any unit vector and $\eta = \xi\cdot\gamma$. Choose K large such that v attends its maximum at a boundary point. By the boundary condition one concludes that $u_{\xi\xi} \le C$. See [LTU] for details.

Once the second derivative is bounded, the equation is uniformly elliptic. Hence by [LT], one obtains higher order derivative estimate. The existence of smooth solutions then follows from the continuity method.

4.2.2 Oblique derivative problem. Consider equation (4.1) subject to the oblique derivative condition

$$\partial_\beta u = \varphi(x,u) \tag{4.7}$$

where β is a unit vector on $\partial\Omega$, satisfying

$$\beta\cdot\gamma \ge \beta_0 > 0.$$

A generalized solution to the oblique derivative problem can be obtained under very mild conditions [W1]. For the regularity, assume that Ω, φ, and f satisfy the same conditions as in Theorem 4.2. Assume φ also satisfies

$$\varphi_u \ge \varphi_0, \tag{4.8}$$

where $\varphi_0 > 0$ is a large constant depending on $\partial\Omega, \beta_0$ and β. Then we have the a priori estimate

$$\sup_{\overline{\Omega}} |D^2 u| \le C \tag{4.9}$$

Note that when (4.9) is proved, the equation becomes uniformly elliptic and higher order regularity follows from [LT].

Estimate (4.9) was obtained in [W1]. In two dimensions it was also obtained by Urbas [U4]. To prove (4.9), one first observes that the gradient estimate and the second derivative estimates (4.5) and (4.6) (with $D_\gamma Du$ in (4.6) replaced by $D_\beta Du$) can be obtained in the same way as for the Neumann problem. Hence it suffices to establish the tangential second derivative estimate on the boundary. The estimate [W1] is obtained by introducing the auxiliary function

$$w(x) = u_{\xi\xi}\,\exp[\alpha|Du|^2 + \tau G(x)]$$

where G is uniformly convex function vanishing on $\partial\Omega$, and ξ is a vector field in $\overline{\Omega}$, tangential to $\partial\Omega$ on the boundary. If w attains its maximum at a boundary point x_0, then $\partial_\beta w \le 0$ and one obtains the estimate (4.9) from the boundary condition. If w attains its maximum at an interior point $\bar{x}$, we must have

$$u_{\xi\xi}(\bar{x}) \ge c_0|D^2u|(\bar{x}) \tag{4.10}$$

for a small, but fixed constant $c_0 > 0$. For if not, the function $\sup_{\zeta,x}\{u_{\zeta\zeta} + A|x|^2\}$ attains its maximum at an interior point, where ζ is a unit vector and A is a large constant. But this is ruled out in the proof of (4.5). By (4.10), one can make a linear transformation such that $u_{\xi\xi}(\bar{x})$ is an eigenvalue of the Hessian matrix D^2u at $\bar{x}$, and conclude that $u_{\xi\xi}(\bar{x}) \le C$, as in §3.4.

In dimension two, instead of the above auxiliary function w, one can use the simpler function $w(x) = u_{\xi\xi} + A|x|^2$ and get a more precise estimate for φ_0 in (4.8), see [U4, W1].

Remark. The condition (4.8) for the oblique derivative problem is necessary. To see this, let u be Pogorelov's function, given in (3.19). Let the domain be a small ball $B_r(0)$. Then near the boundary point $x_b = (r, 0, \cdots, 0)$, let $\beta = (-1, \sigma x_2, \cdots, \sigma x_n)$. Then

$$\partial_\beta u = \big[\sigma(2 - \frac{2}{n}) - \frac{2x_1}{1 + x_1^2}\big]u.$$

When σ is large, $\varphi_u \ge \sigma - 1$ also becomes large. Note that if one chooses β the unit outer normal, one finds that the assumption $C^{1,1}$ smoothness of φ in the Neumann problem cannot be dropped.

4.2.3 Prescribing the image of the gradient mapping. That is the boundary condition

$$Du(\Omega) = \Omega^*, \tag{4.11}$$

where Ω^* is a convex domain in $\mathbb{R}^n$. For this boundary condition, one considers the Monge-Ampère equation of the form

$$\det D^2u = f(x)/g(Du) \ \text{ in } \ \Omega, \tag{4.12}$$

where f, g satisfy the necessary condition

$$\int_\Omega f(x) = \int_{\Omega^*} g(y). \tag{4.13}$$

The existence of a generalized solution in the sense of Aleksandrov was proved in [P3]. Brenier [Br] introduced a different weak solution as follows. For any point $y \in \Omega^*$, denote $V(y) = \{x \in \Omega : \ y \in N_u(x)\}$ and $V(\omega) = \bigcup_{y\in\omega} V(y)$ for any subset $\omega \subset \Omega^*$. From the mapping V, one defines a measure ν_u on Ω^*, by letting

$$\nu_u[\omega] = \int_{V(\omega)} f(x). \tag{4.14}$$

If ν_u coincides with g, then we say u is a generalized solution of Brenier.

When both f and g are positive, Brenier's weak solution is equivalent to that of Aleksandrov. For the regularity of solutions, we have the following theorem.

Theorem 4.3. *Assume that both Ω and Ω^* are uniformly convex with $C^{2,\alpha}$ boundary, $\alpha \in (0,1)$. Assume that $f \in C^\alpha(\overline{\Omega})$, $g \in C^\alpha(\overline{\Omega^*})$, and both f and g are strictly positive. Then a solution to (4.11) (4.12) is $C^{2,\alpha}(\overline{\Omega})$ smooth.*

Theorem 4.3 was established by Caffarelli [C5]. Caffarelli also proved the interior regularity provided Ω^* is convex [C4]. If f, g are $C^{1,1}$ smooth and $\partial\Omega, \partial\Omega^*$ are $C^{3,1}$ smooth, uniformly convex, the global smoothness was also obtained in [D3] for $n = 2$ and in [U7] for $n \geq 2$, by different proofs.

4.3 Entire solutions

In this section we prove the uniqueness of solutions to the Monge-Ampère equation

$$\det D^2 u = 1 \quad \text{in} \quad \mathbb{R}^n. \tag{4.15}$$

Theorem 4.4. *Let u be a locally uniformly convex solution to the equation (4.15). Then u is a quadratic function.*

Proof. By subtracting a linear function, we assume that $u(0) = 0$, $u \geq 0$. For any $h > 0$, by convexity, the set $\{u < h\}$ is bounded, convex. Hence u is smooth and locally uniformly convex. We prove that $D^2 u(x) = D^2 u(0)$ for any $x \in \mathbb{R}^n$. By a linear transform, we may assume that $D^2_{ij} u(0) = \delta_{ij}$. Let $T_h : \ y = A_h x$, where A_h is a unimodular matrix, be a linear transform which normalizes the section $\{u < h\}$. Let $z = y/h^{1/2}$, $u_h = u/h$, and $\Omega_h = h^{-1/2} T_h(\{u < h\})$. Then $B_{1/n}(x_h) \subset \Omega_h \subset B_1(x_h)$, where x_h is the center of Ω_h, and u_h satisfies the equation

$$\det D^2 u_h = 1.$$

The uniform estimate in §3.2 implies that

$$c_1 \leq |\inf_{\Omega_h} u_h| \leq c_2.$$

Hence by the Hölder continuity in §3.2, we have $B_{c_0}(0) \subset \Omega_h \subset B_2(0)$, where c_0, c_1 and c_2 depend only on n. By the interior C^3 estimate,

$$\begin{aligned} &|D^2_{ij} u_h(0)| \leq C, \\ &|D^2_{ij} u_h(x) - D^2_{ij} u_h(0)| \leq C|x|, \end{aligned}$$

where C is independent of h. Observe that $D^2 u_h(0) = A'_h D^2 u(0) A_h = A'_h A_h$. Hence from the first estimate, A_h is uniformly bounded. It follows that for any fixed x, $z = h^{-1/2} A_h x \to 0$ as $h \to \infty$. Hence from the second estimate, we obtain

$$D^2 u(x) - D^2 u(0) = A'_h [D^2 u_h(z) - D^2 u_h(0)] A_h \to 0$$

as $h \to \infty$. □

Remarks

(i) The above proof was due to Pogorelov [P7], a different proof was given by Cheng-Yau

