

# NONCOMMUTATIVE MANIFOLDS FROM GRAPH AND $k$ -GRAPH $C^*$ -ALGEBRAS

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## Abstract

In [PRen] we constructed smooth  $(1, \infty)$ -summable semifinite spectral triples for graph algebras with a faithful trace, and in [PRS] we constructed  $(k, \infty)$ -summable semifinite spectral triples for  $k$ -graph algebras. In this paper we identify classes of graphs and  $k$ -graphs which satisfy a version of Connes' conditions for noncommutative manifolds.

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## 1. INTRODUCTION

The object of this paper is to address the general definition of noncommutative manifolds using new examples. The phrase ‘noncommutative manifold’ is one which is still open to some degree of interpretation. Broadly speaking, a noncommutative manifold is a spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  satisfying some additional conditions, such as those originally proposed by Connes, [C1]. However, the conditions presented in [C1] only make sense when  $\mathcal{A}$  is unital (that is, a compact noncommutative space).

Moreover, the proof of Connes' spin manifold theorem in [C2] uses a version of Connes' conditions which do not have an immediate generalisation to the noncommutative case, since the volume form

is assumed antisymmetric. We suspect that supplementing the conditions in [C2] with the closedness condition discussed at length in [RV] will resolve this issue.

In this paper we consider the set of conditions presented in [RV], and show that there is a natural generalisation of each to noncommutative, nonunital and semifinite spectral triples. In the process, we show that for certain graphs and  $k$ -graphs, the spectral triples constructed in [PRen, PRS] satisfy these conditions, making them reasonable candidates for the title of noncommutative manifolds. Conditions for noncompact noncommutative manifolds have previously been considered in [R1, R2, GGISV].

We have made an effort to generalise the conditions from [RV] in as minimal and stringent a way as possible. Nevertheless, our conditions must be regarded as provisional. Additional examples and applications are required to determine the ‘correct’ conditions characterising noncommutative manifolds.

The vast majority of examples of noncommutative manifolds in this paper come from nonunital algebras (see [PRen, PRS]), so our conditions must address aspects of ‘noncompact noncommutative manifolds.’ Moreover, most of our examples are semifinite, in that the trace employed is not the operator trace on Hilbert space; it is a faithful normal semifinite trace on a different von Neumann algebra. This is not to say that the  $C^*$ -algebras arising in our examples do not admit type I spectral triples. By considering traces which reflect the geometry of the underlying graph (or  $k$ -graph), we are naturally led to semifinite spectral triples.

For simplicity we discuss only graph algebras (i.e. algebras of 1-graphs) in detail; in a final section we summarise the  $k$ -graph situation, since it is largely similar. The main result for  $k$ -graph algebras that is novel is the construction of a Hochschild  $k$ -cycle representing the volume form.

While the proof of the main result of [RV] is incorrect, several important results still hold. For instance, the conditions introduced in Section 3 do not employ Poincaré Duality in  $K$ -theory, but rather the Morita equivalence condition characterising  $\text{spin}^c$  structures, [P]. The equivalence of these two conditions in the compact commutative case (in the presence of the other conditions) follows from [C2] and [RV, Section 7]. In addition, we do not consider the metric condition, since this has been shown to be redundant in the compact case [RV, Appendix B].

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## 2. BACKGROUND ON GRAPH $C^*$ -ALGEBRAS AND SPECTRAL TRIPLES

**2.1. The  $C^*$ -algebras of Graphs.** For a more detailed introduction to graph  $C^*$ -algebras we refer the reader to [BPRS, KPR, R] and the references therein. A directed graph  $E = (E^0, E^1, r, s)$  consists of countable sets  $E^0$  of vertices and  $E^1$  of edges, and maps  $r, s : E^1 \rightarrow E^0$  identifying the range and source of each edge. The graph is *row-finite* if each vertex emits at most finitely many edges and *locally finite* if it is row-finite and each vertex receives at most finitely many edges. We write  $E^n$  for the set of paths  $\mu = \mu_1\mu_2 \cdots \mu_n$  of length  $|\mu| := n$ ; that is, sequences of edges  $\mu_i$  such that  $r(\mu_i) = s(\mu_{i+1})$  for  $1 \leq i < n$ . The maps  $r, s$  extend to  $E^* := \bigcup_{n \geq 0} E^n$  in an obvious way. A *loop* in  $E$  is a path  $L \in E^*$  with  $s(L) = r(L)$ , we say that a loop  $L$  has an *exit* if there is  $v = s(L_i)$  for some  $i$  which emits more than one edge. If  $V \subseteq E^0$  then we write  $V \geq w$  if there is a path  $\mu \in E^*$  with  $s(\mu) \in V$  and  $r(\mu) = w$ . A *sink* is a vertex  $v \in E^0$  with  $s^{-1}(v) = \emptyset$ , a *source* is a vertex  $w \in E^0$  with  $r^{-1}(w) = \emptyset$ . The infinite path space of  $E$  is denoted  $E^\infty$ ; if  $E$  is row-finite then  $E^\infty$  becomes a  $\sigma$ -compact topological space in the topology inherited from the product topology on  $\prod E^1$ .

For a row-finite graph  $E$ , a *Cuntz-Krieger  $E$ -family* in a  $C^*$ -algebra  $B$  consists of mutually orthogonal projections  $\{p_v : v \in E^0\}$  and partial isometries  $\{S_e : e \in E^1\}$  satisfying the *Cuntz-Krieger relations*

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{\{e:s(e)=v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

Theorem 1.2 of [KPR] shows that there is a universal  $C^*$ -algebra  $C^*(E)$  generated by a universal Cuntz-Krieger  $E$ -family  $\{S_e, p_v\}$ . A product  $S_\mu := S_{\mu_1} S_{\mu_2} \dots S_{\mu_n}$  is non-zero precisely when  $\mu = \mu_1 \mu_2 \dots \mu_n$  is a path in  $E^n$ . Since the Cuntz-Krieger relations imply that the projections  $S_e S_e^*$  are also mutually orthogonal, we have  $S_e^* S_f = 0$  unless  $e = f$ , and words in  $\{S_e, S_e^*\}$  collapse to sums of products of the form  $S_\mu S_\nu^*$  for  $\mu, \nu \in E^*$  satisfying  $r(\mu) = r(\nu)$  (cf. [KPR, Lemma 1.1]). Indeed, because the family  $\{S_\mu S_\nu^*\}$  is closed under multiplication and involution, we have

$$(1) \quad C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}.$$

The algebraic relations and the density of  $\text{span}\{S_\mu S_\nu^*\}$  in  $C^*(E)$  play a critical role.

The universal property of  $C^*(E)$  gives rise to a strongly continuous action  $\gamma$ , called the *gauge action* of  $S^1$  on  $C^*(E)$  satisfying  $\gamma_z(S_e) = z S_e$  for all  $e \in E^1$  and  $\gamma_z(P_v) = P_v$  for all  $v \in E^0$ . Because  $S^1$  is compact, averaging over  $\gamma$  with respect to normalised Haar measure gives an expectation  $\Phi$  of  $C^*(E)$  onto the fixed-point algebra  $C^*(E)^\gamma$ :

$$\Phi(a) := \frac{1}{2\pi} \int_{S^1} \gamma_z(a) d\theta \text{ for } a \in C^*(E), \quad z = e^{i\theta}.$$

The map  $\Phi$  is positive, has norm 1, and is faithful in the sense that  $\Phi(a^*a) = 0$  implies  $a = 0$ .

From Equation (1), it is easy to see that a graph  $C^*$ -algebra is unital if and only if the underlying graph is finite. When we consider infinite graphs, formulas which involve sums of projections may contain infinite sums. To interpret these, we use strict convergence in the multiplier algebra of  $C^*(E)$ . The following is proved in [PR].

**Lemma 2.1.** *Let  $E$  be a row-finite graph, let  $A$  be a  $C^*$ -algebra generated by a Cuntz-Krieger  $E$ -family  $\{T_e, q_v\}$ , and let  $\{p_n\}$  be a sequence of projections in  $A$ . If  $p_n T_\mu T_\nu^*$  converges for every  $\mu, \nu \in E^*$ , then  $\{p_n\}$  converges strictly to a projection  $p \in M(A)$ .*

To build spectral triples we will require traces. The following two results from [PRen] characterise the ‘nice’ traces on graph algebras.