[CY3]. When $n = 2$, the result was first proved by Jörgens [Jo]. For $3 \le n \le 5$, the result was established by Calabi [Ca1].
(ii) The local uniform convexity in Theorem 4.4 is not necessary, it suffices to assume that u is a generalized solution of (4.15), because one can show that if φ is a supporting function of u at 0, the contact set $\mathcal{C} = \{u = \varphi\}$ can not contain a straight line, or a ray.
(iii) From the above proof, it is clear that the assumption that u is defined in the entire space $\mathbb{R}^n$ can be replaced by the assumption that u is defined in a convex domain $\Omega \subset \mathbb{R}^n$ and $u \to \infty$ as $x \to \partial\Omega$.
(iv) The existence of entire solutions to the Monge-Ampère equation

$$\det D^2 u = f(x) \ \text{ in } \ \mathbb{R}^n \tag{4.16}$$

was proved in [CW1] provided f satisfies (4.3), and a solution must be of polynomial growth. Condition (4.3) can be relaxed to polynomial growth. For periodic f, it was proved that the solution must be a perturbation of a quadratic function [CL2].
(v) The asymptotic behavior of solutions to (4.15) on the exterior of a domain was investigated in [CL1, FMM]. In the 2-dimension case it was proved in [FMM] that a solution must be a quadratic polynomial plus a logarithmic term at infinity, and in high dimensions a solution must be a quadratic function plus a perturbation of order $O(|x|^{2-n})$ at infinity [CL1]. In other words, at infinity the perturbation has the same order as the fundamental solution of a linear, uniformly elliptic equation.

4.4 Hypersurfaces of prescribed Gauss curvature

4.4.1 The Minkowski problem. As mentioned in the introduction, the existence and uniqueness of solutions to the Minkowski problem were proved by Minkowski [M1, M2] more than a century ago. The regularity was proved by Lewy [L1, L2], Nirenberg [N2], and Pogorelov [P1] in two dimensions, and by Cheng and Yau [CY1] and Pogorelov [P8] in high dimensions. By Caffarelli's strict convexity and $C^{2,\alpha}$ regularity, the solution is $C^{2,\alpha}$ smooth if the Gauss curvature is Hölder continuous. We record the result as follows.

Theorem 4.5. *Given a bounded, positive function K on the unit sphere S^n, satisfying the integral condition*

$$\int_{S^n} x_i K^{-1} = 0, \tag{4.17}$$

there exists a unique (up to translation), closed convex hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$ such that its (generalized) Gauss curvature at $p \in \mathcal{M}$ is equal to $K(\gamma_p)$, where x_i is the coordinate functions, and γ_p is the unit outer normal of $\mathcal{M}$ at p. If furthermore K is Hölder continuous, then $\mathcal{M} \in C^{2,\alpha}$.

Let H be the support function of $\mathcal{M}$. It is a function on the unit sphere S^n, given by

$$H(x) = \sup\{x \cdot p : \ p \in \mathcal{M}\}$$

where $x \cdot p$ denotes the inner product. The sup is attained at a point $p \in \mathcal{M}$ with outer normal x, and the Gauss curvature $K(x)$ at p is given by

$$\det(\nabla^2 H + HI) = \frac{1}{K(x)} \quad \text{on} \quad S^n, \tag{4.18}$$

where I is the unit matrix, and ∇ is the covariant derivative in a local orthonormal frame. Extend H to R^{n+1} such that it is homogeneous of degree one. Then the hypersurface $\mathcal{M}$ can be recovered from H by

$$\mathcal{M} = \{DH(x) = (\partial_1 H, \cdots, \partial_{n+1} H)(x) : \ x \in S^n\},$$

Theorem 4.5 asserts that for any function $K > 0$ on S^n satisfying (4.17), there is a solution H to (4.18), which is unique up to a linear function, and is $C^{2,\alpha}$ smooth if $K \in C^\alpha(S^n)$. From (4.18) one also sees that (4.17) is a necessary condition for the solvability of the Minkowski problem.

A parabolic approach to the existence and regularity of solutions to the Minkowski problem was used in [CW2], where the authors proved that for any closed, convex hypersurface $\mathcal{M}_0$, there is a constant θ such that the parabolic equation

$$\frac{\partial H}{\partial t} = \log[K(x)\det(\nabla^2 H + HI)] \tag{4.19}$$

with initial condition $\theta\mathcal{M}_0$ has a global smooth solution which converges to a solution of the Minkowski problem.

Lutwak [Lu2] introduced a related p-Minkowski problem, which concerns the existence and regularity of solutions to the equation

$$\det(\nabla^2 H + HI) = \frac{H^{p-1}}{K(x)}, \tag{4.20}$$

where p is a constant. When $p = 1$, it reduces to the Minkowski problem. The existence and regularity of solutions have been obtained when $p > -n-1$ [CW3], see also [Lu2, LO, LYZ1, GL]. The p-Minkowski problem was used in [LYZ2] to establish the Sobolev inequality.

The case $p = -n-1$ is special, as it corresponds to the critical exponent in the Blaschke-Santalo inequality [Sch], and equation (4.20) has various geometric interpretations. It is the equation for the Minkowski problem in centro-affine geometry [CW3], and is also the prescribed affine distance problem [HS]. When $K \equiv 1$, a solution to (4.20) is an elliptic affine sphere, and it must be an ellipsoid by Theorem 5.2 below. For a general positive function K, a Kazdan-Warner type obstruction for the existence of solutions was found in [CW3].

Another related problem is the following: given a positive function $f(x) \in \mathbb{R}^{n+1}$, find a closed convex hypersurface $\mathcal{M}$ such that its Gauss curvature is equal to f, namely for any $x \in \mathcal{M}$,

$$K(x) = f(x). \tag{4.21}$$

Problems of this nature were raised by S.T. Yau [Y2], and in particular (4.21) was studied by Oliker [O], who proved the existence of solutions if there exist $0 < R_1 < R_2$ such that

$$\begin{aligned} f(x) &> R_1^{-n} \text{ for } |x| = R_1, \\ f(x) &< R_2^{-n} \text{ for } |x| = R_2, \end{aligned} \tag{4.22}$$

and $\frac{\partial}{\partial\rho}\rho^n f(\rho x) \le 0$ for $x \in S^n$ and $\rho > 0$. The latter assumption was removed by Delanoë [D2]. This problem was also studied by K. Chou [Cu2] via a variational approach. Let σ_k, $k = 1, 2, \cdots$, denote the kth elementary symmetric polynomial of the principal curvature of $\mathcal{M}$. Let $I_k = \frac{1}{n-k}\int_{\mathcal{M}} \sigma_k$. Then the first variation of I_k is given by

$$\langle \delta I_k(\mathcal{M}), \xi\rangle = \int_{\mathcal{M}} \sigma_{k+1}(\xi \cdot \gamma), \tag{4.23}$$

where ξ is any smooth vector on $\mathcal{M}$ and γ is the unit outer normal. Therefore a solution to (4.21) is a critical point of the functional

$$J(\mathcal{M}) = I_{n-1} - \int_{Cl(\mathcal{M})} f(x), \tag{4.24}$$

where $Cl(\mathcal{M})$ denotes the convex body enclosed by $\mathcal{M}$.

Chou proved that if there exists a hypersurface $\mathcal{M}_0$ such that $J(\mathcal{M}_0) < 0$, there is a minimizer of the functional which is a solution of (4.21). In particular if f satisfies (4.22), there is a minimizer of J.

The variational approach was adopted in [W3], where the second author proved that if there is a minimizer, there must be another (unstable) critical point of J, and so (4.21) has at least two solutions. A critical point is unstable if it is neither a minimizer nor a maximizer. In particular he proved that if

$$f(x) \to f_0 \text{ as } |x| \to \infty, \tag{4.25}$$

where f_0 is any positive constant, then (4.21) has an unstable solution.

4.4.2 Boundary value problems. Let Ω be a uniformly convex domain, $\varphi \in C^{1,1}$, and $K \in C^{1,1}(\Omega) \cap C^{0,1}(\overline{\Omega})$ is positive in Ω. The Dirichlet problem

$$\begin{aligned} \det D^2 u &= K(x)(1 + |Du|^2)^{(n+2)/2} \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned} \tag{4.26}$$

has been studied by many authors. The following necessary and sufficient condition for the classical solvability of (4.26) is proved in [TU].

Theorem 4.6. *There is a classical solution $u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ to the Dirichlet problem (4.26) if and only if the following two conditions hold,*

$$\begin{aligned} \int_\Omega K &< \int_{\mathbb{R}^n} (1 + |p|^2)^{-1/(n+2)}, \\ K &= 0 \quad \textit{on } \partial\Omega. \end{aligned} \tag{4.27}$$

The existence and regularity of solutions can also be found in Bakelman [B2] and Lions [Ls1]. Lions proved that (4.26) has a classical solution if there is a subsolution. For the Dirichlet problem (4.26) under conditions (4.27), by constructing proper subbarriers one obtains the global gradient estimate $\sup_\Omega |Du| \le C$, and the existence of solutions follows. If the Gauss curvature does not vanish on the boundary, there exists a smooth function φ such that (4.26) has no solution satisfying the boundary condition [TU].

The limit case

$$\int_\Omega K = \int_{\mathbb{R}^n} (1+|p|^2)^{-1/(n+2)} \tag{4.28}$$

was studied by Urbas in [U1, U5, U6]. He proved the existence of smooth solutions to (4.26) which satisfy the Dirichlet boundary condition $u = \varphi$ in the sense that $u \le \varphi$ on $\partial\Omega$ and if $v \in C^2(\Omega)$ is another solution of (4.26) satisfying $v \le \varphi$ on $\partial\Omega$, then $v \le u$ on $\partial\Omega$.

Under appropriate structural conditions he also proved the global Hölder continuity of the solution and smoothness of the graph of the solution. His results can be stated as follows.

Theorem 4.7. *Let Ω be a uniformly convex domain with $C^{2,1}$ smooth boundary, and K a positive, C^2 smooth function, satisfying (4.28). Let u be a solution to the Dirichlet problem (4.26). Then*
(i) $u \in C^{1/2}(\overline{\Omega})$;
(ii) the graph of u is $C^{2,\alpha}$ smooth for some $\alpha \in (0,1)$;
(iii) the restriction of u on $\partial\Omega$ is $C^{1,\alpha}$ smooth;
(iv) If $\partial\Omega \in C^{k+1,\alpha}$ and $K \in C^{k-1,\alpha}$, $k \ge 2$, then the graph of u is $C^{k+1,\alpha}$ smooth and the restriction of u on $\partial\Omega$ is $C^{k,\alpha}$ smooth.

4.4.3 A Plateau type problem for the Gauss curvature. This is a natural extension of the Dirichlet problem. In this problem we are concerned with the existence of a locally convex hypersuraface of prescribed Gauss curvature with given boundary. This type of problem, for more general curvatures, was suggested by S.T. Yau [Y2]. For the Gauss curvature, we have the following result.

Theorem 4.8. *Let Γ be a smooth disjoint finite collection of closed codimension 2 submanifolds in $\mathbb{R}^{n+1}$. Suppose Γ bounds a locally strictly convex hypersurface $\mathcal{M}_0$ with Gauss curvature $K(\mathcal{M}_0) \ge K_0 > 0$. Then Γ bounds a locally convex hypersurface of Gauss curvature K_0.*

Theorem 4.8 was proved by the authors in [TW4], and also by Guan and Spruck [GS2]. Theorem 4.8 also gave an affirmative answer to a specific conjecture of Spruck [Sp]. Theorem 4.8 was extended to other curvatures, such as the harmonic curvature, in [SUW], by establishing the interior second derivative estimates for corresponding curvature equations.

Our proof of Theorem 4.8 was based on Lemma 2.1, which was also proved in [TW4]. By Lemma 2.1, we infer that for any locally convex hypersurface $\mathcal{M} \subset B_R(0)$

with C^2 boundary $\partial\mathcal{M}$, there exists $r > 0$ which depends only on n, R, $\partial\mathcal{M}$, and the upper and lower bounds of the principal curvatures of $\mathcal{M}$ on $\partial\mathcal{M}$, such that for any point $x \in \mathcal{M}$, $\omega_{r,x}$ is a convex graph with gradient bounded by r^{-1}, where $\omega_{r,x}$ is the component of $\mathcal{M} \cap B_r(x)$ containing x. Consequently one can prove Theorem 4.8 by the Perron method, or by a deformation argument, as the above property implies the compactness in the deformation.

4.5 Variational problems for the Monge-Ampère equation

The Monge-Ampère equation

$$\det D^2 u = f(x, u) \quad \text{in} \quad \Omega \tag{4.29}$$

is the Euler equation of the functional

$$\begin{aligned} J(u) &= \frac{1}{n+1} \int_\Omega (-u) \det D^2 u dx - \int_\Omega F(x, u) \\ &= \frac{1}{n(n+1)} \int_\Omega U^{ij} u_i u_j dx - \int_\Omega F(x, u) \end{aligned} \tag{4.30}$$

where $F(x, u) = \int_0^u f(x, u)$. The functional J has been studied by Gillis in the 1950's, Bakel'man in the 1960's, and later also by Aubin [Au1]. Bakel'man [B3] proved that when f is independent of u, there is a minimizer of the functional J in the set of convex functions. Chou [Cu1] employed a gradient flow method to investigate the functional J. By establishing the global existence and regularity of solutions to the initial boundary value problem for the parabolic Monge-Ampère equation

$$u_t - \log \det D^2 u = \log f(x, u) \tag{4.31}$$

he obtained smooth minimizers when f is sub-linear, namely

$$\lim_{t \to \infty} f(x, t)/|t|^n = 0;$$

and mountain pass solutions when f is super-linear, namely

$$\lim_{t \to \infty} f(x, t)/|t|^n = \infty.$$

The case when

$$f(x, u) = \lambda |u|^n$$

was studied by Lions [Ls2]. He proved that there is a unique positive constant λ_1 such that (4.29) has a nonzero convex solution. The constant λ_1 is called the eigenvalue of the Monge-Ampère operator.

The variational method was also used in [Cu2, CW2,CW3, W3] for Minkowski type problems.

4.6 Application to the isoperimetric inequality

The Monge-Ampère equation has found various applications in analysis and geometry. For example, the Aleksandrov-Bakelman maximum principle, obtained by virtue of the Monge-Ampère equation, plays a fundamental role in the study of non-divergence elliptic equations [GT]. In this section, we use the Monge-Ampère equation to give a new proof of the sharp isoperimetric inequality for general domains in $\mathbb{R}^n$,

$$|\Omega|^{1-1/n} \le \frac{1}{n\omega_n^{1/n}} |\partial\Omega|_{n-1}, \tag{4.32}$$

where ω_n is the volume of the unit ball. The proof was found by the first author [T2], which we include here.

Let $\Omega \subset \mathbb{R}^n$ be any bounded domain and let χ be the characteristic function of the domain. Let u be a generalized solution to the Monge-Ampère equation

$$\det D^2 u = \chi \quad \text{in} \quad B_R(0).$$

The solution may not be smooth but we can choose a sequence of positive, smooth functions χ_k to approximate χ. We choose $R > r > 0$ such that $\Omega \subset B_r(0)$. Similar to the proof of (3.4) we have

$$\omega_n \sup_{B_R} |u|^n \le (R+r)^n \int_{B_R} \det D^2 u = (R+r)^n \int_{B_R} |\Omega|.$$

By convexity, we then obtain

$$\omega_n \sup |Du|^n \le \left(\frac{R+r}{R-r}\right)^n |\Omega|.$$

Hence

$$\sup |Du| \le \omega_n^{-1/n} \left(\frac{R+r}{R-r}\right)^n |\Omega|^{1/n}.$$

By the arithmetic-geometric mean inequality,

$$\frac{1}{n}\Delta u \ge [\det D^2 u]^{1/n} = 1 \quad \text{in } \Omega.$$

We obtain

$$\begin{aligned} |\Omega| &\le \frac{1}{n}\int_\Omega \Delta u = \frac{1}{n}\int_{\partial\Omega} \gamma \cdot Du \\ &\le \frac{|\partial\Omega|_{n-1}}{n} \omega_n^{-1/n} \left(\frac{R+r}{R-r}\right)^n |\Omega|^{1/n}, \end{aligned}$$

where γ is the unit outer normal. Letting $R \to \infty$ we obtain the isoperimetric inequality. The inequality (4.32) is sharp as all inequalities above optimal when Ω is a ball. We remark that a similar proof may be made using the Gauss curvature equation.