**Lemma 2.2.** *Let  $E$  be a row-finite directed graph.*

- (i) *If  $C^*(E)$  has a faithful, semifinite trace then no loop can have an exit.*
- (ii) *If  $C^*(E)$  has a faithful, semifinite, lower semicontinuous, gauge-invariant trace  $\tau$  then  $\tau \circ \Phi = \tau$  and*

$$\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu} \tau(p_{r(\mu)}).$$

This result was sharpened using the notion of a graph trace, which we also require in this paper.

**Definition 2.3** (See [T]). *If  $E$  is a row-finite directed graph, then a graph trace on  $E$  is a function  $g : E^0 \rightarrow \mathbf{R}^+$  such that for any  $v \in E^0$  which is not a sink we have*

$$(2) \quad g(v) = \sum_{s(e)=v} g(r(e)).$$

**Proposition 2.4.** *Let  $E$  be a row-finite directed graph. Then there is a one-to-one correspondence between faithful graph traces on  $E$  and faithful, semifinite, lower semicontinuous, gauge invariant traces on  $C^*(E)$ .*

Another graph theoretic concept useful for the graphs we will be dealing with is the following.

**Definition 2.5.** *Let  $E$  be a row-finite directed graph. An end will mean either a sink or the shift-tail equivalence class of an infinite path with no exits.*

For more information about shift-tail equivalence see [KPRR, §2].

**Lemma 2.6.** *Let  $A = C^*(E)$  be a graph  $C^*$ -algebra such that no loop in the locally finite graph  $E$  has an exit. Then,*

$$K_0(C^*(E)) = \mathbf{Z}^{\#\text{ends}}, \quad K_1(C^*(E)) = \mathbf{Z}^{\#\text{loops}}.$$

*In particular,  $K_*(C^*(E))$  is finitely generated if there are finitely many ends in  $E$ .*

**2.2. Semifinite Spectral Triples.** We begin with some semifinite versions of standard definitions and results. Let  $\tau$  be a fixed faithful, normal, semifinite trace on the von Neumann algebra  $\mathcal{N}$ . Let  $\mathcal{K}_{\mathcal{N}}$  be the  $\tau$ -compact operators in  $\mathcal{N}$  (that is the norm closed ideal generated by the projections  $E \in \mathcal{N}$  with  $\tau(E) < \infty$ ).

**Definition 2.7.** *A semifinite spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  consists of a Hilbert space  $\mathcal{H}$ , a  $*$ -algebra  $\mathcal{A} \subset \mathcal{N}$  where  $\mathcal{N}$  is a semifinite von Neumann algebra acting on  $\mathcal{H}$ , and a densely defined unbounded self-adjoint operator  $\mathcal{D}$  affiliated to  $\mathcal{N}$  such that*

- 1)  $[\mathcal{D}, a]$  is densely defined and extends to a bounded operator in  $\mathcal{N}$  for all  $a \in \mathcal{A}$ , and
- 2)  $a(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_{\mathcal{N}}$  for all  $\lambda \notin \mathbf{R}$  and all  $a \in \mathcal{A}$ .

*The triple is said to be even if there is  $\Gamma \in \mathcal{N}$  such that  $\Gamma^* = \Gamma$ ,  $\Gamma^2 = 1$ ,  $a\Gamma = \Gamma a$  for all  $a \in \mathcal{A}$  and  $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$ . Otherwise it is odd.*

**Definition 2.8.** *A semifinite spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is  $QC^k$  for  $k \geq 1$  ( $Q$  for quantum) if for all  $a \in \mathcal{A}$  the operators  $a$  and  $[\mathcal{D}, a]$  are in the domain of  $\delta^k$ , where  $\delta(T) = [|\mathcal{D}|, T]$  is the partial derivation on  $\mathcal{N}$  defined by  $|\mathcal{D}|$ . We say that  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is  $QC^\infty$  if it is  $QC^k$  for all  $k \geq 1$ .*

**Note.** The notation is meant to be analogous to the classical case, but we introduce the  $Q$  so that there is no confusion between quantum differentiability of  $a \in \mathcal{A}$  and classical differentiability of functions.

**Remarks concerning derivations and commutators.** By partial derivation we mean that  $\delta$  is defined on some subalgebra of  $\mathcal{N}$  which need not be (weakly) dense in  $\mathcal{N}$ . More precisely,  $\text{dom } \delta = \{T \in \mathcal{N} : \delta(T) \text{ is bounded}\}$ . We also note that if  $T \in \mathcal{N}$ , one can show that  $[|\mathcal{D}|, T]$  is bounded if and only if  $[(1 + \mathcal{D}^2)^{1/2}, T]$  is bounded, by using the functional calculus to show that  $|\mathcal{D}| - (1 + \mathcal{D}^2)^{1/2}$  extends to a bounded operator in  $\mathcal{N}$ . In fact, writing  $|\mathcal{D}|_1 = (1 + \mathcal{D}^2)^{1/2}$  and  $\delta_1(T) = [|\mathcal{D}|_1, T]$  we have

$$\text{dom } \delta^n = \text{dom } \delta_1^n \quad \text{for all } n.$$

We also observe that if  $T \in \mathcal{N}$  and  $[\mathcal{D}, T]$  is bounded, then  $[\mathcal{D}, T] \in \mathcal{N}$ . Similar comments apply to  $[|\mathcal{D}|, T]$ ,  $[(1 + \mathcal{D}^2)^{1/2}, T]$ . The proofs of these statements can be found in [CPRS2].

**Definition 2.9.** *A  $*$ -algebra  $\mathcal{A}$  is smooth if it is Fréchet and  $*$ -isomorphic to a proper dense subalgebra  $i(\mathcal{A})$  of a  $C^*$ -algebra  $A$  which is stable under the holomorphic functional calculus.*

Thus saying that  $\mathcal{A}$  is *smooth* means that  $\mathcal{A}$  is Fréchet and a pre- $C^*$ -algebra. Asking for  $i(\mathcal{A})$  to be a *proper* dense subalgebra of  $A$  immediately implies that the Fréchet topology of  $\mathcal{A}$  is finer than the  $C^*$ -topology of  $A$  (since Fréchet means locally convex, metrizable and complete.)

It has been shown that if  $\mathcal{A}$  is smooth in  $A$  then  $M_n(\mathcal{A})$  is smooth in  $M_n(A)$ , [GVF, S]. This ensures that the  $K$ -theories of  $\mathcal{A}$  and  $A$  are isomorphic, the isomorphism being induced by the inclusion map.

Stability under the holomorphic functional calculus extends to nonunital algebras, since the spectrum of an element in a nonunital algebra is defined to be the spectrum of this element in the ‘one-point’ unitization, though we must of course restrict to functions satisfying  $f(0) = 0$ . Likewise, the definition of a Fréchet algebra does not require a unit. The point of contact between smooth algebras and  $QC^\infty$  spectral triples is the following Lemma, proved in [R1].

**Lemma 2.10.** *If  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a  $QC^\infty$  spectral triple, then  $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$  is also a  $QC^\infty$  spectral triple, where  $\mathcal{A}_\delta$  is the completion of  $\mathcal{A}$  in the locally convex topology determined by the seminorms*

$$q_{n,i}(a) = \|\delta^n d^i(a)\|, \quad n \geq 0, \quad i = 0, 1,$$

where  $d(a) = [\mathcal{D}, a]$ . Moreover,  $\mathcal{A}_\delta$  is a smooth algebra.

We call the topology on  $\mathcal{A}$  determined by the seminorms  $q_{ni}$  of Lemma 2.10 the  $\delta$ -topology.

Whilst smoothness does not depend on whether  $\mathcal{A}$  is unital or not, many analytical problems arise because of the lack of a unit. As in [R1, R2, GGISV], we make two definitions to address these issues.