5 The affine metric

5.1 Affine completeness

Let $\mathcal{M}$ be a smooth, locally convex hypersurface in $\mathbb{R}^{n+1}$. The affine metric, which is also called the Berwald-Blaschke metric, is given by

$$g = K^{-1/(n+2)} II, \tag{5.1}$$

where K is the Gauss curvature and II is the second fundamental form on $\mathcal{M}$. If locally $\mathcal{M}$ is a graph given by $x_{n+1} = u(x)$. Then

$$g = \rho u_{x_i x_j} dx_i dx_j, \tag{5.2}$$

where $\rho = (\det D^2 u)^{-1/(n+2)}$.

Recall that a locally convex hypersurface $\mathcal{M}$ is an immersion of a manifold $\mathcal{N}$ in $\mathbb{R}^{n+1}$, there is a natural metric g_e induced from the standard metric in $\mathbb{R}^{n+1}$, which we call the Euclidean metric.

A Euclidean complete convex hypersurface may not be affine complete. For example, by the formula (5.4) below, it is easy to compute that the graph of the convex function

$$u(x) = (1 + |x|^k)^{1/k} \tag{5.3}$$

is not affine complete for any $k > 2$. A fundamental problem is whether affine completeness implies Euclidean completeness for $n > 1$ (when $n = 1$, it is obviously false). This problem appears not only in the classification of hyperbolic affine spheres but also in the study of the affine Bernstein problem. The following theorem from [TW2] gives an affirmative answer to the problem.

Theorem 5.1. *Let $\mathcal{M}$ be an affine complete locally uniformly convex hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$. Then $\mathcal{M}$ is Euclidean complete.*

We sketch the proof here. If $\mathcal{M}$ is not Euclidean complete, then $\partial\mathcal{M} \neq \emptyset$. For any $q \in \mathcal{M}$, we choose coordinates such that $-e_{n+1} = (0, \cdots, 0, -1)$ is the normal of $\mathcal{M}$ at q. Let $\Gamma = \Gamma_q$ be the connected domain of $\mathcal{M}$ containing q such that $p \in \Gamma$ if and only if there is a curve $\gamma = \gamma_p$ on $\mathcal{M}$ connecting q and p such that $G(\gamma)$ is a geodesic line on the south hemisphere, where G is the Gauss mapping. By Lemma 2.1, one can show that there is a point $q \in \mathcal{M}$ such that Γ_q contains a boundary point of $\mathcal{M}$, that is, there is a curve $\gamma^* \subset \Gamma_q$ from q to a point $p^* \in \partial\mathcal{M}$ such that $G(\gamma^*)$ is a geodesic line strictly contained in the south hemisphere.

Let Ω be the projection of Γ on $\{x_{n+1} = 0\}$. Then Γ is the graph of a locally (possibly multi-valued) convex function u in Ω, and $Du(\ell)$ is a line segment in Ω^*, where ℓ is the projection of γ on $\{x_{n+1} = 0\}$, and $\Omega^* = Du(\Omega)$. It follows that Ω^* is a star-shaped domain, and Ω^* is not the entire space $\mathbb{R}^n$.

By (5.2), the affine arc-length of a curve $\gamma \subset \Gamma$ is given by

$$L = \int_\ell (\rho u_{\xi\xi})^{1/2} ds, \tag{5.4}$$

where s is the (Euclidean) arc-length parameter on ℓ, ξ is the unit tangent vector on ℓ. Let u^* be the Legendre transformation of u. Then

$$L = \int_{\ell^*} (\rho^* u^*_{\eta\eta})^{1/2} ds, \tag{5.5}$$

where $\rho^* = [\det D^2 u^*]^{1/(n+2)}$, (observe the power in ρ^* is *positive*).

Let $\mathcal{C}_\theta = \mathcal{C}_\theta(z, r, \xi)$ denote the round cone with vertex at z, radius r, aperture θ, and axial direction ξ. One can show that if $\partial_\xi u^*(z)$ is finite, then $Du^*(y)$ is bounded on the cone $\mathcal{C}_{\theta/2}(y_0, \frac{1}{2}r, \xi)$. It follows that the integral (5.5) is finite for some line segment ℓ from the vertex z to some interior point in the cone. Therefore if there is a boundary point $y_0 \in \partial\Omega^*$ such that Ω^* satisfies the inner cone condition at y_0 and $\partial_\xi u^*(y_0)$ is finite, where ξ is the axial direction of the cone, then $\mathcal{M}$ is not affine complete.

To proceed further we need a technical lemma, which was proved in [TW3], a different proof was given in [Scn].

Lemma 5.1. *Let D be a star shaped domain in $\mathbb{R}^n$ and w be a convex function defined on D. Suppose $\partial_\xi w(y) = \infty$ at any boundary point $y \in \partial D$ at which D satisfies the inner cone condition, where ξ is the axial direction of the cone. Then D is a convex domain.*

From Lemma 5.1, we conclude that $|Du(x)| \to \infty$ as $x \to \partial\Omega$. Indeed, since $\mathcal{M}$ is affine complete, by Lemma 5.1, Ω^* is convex and $|Du^*(y)| = \infty$ for any $y \in \partial\Omega^*$. Let $\{x_k\} \subset \Omega$ be a sequence of points converging to a point $x_0 \in \partial\Omega$. If $y_k = Du(x_k)$ converges to a boundary point, we have $x_k = Du^*(y_k) \to \infty$, which is a contradiction by Lemma 2.1. Hence $|Du(x_k)| \to \infty$ and we also reach a contradiction near the boundary point p^*.

5.2 Affine spheres

Affine spheres were first studied by Tzitzeica [Tz] and by Blaschke and his school [Bls], and were later studied by Calabi, Cheng and Yau, Nirenberg, Pogorelov, and many others [Ca1, Ca2, P7, CY3]. Let $x : \mathcal{M} \to \mathbb{R}^{n+1}$ be a locally convex hypersurface. The affine normal of $\mathcal{M}$ is defined by

$$Y = \frac{1}{n}\Delta x = \frac{1}{n}(\Delta x_1, \cdots, \Delta x_{n+1}), \tag{5.6}$$

where Δ is the Laplacian with respect to the affine metric.

We say a locally uniformly convex hypersurface $\mathcal{M}$ is an *affine sphere*, if the affine normal lines through each point of $\mathcal{M}$ either all intersect at a point, called its center, or else are mutually parallel (center at infinity). An affine sphere is called elliptic, parabolic (center at infinity), or hyperbolic type, according to whether the center is, respectively on the concave side, at infinity, or the convex side.

An affine sphere of elliptic or hyperbolic type is called proper, and a parabolic affine sphere is called improper. If $\mathcal{M}$ is a proper affine sphere with center 0, then

$$Y = -H\,X \tag{5.7}$$

where H is the affine mean curvature, which is a constant. The following proposition relates the affine sphere problem to the Monge-Ampère equation [Ca2].

Proposition 5.1. *Let $\mathcal{M}$ be an affine sphere, given by $x_{n+1} = f(x)$. Let u be the Legendre transform of f. Then $\mathcal{M}$ is an affine sphere of elliptic or hyperbolic type with center at 0 and mean curvature H if and only if u satisfies*

$$detD^2u = (Hu)^{-n-2}; \tag{5.8}$$

and $\mathcal{M}$ is a parabolic affine sphere with all affine normal vector $(0, \cdots, 0, 1)$ if and only if u satisfies

$$detD^2u = 1. \tag{5.9}$$

A basic problem is to classify all complete affine spheres. An elliptic affine sphere has positive affine mean curvature and must be a closed hypersurface. The following result is now classical.