**Definition 2.11.** *An algebra  $\mathcal{A}$  has local units if for every finite subset of elements  $\{a_i\}_{i=1}^n \subset \mathcal{A}$ , there exists  $\phi \in \mathcal{A}$  such that for each  $i$*

$$\phi a_i = a_i \phi = a_i.$$

**Definition 2.12.** *Let  $\mathcal{A}$  be a Fréchet algebra and  $\mathcal{A}_c \subseteq \mathcal{A}$  be a dense subalgebra with local units. Then we call  $\mathcal{A}$  a quasi-local algebra (when  $\mathcal{A}_c$  is understood.) If  $\mathcal{A}_c$  is a dense ideal with local units, we call  $\mathcal{A}_c \subset \mathcal{A}$  local.*

Separable quasi-local algebras have an approximate unit  $\{\phi_n\}_{n \geq 1} \subset \mathcal{A}_c$  such that  $\phi_{n+1}\phi_n = \phi_n$ , [R1].

**Example** For a graph  $C^*$ -algebra  $A = C^*(E)$ , Equation (1) shows that

$$A_c = \text{span}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}$$

is a dense subalgebra. It has local units because

$$p_\nu S_\mu S_\nu^* = \begin{cases} S_\mu S_\nu^* & v = s(\mu) \\ 0 & \text{otherwise} \end{cases}.$$

Similar comments apply to right multiplication by  $p_{s(\nu)}$ . By summing the source and range projections (without repetitions) of all  $S_{\mu_i} S_{\nu_i}^*$  appearing in a finite sum

$$a = \sum_i c_{\mu_i, \nu_i} S_{\mu_i} S_{\nu_i}^*$$

we obtain a local unit for  $a \in A_c$ . By repeating this process for any finite collection of such  $a \in A_c$  we see that  $A_c$  has local units.

We also require that when we have a spectral triple the operator  $\mathcal{D}$  is compatible with the quasi-local structure of the algebra, in the following sense.

**Definition 2.13.** *If  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a spectral triple, then we define  $\mathcal{C}_\mathcal{D}(\mathcal{A})$  to be the algebra generated by  $\mathcal{A}$  and  $[\mathcal{D}, \mathcal{A}]$ .*

**Definition 2.14.** A local spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a spectral triple with  $\mathcal{A}$  quasi-local such that there exists an approximate unit  $\{\phi_n\} \subset \mathcal{A}_c$  for  $\mathcal{A}$  satisfying

$$\mathcal{C}_{\mathcal{D}}(\mathcal{A}_c) = \bigcup_n \mathcal{C}_{\mathcal{D}}(\mathcal{A})_n,$$

where

$$\mathcal{C}_{\mathcal{D}}(\mathcal{A})_n = \{\omega \in \mathcal{C}_{\mathcal{D}}(\mathcal{A}) : \phi_n \omega = \omega \phi_n = \omega\}.$$

**Remark** A local spectral triple has a local approximate unit  $\{\phi_n\}_{n \geq 1} \subset \mathcal{A}_c$  such that  $\phi_{n+1} \phi_n = \phi_n \phi_{n+1} = \phi_n$  and  $\phi_{n+1} [\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n] \phi_{n+1} = [\mathcal{D}, \phi_n]$ , [R2]. This is the crucial property we require to prove most of our summability results, to which we now turn.

**2.3. Summability.** In the following, let  $\mathcal{N}$  be a semifinite von Neumann algebra with faithful normal trace  $\tau$ . Recall from [FK] that if  $S \in \mathcal{N}$ , the  $t$ -th generalized singular value of  $S$  for each real  $t > 0$  is given by

$$\mu_t(S) = \inf\{\|SE\| : E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leq t\}.$$

The ideal  $\mathcal{L}^1(\mathcal{N})$  consists of those operators  $T \in \mathcal{N}$  such that  $\|T\|_1 := \tau(|T|) < \infty$  where  $|T| = \sqrt{T^*T}$ . In the Type I setting this is the usual trace class ideal. We will simply write  $\mathcal{L}^1$  for this ideal in order to simplify the notation, and denote the norm on  $\mathcal{L}^1$  by  $\|\cdot\|_1$ . An alternative definition in terms of singular values is that  $T \in \mathcal{L}^1$  if  $\|T\|_1 := \int_0^\infty \mu_t(T) dt < \infty$ .

Note that in the case where  $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$ ,  $\mathcal{L}^1$  is not necessarily complete in this norm but it is complete in the norm  $\|\cdot\|_1 + \|\cdot\|_\infty$ . (where  $\|\cdot\|_\infty$  is the uniform norm). Another important ideal for us is the domain of the Dixmier trace:

$$\mathcal{L}^{(1, \infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} : \|T\|_{\mathcal{L}^{(1, \infty)}} := \sup_{t > 0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds < \infty \right\}.$$

We will suppress the  $(\mathcal{N})$  in our notation for these ideals, as  $\mathcal{N}$  will always be clear from context. The reader should note that  $\mathcal{L}^{(1, \infty)}$  is often taken to mean an ideal in the algebra  $\tilde{\mathcal{N}}$  of  $\tau$ -measurable operators affiliated to  $\mathcal{N}$ . Our notation is however consistent with that of [C] in the special case  $\mathcal{N} = \mathcal{B}(\mathcal{H})$ . With this convention the ideal of  $\tau$ -compact operators,  $\mathcal{K}(\mathcal{N})$ , consists of those  $T \in \mathcal{N}$  (as opposed to  $\tilde{\mathcal{N}}$ ) such that

$$\mu_\infty(T) := \lim_{t \rightarrow \infty} \mu_t(T) = 0.$$

**Definition 2.15.** A semifinite local spectral triple is  $(1, \infty)$ -summable if

$$a(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(1, \infty)} \quad \text{for all } a \in \mathcal{A}_c, \quad \lambda \in \mathbf{C} \setminus \mathbf{R}.$$

**Remark** If  $\mathcal{A}$  is unital,  $\ker \mathcal{D}$  is  $\tau$ -finite dimensional. Note that the summability requirements are only for  $a \in \mathcal{A}_c$ . We do not assume that elements of the algebra  $\mathcal{A}$  are all integrable in the nonunital case. Strictly speaking, this definition describes *local*  $(1, \infty)$ -summability, however we use the terminology  $(1, \infty)$ -summable to be consistent with the unital case.

We need to briefly discuss the Dixmier trace, but fortunately we will usually be applying it in reasonably simple situations. For more information on semifinite Dixmier traces, see [CPS2]. For  $T \in \mathcal{L}^{(1, \infty)}$ ,  $T \geq 0$ , the function

$$F_T : t \rightarrow \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds$$

is bounded. For certain generalised limits  $\omega \in L^\infty(\mathbf{R}_*^+)^*$ , we obtain a positive functional on  $\mathcal{L}^{(1,\infty)}$  by setting

$$\tau_\omega(T) = \omega(F_T).$$

This is the Dixmier trace associated to the semifinite normal trace  $\tau$ , denoted  $\tau_\omega$ , and we extend it to all of  $\mathcal{L}^{(1,\infty)}$  by linearity, where of course it is a trace. The Dixmier trace  $\tau_\omega$  is defined on the ideal  $\mathcal{L}^{(1,\infty)}$ , and vanishes on the ideal of trace class operators. Whenever the function  $F_T$  has a limit at infinity, all Dixmier traces return the value of the limit. We denote the common value of all Dixmier traces on measurable operators by  $\int$ . So if  $T \in \mathcal{L}^{(1,\infty)}$  is measurable, for any allowed functional  $\omega \in L^\infty(\mathbf{R}_*^+)^*$  we have

$$\tau_\omega(T) = \omega(F_T) = \int T.$$

**Example** Let  $\mathcal{D} = \frac{1}{i} \frac{d}{d\theta}$  act on  $L^2(S^1)$ . Then it is well known that the spectrum of  $\mathcal{D}$  consists of eigenvalues  $\{n \in \mathbf{Z}\}$ , each with multiplicity one. So, using the standard operator trace, Trace, the function  $F_{(1+\mathcal{D}^2)^{-1/2}}$  is

$$\frac{1}{\log 2N} \sum_{n=-N}^N (1+n^2)^{-1/2}$$

and this is bounded. Hence  $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$  and

$$(3) \quad \text{Trace}_\omega((1+\mathcal{D}^2)^{-1/2}) = \int (1+\mathcal{D}^2)^{-1/2} = 2.$$