Theorem 5.2. *Let $\mathcal{M}$ be a closed affine sphere of elliptic type. Then $\mathcal{M}$ is an ellipsoid.*

Theorem 5.2 was proved by Blaschke [Bls] for $n = 2$, and by Deicke [De] for $n \geq 2$ and also by Calabi [Ca2]. To prove Theorem 5.2, one computes the Fubini-Pick form J, and find that $\Delta J \geq 0$. Hence when $\mathcal{M}$ is closed, $J \equiv$constant, which implies $\mathcal{M}$ is an ellipsoid [Ca2]. A geometric flow proof was given by Andrews [An].

For parabolic affine spheres, by Theorem 4.4 and the remark thereafter, we have the following classifications.

Theorem 5.3. *Let $\mathcal{M}$ be a complete affine sphere of parabolic type. Then $\mathcal{M}$ is an elliptic paraboloid.*

Theorem 5.3 was first proved by Jörgens [J] for $n = 2$, using complex variable theory, and by Calabi [Ca1] for $3 \leq n \leq 5$, by establishing the third derivative estimate, and for all dimensions by Pogorelov, whose proof was presented in §4.3 above. A different proof was given by Cheng and Yau [CY3].

For hyperbolic affine sphere, we have the following results.

Theorem 5.4. *(i) Every complete, n-dimensional affine sphere with mean curvature $H < 0$ is asymptotic to the boundary of a convex cone with vertex at the center. (ii) Every uniformly convex cone K determines an affine sphere of hyperbolic type, which is asymptotic to the cone K, and uniquely determined by the mean curvature.*

Theorem 5.4 is a conjecture of Calabi [Ca2]. For the first part, one needs to prove that an affine complete hyperbolic sphere is also Euclidean complete, which was achieved by Cheng and Yau [CY3], and was also included in Theorem 5.1 above. The second part, by Proposition 5.1, is equivalent to proving that for any uniformly convex domain Ω, there exists a unique smooth solution to (5.8) subject to the boundary condition $u = 0$ on $\partial\Omega$, and the results were obtained by Cheng and Yau [CY2], in which they proved the existence of (interior) smooth solutions to the Dirichlet problem of the Monge-Ampère equation. By constructing suitable barriers it is not hard to verify that the gradient of the solution converges to infinity near the boundary, so the associated hypersurface is

Euclidean complete. Cheng and Yau also proved that the hyperbolic affine sphere is affine complete [CY3].

6 Affine maximal surfaces

6.1 The affine maximal surface equation

The affine maximal surface equation is a fourth order partial differential equation, given by

$$U^{ij}w_{ij} = 0, \tag{6.1}$$

where U^{ij} is the cofactor of the matrix D^2u, and

$$\det D^2 u = w^{-\frac{n+2}{n+1}}. \tag{6.2}$$

Equation (6.1) is the Euler equation of the affine area functional

$$A(u) = \int_\Omega [\det D^2 u]^{\frac{1}{n+2}}. \tag{6.3}$$

Recall that $\{U^{ij}\}$ is of divergence free, $\sum_i \partial_i U^{ij} = 0 \; \forall\, j$ (see (1.9)). Consequently (6.1) can also be written in the form

$$\Delta h = 0,$$

where Δ is the Laplace-Beltrami operator with respect to the affine metric, and $h = [\det D^2 u]^{-1/(n+2)}$.

The functional A is obviously concave in the set of convex functions. So if u is convex and is a critical point of the functional, it is a local maximizer under local perturbation. Therefore Calabi suggested to call a surface satisfying (6.1) affine maximal, instead of affine minimal, as originally suggested by Chern [Ch]. Accordingly equation (6.1) is called the affine maximal surface equation.

The affine maximal surface equation is invariant under unimodular linear transformations in $\mathbb{R}^{n+1}$. Indeed, the affine area functional can also be written as

$$A(\mathcal{M}) = \int_{\mathcal{M}} K^{\frac{1}{n+2}}. \tag{6.4}$$

Note that a unimodular matrix M can be decomposed as the product of an orthogonal matrix and a diagonal one. From the functional (6.3), the equation (6.1) is invariant under a linear transform determined by a diagonal matrix; and from (6.4), it is also invariant under rotation of axes.

The fourth order equation (6.1) can also be viewed as a system of two second order pdes, one is the linearized Monge-Ampère equation (6.1), regarded as a linear elliptic equation for w, and the other one is the Monge-Ampère equation (6.2). In our a priori estimates in §6.2 below, we will need the $C^{2,\alpha}$ regularity for the Monge-Ampère equation in §3.5, and the Hölder continuity of solutions of the linearized Monge-Ampère equation in §3.7.

6.2 A priori estimates

For the Monge-Ampère equation, we have the interior $W^{2,p}$ estimate and $C^{2,\alpha}$ estimates for strictly convex solutions (Theorems 3.2 and 3.3). For the affine maximal surface equation, we have corresponding $W^{4,p}$ estimate and $C^{4,\alpha}$ estimates. Let us consider the inhomogeneous equation

$$U^{ij}w_{ij} = f. \tag{6.5}$$

Theorem 6.1. *($W^{4,p}$ estimate) Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of (6.5). Then for any $\Omega' \subset\subset \Omega$, $p \geq 1$, we have the estimate*

$$\|u\|_{W^{4,p}(\Omega')} \leq C, \tag{6.6}$$

where C depends on $n, p, \sup_\Omega |f|$, dist$(\Omega', \partial\Omega)$, and the modulus of convexity of u (defined in (3.21)).

Theorem 6.2. *($C^{4,\alpha}$ estimate) Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of (6.5) with $f \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$. Then $u \in C^{4,\alpha}(\Omega)$ and for any $\Omega' \subset\subset \Omega$,*

$$\|u\|_{C^{4,\alpha}(\Omega')} \leq C, \tag{6.7}$$

where C depends on $n, \alpha, \|f\|_{C^\alpha(\Omega)}$, dist$(\Omega', \partial\Omega)$, and the modulus of convexity of u.

To prove the above theorems, we first establish an upper bound for $\det D^2u$, namely if $u \in C^4(\Omega) \cap C^{0,1}(\overline{\Omega})$ is a locally uniformly convex solution of (6.5) with $u = 0$ on $\partial\Omega$, we have, for any point $y \in \Omega$,

$$\det D^2u(y) \leq C, \tag{6.8}$$

where C depends on n, $\text{dist}(y, \partial\Omega)$, $\sup_\Omega |Du|$, $\sup_\Omega f$, and $\sup_\Omega |u|$. To prove (6.8), we introduce the auxiliary function

$$z = \log[w/(-u)^\beta] - A|Du|^2,$$

where $\beta = n(n+1)/(n+2)$ and A is also a positive constant, and show that at the minimum point z, $|u|\Delta u \leq C(1 + |Du|^2)$.

Next we prove that $\det D^2u$ is also bounded from below by a positive constant. For this estimate we assume that there exists an open set $\omega \subset \Omega$ such that $x \cdot Du < u$ in ω and $x \cdot Du = u$ on $\partial\omega$. This condition is satisfied if u is strictly convex. Then for any $y \in \omega$,

$$\det D^2u(y) \geq C, \tag{6.9}$$

where $C > 0$ depends on n, $\text{dist}(y, \partial\omega)$, $\sup_\Omega |Du|$, $\inf_\Omega f$ and $\sup_\omega |u - x \cdot Du|$. The proof of (6.9) is again by proper construction of auxiliary function. Indeed, let

$$z = \log w + \beta \log(u - x \cdot Du) + A|x|^2,$$

where β and A are proper positive constants. Then at the maximum point of z, $|u - x_i u_i| u^{ii} \le C$ and hence (6.9) holds. Note that by the Legendre transform, the second auxiliary function is the same as the first one.