In [R1, R2] we proved numerous properties of local algebras. The introduction of quasi-local algebras in [GGISV] led us to review the validity of many of these results for quasi-local algebras. Most of the summability results of [R2] are valid in the quasi-local setting. In addition, the summability results of [R2] are also valid for general semifinite spectral triples since they rely only on properties of the ideals  $\mathcal{L}^{(p,\infty)}$ ,  $p \geq 1$ , [C, CPS2], and the trace property. We quote the version of the summability results from [R2] that we require below.

**Proposition 2.16** ([R2]). *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a  $QC^\infty$ , local  $(1, \infty)$ -summable semifinite spectral triple relative to  $(\mathcal{N}, \tau)$ . Let  $T \in \mathcal{N}$  satisfy  $T\phi = \phi T = T$  for some  $\phi \in \mathcal{A}_c$ . Then*

$$T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}.$$

For  $\text{Re}(s) > 1$ ,  $T(1+\mathcal{D}^2)^{-s/2}$  is trace class. If the limit

$$(4) \quad \lim_{s \rightarrow 1/2^+} (s-1/2)\tau(T(1+\mathcal{D}^2)^{-s}) \text{ exists, then it is equal to } \frac{1}{2} \int T(1+\mathcal{D}^2)^{-1/2}.$$

In addition, for any Dixmier trace  $\tau_\omega$ , the function  $a \mapsto \tau_\omega(a(1+\mathcal{D}^2)^{-1/2})$  defines a trace on  $\mathcal{A}_c \subset \mathcal{A}$ .

**2.4. The Gauge Spectral Triple for a Graph  $C^*$ -Algebra.** In this section we summarise the construction of a Kasparov module and a semifinite spectral triple for locally finite directed graphs with no sources. This material is based on [PRen]. We begin by constructing a Kasparov module.

For  $E$  a row finite directed graph, we set  $A = C^*(E)$ ,  $F = C^*(E)^\gamma$ , the fixed point algebra for the  $S^1$  gauge action. The algebras  $A_c, F_c$  are defined as the finite linear span of the generators. Right multiplication makes  $A$  into a right  $F$ -module, and similarly  $A_c$  is a right module over  $F_c$ . We define an  $F$ -valued inner product  $(\cdot|\cdot)_R$  on both these modules by

$$(a|b)_R := \Phi(a^*b),$$

where  $\Phi$  is the expectation  $A \rightarrow F$ . Completing  $A$  in the norm  $\|x\|_X^2 := \|(x|x)_R\|_F = \|\Phi(x^*x)\|_F$  gives us a right  $C^*$ - $F$ -module denoted  $X$ . The algebra  $A$  acting by multiplication on the left of  $X$  provides a representation of  $A$  as adjointable operators on  $X$ . We let  $X_c$  be the copy of  $A_c \subset X$ . For each  $k \in \mathbf{Z}$ , define an adjointable endomorphism  $\Phi_k$  on  $X$  by

$$(5) \quad \Phi_k(x) = \frac{1}{2\pi} \int_{S^1} z^{-k} \gamma_z(x) d\theta, \quad z = e^{i\theta}, \quad x \in X, \quad \text{so} \quad \Phi_k(S_\alpha S_\beta^*) = \begin{cases} S_\alpha S_\beta^* & |\alpha| - |\beta| = k \\ 0 & |\alpha| - |\beta| \neq k \end{cases}.$$

**Proposition 2.17** ([PRen]). *Let  $X$  be the right  $C^*$ - $F$ -module defined above. Let*

$$X_{\mathcal{D}} = \{x \in X : \sum_{k \in \mathbf{Z}} k^2 (x_k | x_k)_R < \infty\},$$

and define  $\mathcal{D} : X_{\mathcal{D}} \rightarrow X$  by  $\mathcal{D} \sum_{k \in \mathbf{Z}} x_k = \sum_{k \in \mathbf{Z}} k x_k$ . Then  $\mathcal{D}$  is closed, self-adjoint and regular.