Now regard equation (6.5) as a system of two second order equations, that is regard (6.5) as a linear equation for w, and (6.2) as a Monge-Ampère equation for u. By Caffarelli and Gutierrez's Hölder estimate for the linearized Monge-Ampère equation (Theorem 3.4), we conclude the Hölder continuity of the function w. By Caffarelli's interior Schauder estimate for the Monge-Ampère equation (Theorem 3.2), we then obtain interior Hölder estimate for the second derivatives of u. Hence (6.5) becomes a linear uniformly elliptic equation with Hölder continuous coefficients. It follows that $w \in W^{2,p}_{loc}(\Omega)$ (for any $p < \infty$). The $W^{4,p}_{loc}(\Omega)$ and $C^{4,\alpha}(\Omega)$ estimates for u now follow from standard elliptic regularity theory.

6.3 The affine Bernstein problem

The following Bernstein type result was proved in [TW2].

Theorem 6.3. *A Euclidean complete, affine maximal, locally uniformly convex surface $\mathcal{M} \subset \mathbb{R}^3$ must be an elliptic paraboloid.*

The proof is as follows. By Corollary 2.1, we may assume that $\mathcal{M}$ is the graph of a convex function u, defined in a domain $\Omega \subset \mathbb{R}^2$, with $u(x) \to \infty$ as $x \to \partial\Omega$. By a translation of coordinates, we may assume $u(0) = 0$, $Du(0) = 0$. For any constant $h > 1$, let $\mathcal{M}_h = \{(x_1, x_2, x_3) \in \mathcal{M} : \ x_3 < h\}$ and let Q_h be the convex body enclosed by $\mathcal{M}_h$ and the plane $\{x_3 = h\}$. Let T_h be a linear transform in $\mathbb{R}^3$ which normalizes Q_h such that $B_{1/3} \subset T_h(Q_h) \subset B_1$ (see Lemma 3.1). Let $h \to \infty$, so that $T_h(\mathcal{M}_h)$ sub-converges to a convex surface $\mathcal{M}^*$. If $\mathcal{M}^*$ is strictly convex, then by the a priori estimates in §6.2, $T_h(\mathcal{M}_h)$ is uniformly convex and smooth. Therefore the proof of Theorem 4.4 implies that u is a quadratic function.

Therefore it suffices to show that $\mathcal{M}^*$ is strictly convex. If this is not the case, there is a supporting plane P such that the contact set $\mathcal{C} = \mathcal{M}^* \cap P$ contains more than one point. Since $\partial\mathcal{M}^*$ is on a plane, there is an interior point p of $\mathcal{M}^*$ which is an extreme point of $\mathcal{C}$, namely there is a unit vector γ such that $(q - p) \cdot \gamma > 0$ for all $q \in \mathcal{C}$, $q \ne p$. We may assume that p is the origin and $\gamma = e_1$. Suppose P is given by $x_3 = \varphi(x)$, where $x = (x_1, x_2)$. Let $p^* = (x_1^*, x_2^*, x_3^*)$ ($\ne p$) be a point in $\mathcal{C}$. Let $\varphi_\varepsilon(x) = \varphi(x) - \varepsilon(x_1 - x_1^*)$. Then $\mathcal{M}^*$ and φ_ε enclose a convex body $G_\varepsilon \subset \mathbb{R}^3$. Let T_ε be a linear transform such that $B_{1/3} \subset T_\varepsilon(G_\varepsilon) \subset B_1$. Then $T_\varepsilon(G_\varepsilon)$ sub-converges as $\varepsilon \to 0$ to a convex surface G^*, and G^* is not C^1 at the origin. Hence we may rotate the axes (x_1, x_2, x_3) such that G^* is contained in $x_3 \ge a|x_1|$ for some $a > 0$ and the set $G^* \cap \{x_3 = 0\}$ is either a single point, or a segment.

In the former case, by (6.8) and Lemma 6.3 below we get a contradiction. In the latter case, if at both endpoints of the segment, all supporting planes of G^* have uniformly bounded gradient, by 6.8) and Lemma 6.3 we also get a contradiction. Otherwise at an endpoint, G^* is contained in a round cone with vertex at the point, namely $G^* \subset \{x_3 \ge \bar{a}|x|\}$ for some $\bar{a} > 0$, after a proper rotation of the coordinates. Therefore again we reach a contradiction by (6.8).

Lemma 6.3. *Let u be a convex function defined in a convex domain $\Omega \subset \mathbb{R}^2$. Suppose $u(0) = 0$, $u > 0$ on $\partial\Omega$, and $u(x) \geq a|x_1|$ for some constant $a > 0$. Then the density of the Monge-Ampère measure μ_u cannot be a bounded function.*

Under the assumptions of Lemma 6.3, there is a section S_h, which contains the origin 0, such that $|N_u(S_h^0)|/|S_h^0| \to \infty$, which implies the density of μ_u cannot be a bounded function. We note that Lemma 6.3 goes back to Aleksandrov [A2], in which he established the C^1 smoothness of solutions to the Minkowski problem.

The affine Bernstein problem was proposed by Chern [Ch], who conjectured that an entire convex solution to (6.1) in $\mathbb{R}^2$ must be a quadratic function. The Bernstein problem was investigated by Calabi [Ca3], who proved Chern's conjecture if in addition the surface is affine complete. We proved the Chern conjecture in [TW2] by an argument outlined above. A related problem, called the Calabi conjecture [Si], is whether the Bernstein property holds under affine completeness alone. We have an affirmative answer by our Theorem 5.1. That is

Theorem 6.4. *An affine complete, affine maximal, locally uniformly convex surface $\mathcal{M} \subset \mathbb{R}^3$ is an elliptic paraboloid.*

We remark here that in our proof of Theorem 6.3, by using Bernstein's original result that a bounded entire solution of a homogeneous elliptic equation in two dimensions is a constant, we can use Jörgens's theorem rather than Caffarelli-Gutierrez theory in the proof, (see [T4, T5]). We also note that Theorem 6.4 was proved by Li and Jia in [LJ] by a different method. The affine Bernstein problem in high dimensions is an open problem. We would like to make the following conjecture.

Conjecture. When $n \leq 9$, a Euclidean complete, locally uniformly convex affine maximal hypersurface in $\mathbb{R}^{n+1}$ must be an elliptic paraboloid. When $n \geq 10$, there is a smooth, locally convex affine maximal hypersurface in $\mathbb{R}^{n+1}$ which is not an elliptic paraboloid.

An interesting question is whether an affine complete, locally uniformly convex affine maximal hypersurface is an elliptic paraboloid in dimensions $n \geq 3$. In [TW2] we proved the function

$$u(x) = \sqrt{|x'|^9 + x_{10}^2},$$

where $x' = (x_1, \cdots, x_9)$, is affine maximal. This function has a singular point, and is affine invariant and is analogous to a minimal cone in minimal surface theory. A possible approach to prove the conjecture is to classify all affine maximal cones, and to prove that the affine Bernstein theorem holds if and only if there is an affine maximal cone.

6.4 The first boundary value problem

The affine maximal surface equation is a nonlinear, fourth order partial differential equation. One needs to impose two boundary conditions. The first boundary value problem is to prescribe the solution and its gradient, namely

$$u = \varphi \quad \text{on} \quad \partial\Omega, \tag{6.10}$$

$$Du = D\varphi \quad \text{on} \quad \partial\Omega, \tag{6.11}$$

where Ω is a bounded, Lipschitz domain in $\mathbb{R}^n$, and $\varphi \in C^2(\overline{\Omega})$ is a locally uniformly convex function in Ω.

The first boundary value problem has a corresponding variational problem. Denote by $\overline{S}[\varphi, \Omega]$ the set of convex functions v which satisfy $v = \varphi$ on $\partial\Omega$ and $Dv(\Omega) \subset D\varphi(\overline{\Omega})$. Then a solution to the first boundary value problem is a maximizer of the affine area functional A in the set $\overline{S}[\varphi, \Omega]$. Obviously the set $\overline{S}[\varphi, \Omega]$ is compact.

For the existence of maximizers, we need to extend the functional A to nonsmooth convex functions. There are several extended affine area functionals which are all equivalent [Hug, Le]. But here we adopt the definition from [TW2, TW6]. Recall that the Monge-Ampère measure μ_u can be decomposed as the sum of the singular part $\mu_u^{(s)}$ and the regular part $\mu_u^{(r)}$, and the regular part is given by (see Lemma 2.3)

$$\mu_u^{(r)} = \det \partial^2 u \, dx, \tag{6.12}$$

where $\partial^2 u(x) = \partial_{ij} u(x)$ if u is twice differentiable at x and $\partial^2 u(x) = 0$ otherwise. We extend the affine area functional A to nonsmooth functions by

$$A(u) = \int_\Omega [\det \partial^2 u]^{\frac{1}{n+2}}. \tag{6.13}$$

We say a convex function u is *affine maximal* if it is a maximizer of the extended functional A under local convex perturbation. That is for any convex function v such that $u - v$ has compact support in $\Omega' \subset\subset \Omega$, $A(v) \le A(u)$. A locally convex hypersurface $\mathcal{M}$ is called affine maximal if locally it is the graph of an affine maximal function.

Let u be a convex function and η be a continuous function such that $u + t\eta$ is convex for sufficiently small $t \ge 0$. Then

$$\frac{d}{dt} A(u + t\eta)\big|_{t=0} = \frac{1}{n+2} \int_\Omega w U^{ij} \partial_{ij} \eta.$$

Therefore u is affine maximal if and only if for any such η with compact support in Ω,

$$\int_\Omega w U^{ij} \partial_{ij} \eta \le 0. \tag{6.14}$$

For the existence and regularity of maximizers we may consider the more general functional

$$\sup\{A_f(u, \Omega) : \ u \in \overline{S}[\varphi, \Omega]\}, \tag{6.15}$$

where

$$A_f(u, \Omega) = \int_\Omega \{[\det \partial^2 u]^{\frac{1}{n+2}} - f u\} \tag{6.16}$$

and f is a bounded, measurable function.

Theorem 6.5. *Let Ω be a bounded, Lipschitz domain in $\mathbb{R}^n$. Suppose φ is a locally uniformly convex, Lipschitz continuous function, and f is a bounded measurable function. Then there is a unique maximizer u.*

The existence of maximizers follows from the upper semi-continuity of affine area functional A and the compactness of the set $\overline{S}[\varphi, \Omega]$. The upper semi-continuity was first proved in [Lu1] and a different proof was given in [TW2]. A simple proof was presented in [TW6], which we repeat below.

Lemma 6.4. *The affine area functional A is upper semi-continuous.*

Indeed, by the Hölder inequality we have

$$A(u,\Omega) \le \Big(\int \frac{\det\partial^2 u}{\rho^{n+1}}\Big)^{1/(n+2)} \Big(\int \rho\Big)^{(n+1)/(n+2)}$$

for any positive function ρ. It follows

$$\begin{aligned} A(u,\Omega) &= \inf\Big\{\Big(\int \frac{\det\partial^2 u}{\rho^{n+1}}\Big)^{1/(n+2)}, \quad \rho \in C^0, \rho > 0, \int \rho = 1\Big\} \\ &= \inf\Big\{\Big(\int \frac{d\mu_u^{(r)}}{\rho^{n+1}}\Big)^{1/(n+2)}, \quad \rho \in C^0, \rho > 0, \int \rho = 1\Big\}. \end{aligned}$$

Since the singular part is defined on a set of measure zero, we have

$$A(u,\Omega) = \inf\Big\{\Big(\int \frac{d\mu_u}{\rho^{n+1}}\Big)^{1/(n+2)}, \quad \rho \in C^0, \rho > 0, \int \rho = 1\Big\}.$$

The upper semi-continuity then follows from the weak continuity of the Monge-Ampère measure (Lemma 2.2).

The uniqueness follows from Lemma 6.5 below. Indeed, if u and v are two maximizers, by the concavity of A we have $\det\partial^2 u = \det\partial^2 v$. Hence by Lemma 6.5, $\mu_u = \mu_v$, and so $u = v$ by the uniqueness of generalized solutions to the Dirichlet problem of the Monge-Ampère equation. See Theorem 2.1.

Lemma 6.5. *Let u be a maximizer of (6.15). Then the Monge-Ampère measure μ_u has no singular part.*

Suppose to the contrary that $\mu_u^{(s)} \not\equiv 0$. Since $\mu_u^{(s)}$ is supported on a set of measure zero and $\mu_u^{(r)}$ is an integrable function, it follows that for any positive constant $K \ge 1$, there is a ball $B_r \subset \Omega$ such that

$$\mu_u^{(s)}(B_r) \ge K\mu_u^{(r)}(B_r) + 2K^2|B_r|.$$

Let v be the solution to the Dirichlet problem

$$\begin{aligned} \mu_v &= K\mu_u^{(r)} + 2K^2 \quad \text{in } B_r, \\ v &= u \quad \text{on } \partial B_r. \end{aligned}$$

Since $\mu_u(B_r) \ge K\mu_u^{(r)}(B_r) + 2K^2|B_r|$, the set $E = \{v > u\}$ is not empty. We have

$$\begin{aligned} A_f(v,E) - A_f(u,E) &= \int_E (\det\partial^2 v)^{1/(n+2)} - \int_E (\det\partial^2 u)^{1/(n+2)} - \int_E f(v-u) \\ &\ge \int_E (K\det\partial^2 u + 2K^2)^{1/(n+2)} - \int_E (\det\partial^2 u)^{1/(n+2)} - C|E| \\ &> 0 \end{aligned}$$

if K is sufficiently large.

Let $\widetilde{u} = u$ in $\Omega - E$ and $\widetilde{u} = v$ in E. Then $\widetilde{u} \in \overline{S}[\varphi, \Omega]$ and

$$A_f(\widetilde{u}, \Omega) = A_f(u, \Omega - E) + A_f(v, E) > A_f(u, \Omega),$$

which is a contradiction as u is a maximizer.

For the regularity of maximizers, we have the following theorem.

Theorem 6.6.
(i) If u is a strictly convex maximizer, then u is smooth, in the sense that if $f \in L^\infty(\Omega)$, then $u \in W^{4,p}_{loc}(\Omega)$ $\forall$ $1 < p < \infty$; if $f \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$, then $u \in C^{4,\alpha}(\Omega)$; if $f \in C^\infty$, then $u \in C^\infty(\Omega)$.
(ii) Assume that Ω is a smooth domain and φ is a uniformly convex function. Then a maximizer u is strictly convex if $n = 2$.

To derive the regularity in Theorem 6.6 from the a priori estimates in §6.2, we must prove that the maximizer can be approximated by smooth ones. Our proof is very technical. We use a penalty argument and employ the existence and regularity of solutions to the second boundary problem, treated in §6.5 below.

There are still many interesting problems unresolved. One is whether the maximizer is strictly convex when $n \ge 3$. Another one is the regularity of the maximizer near the boundary. Concerning the second question, one may also ask whether the maximizer in Theorem 6.5 satisfies the boundary condition (6.11). Recall that for the minimal surface equation, the Dirichlet boundary condition is satisfied for arbitrary smooth boundary value if and only if the boundary is mean convex (Theorem 16.8 [GT]), and for the prescribed Gauss curvature equation (4.26), Dirichlet boundary condition is satisfied if and only if the Gauss curvature vanishes on $\partial\Omega$ (Theorem 4.6 above).

6.5 The second boundary value problem

The second boundary problem for the affine maximal surface equation is to prescribe the solution and its Hessian determinant,

$$u = \varphi \quad \text{on} \quad \partial\Omega, \tag{6.17}$$

$$w = \psi \quad \text{on} \quad \partial\Omega, \tag{6.18}$$

where w is given in (6.2), and

$$C_0^{-1} \le \psi \le C_0. \tag{6.19}$$

For the approximation used in the proof of Theorem 6.6, we need to consider the affine maximal surface equation with the inhomogeneous term f depending on u, namely

$$U^{ij}w_{ij} = f(x,u). \tag{6.20}$$

Theorem 6.7. *Let Ω be a uniformly convex domain in $\mathbb{R}^n$, with $\partial\Omega \in C^{3,1}$, $\varphi \in C^{3,1}(\overline{\Omega})$, $\psi \in C^{3,1}(\overline{\Omega})$. Assume that $f \in L^\infty$, f is nondecreasing in u and $f(x,t) \le 0$ when $t \le t_0$ for some $t_0 \le 0$ and (6.19) holds. Then the boundary value problem (6.20), (6.17), (6.18) admits a solution $u \in W^{4,p}_{loc}(\Omega) \cap C^{0,1}(\overline{\Omega})$, $\forall\, p > 1$, with $\det D^2u \in C^0(\overline{\Omega})$. If furthermore $f \in C^\alpha$ for some $\alpha \in (0,1)$, then $u \in C^{4,\alpha}(\Omega)$; if $f \in C^\infty$, then $u \in C^\infty(\Omega)$;*

To prove Theorem 6.7, one regards (6.20) as a system of two second order equations and considers the approximation problem

$$\begin{aligned} U^{ij}w_{ij} &= f \qquad \text{in } \ \Omega, \\ \det D^2u &= \eta_k w^{-(n+2)/(n+1)} + (1-\eta_k) \ \text{ in } \ \Omega, \end{aligned}$$

where $\eta_k \in C_0^\infty(\Omega)$ is a nonnegative cut-off function satisfying $\eta = 1$ in $\Omega_k = \{x \in \Omega : \text{dist}(x,\partial\Omega) < 1/k\}$. For the approximation problem, one proves there exists a solution $u_k \in C^{2,\alpha}(\overline{\Omega})$, $w_k \in W^{2,p}(\Omega)$ $(p > n)$. Moreover, by (6.19), $\det D^2u_k$ has uniform positive upper and lower bounds. Hence u_k is strictly convex and so by Theorem 6.1, $u_k \in W^{4,p}_{loc}(\Omega)$, with uniform upper bound. Sending $k \to \infty$, one obtains a smooth solution to (6.20).

Theorem 6.7 is proved in [TW6], and is sufficient for the proof of Theorem 6.6. The interior regularity in Theorem 6.7 was strengthened to the global regularity in [TW7].

Theorem 6.8. *Under the assumptions in Theorem 6.7, there is a unique uniformly convex solution $u \in W^{4,p}(\Omega)$ ($\forall\ 1 < p < \infty$) to the boundary value problem (6.20), (6.17),(6.18). If furthermore $f \in C^\alpha(\overline{\Omega}\times\mathbb{R})$, $\varphi \in C^{4,\alpha}(\overline{\Omega})$, $\psi \in C^{4,\alpha}(\overline{\Omega})$, and $\partial\Omega \in C^{4,\alpha}$ for some $\alpha \in (0,1)$, then the solution $u \in C^{4,\alpha}(\overline{\Omega})$*

The uniqueness was proved in Lemma 7.1 [TW7]. The condition $f(x,t) \le 0$ when $t \le t_0$ is needed for the lower bound of u. The proof of Theorem 6.8 involves very complicated convexity analysis. The key estimate is the global $C^{2,\alpha}$ regularity for the Monge-Ampère equation with Hölder continuous inhomogeneous term f (Theorem 4.1).

6.6 The affine Plateau problem

The Plateau problem for affine maximal hypersurfaces is the affine invariant analogue of the classical Plateau problem for minimal surfaces, and can be formulated as follows. Let $\mathcal{M}_0$ be a bounded, connected, smooth, locally uniformly convex hypersurface in $\mathbb{R}^{n+1}$, with smooth boundary $\Gamma = \partial\mathcal{M}_0$. Let $S[\mathcal{M}_0]$ denote the set of locally uniformly convex hypersurfaces $\mathcal{M}$ with boundary Γ, which can be smoothly deformed from $\mathcal{M}_0$ in the family of locally uniformly convex hypersurfaces whose Gauss mapping images lie in that of $\mathcal{M}_0$. The affine Plateau problem, proposed by Chern [Ch], Calabi

[Ca4], is to determine a hypersurface $\mathcal{M} \in S[\mathcal{M}_0]$, maximizing the functional A over $S[\mathcal{M}_0]$, that is

$$A(\mathcal{M}) = \sup\{A(\mathcal{M}') : \quad \mathcal{M}' \in S[\mathcal{M}_0]\}. \tag{6.21}$$

A special case is the first boundary value problem studied in §6.4, that is when $\mathcal{M}_0$ is the graph of a smooth, locally uniformly convex function φ.

As with the first boundary value problem, we need to deal with non-smooth, locally convex hypersurfaces. A crucial ingredient of our treatment is Lemma 2.1, from which we have the following two conclusions.

- Suppose the image of the Gauss mapping of $\mathcal{M}_0$ does not cover any hemi-sphere. Then there exists $R > 0$ such that $\mathcal{M} \subset B_R(0)$ for any $\mathcal{M} \in S[\mathcal{M}_0]$.
- The set $S[\mathcal{M}_0]$ is precompact. To see this, extend $\mathcal{M}_0$ to a smooth, locally uniformly convex hypersurface $\widetilde{\mathcal{M}}_0$ such that $\partial\mathcal{M}_0 \subset \widetilde{\mathcal{M}}_0$ and $\widetilde{\mathcal{M}}_0 - \mathcal{M}_0$ is a thin strip. Also extend $\mathcal{M} \in S[\mathcal{M}_0]$ to $\widetilde{\mathcal{M}} = \mathcal{M} \cap \{\widetilde{\mathcal{M}}_0 - \mathcal{M}_0\}$. Then for any point $p \in \widetilde{M}$, there exists $r > 0$, depending only on n, R, and the extended part $\widetilde{\mathcal{M}}_0 - \mathcal{M}_0$ such that the r-neighborhood of p in $\mathcal{M}$ is a convex graph.

Therefore by the upper semi-continuity of the affine area functional, there is a maximizer of the variational problem (6.21). That is

Theorem 6.9. *Let $\mathcal{M}_0$ be a bounded, connected, smooth, locally uniformly convex hypersurface in $\mathbb{R}^{n+1}$, with smooth boundary $\Gamma = \partial\mathcal{M}_0$. Suppose the image of the Gauss mapping of $\mathcal{M}_0$ does not cover any hemi-sphere in S^n. Then there is a locally convex maximizer to (6.21).*

The assumption that the image of the Gauss mapping of $\mathcal{M}_0$ does not cover any hemi-sphere is necessary, otherwise the sup in (6.21) is unbounded.

For the regularity, by the a priori estimates in §6.2 and the approximation, it suffices to establish the local strict convexity of maximizers. Unlike the proof for affine Bernstein theorem, where we need only to consider sections of the solution, in the present situation we must rule out the possibility that the affine maximal hypersurface contains a line segment with both endpoints on the boundary. We were able to prove it in dimension two [TW6]. Therefore we have the following existence and regularity in dimension two.

Theorem 6.10. *Let $\mathcal{M}_0$ be as above. There exists a smooth, locally uniformly convex hypersurface $\mathcal{M} \in S[\mathcal{M}_0]$ solving the variational Plateau problem (6.21) in the two dimensional case if and only if the image of the Gauss mapping of $\mathcal{M}_0$ does not cover any hemisphere.*

The affine maximal surface equation is the Euler equation of the affine volume functional. There are many other related fourth order equations. For example, one may study the Euler equation of the more general functional

$$A(u) = \int_\Omega F(\det D^2 u), \tag{6.22}$$

where F is a concave function satisfying $F(t)/t \to 0$ as $t \to +\infty$. The affine area functional corresponds to the case $F(t) = t^{1/(n+2)}$. An interesting case is when $F(t) = \log t$, and the corresponding Euler equation is called Abreu's equation arising in complex geometry [Do]. We remark that Lemmas 6.4 and 6.5 carry over to the general case for $F \in C^1[0,\infty)$, $F' > 0$ and $F(t) \to \infty$ as $t \to \infty$.

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