**Theorem 2.18.** *Suppose that the graph  $E$  is locally finite and has no sources. Let  $V = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$ . Then  $(X, V)$  is an odd Kasparov module for  $A$ - $F$  and so defines an element of  $KK^1(A, F)$ .*

To construct a semifinite spectral triple, we suppose that our graph  $C^*$ -algebra also has a faithful gauge invariant trace  $\tau : A \rightarrow \mathbf{C}$ . Using  $\tau$ , we define a  $\mathbf{C}$ -valued inner product  $\langle \cdot, \cdot \rangle$  on  $X_c$  by

$$\langle x, y \rangle := \tau((x|y)_R) = \tau(\Phi(x^*y)) = \tau(x^*y),$$

the last equality following from the gauge invariance of  $\tau$ . Denote the Hilbert space completion of  $X_c$  by  $\mathcal{H} = L^2(X, \tau)$ . The operator  $\mathcal{D}$  extends to a self-adjoint operator on  $\mathcal{H}$ , [PRen, Lemma 5.5], and for all  $a \in A_c$  the commutator  $[\mathcal{D}, a]$  extends to a bounded operator on  $\mathcal{H}$ .

**Lemma 2.19.** *The algebra  $A_c$  and the linear space  $[\mathcal{D}, A_c]$  are contained in the smooth domain of the derivation  $\delta$  where for  $T \in \mathcal{B}(\mathcal{H})$ ,  $\delta(T) = [[\mathcal{D}, T]]$ . So the completion of  $A_c$  in the  $\delta$ -topology, which we denote by  $\mathcal{A}$ , is a Fréchet pre- $C^*$ -algebra. Moreover  $\mathcal{A}$  is a quasi-local algebra with dense subalgebra  $A_c$ , and  $\mathcal{C}_{\mathcal{D}}(A_c) \subset \mathcal{C}_{\mathcal{D}}(\mathcal{A})$  is also quasi-local.*

The last piece of information we require is the von Neumann algebra and trace which give us a semifinite spectral triple. Let  $\text{End}_F^{00}(X_c)$  be the finite rank endomorphisms of the pre- $C^*$ -module  $X_c$ .

**Proposition 2.20.** *Let  $\mathcal{N} = (\text{End}_F^{00}(X_c))''$ . Then there exists a faithful, normal, semifinite trace  $\tilde{\tau} : \mathcal{N} \rightarrow \mathbf{C}$  such that for all rank one endomorphisms  $\Theta_{x,y}$  of  $X_c$  we have*

$$\tilde{\tau}(\Theta_{x,y}) = \tau((y|x)_R), \quad x, y \in X_c.$$

Moreover, for all  $a \in \mathcal{A}$  and  $\lambda \in \mathbf{C} \setminus \mathbf{R}$  the operator  $a(\lambda - \mathcal{D})^{-1}$  lies in  $\mathcal{K}_{\mathcal{N}}$ .

Hence we obtain a semifinite spectral triple. However, more is true.

**Theorem 2.21.** *Let  $E$  be a locally finite graph with no sources, and let  $\tau$  be a faithful, semifinite, norm lower-semicontinuous, gauge invariant trace on  $C^*(E)$ . Then  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a  $QC^\infty(1, \infty)$ -summable odd local semifinite spectral triple (relative to  $(\mathcal{N}, \tilde{\tau})$ ). For all  $a \in \mathcal{A}$  the operator  $a(1 + \mathcal{D}^2)^{-1/2}$  is not trace class. For any  $v \in E^0$  which does not connect to a sink we have*

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-1/2}) = 2\tau(p_v),$$

where  $\tilde{\tau}_\omega$  is any Dixmier trace associated to  $\tilde{\tau}$ .

The main point is that for  $v \in E^0$  such that  $v$  does not connect to a sink, and for  $k \in \mathbf{Z}$  we have

$$\tilde{\tau}(p_v \Phi_k) = \tau(p_v).$$

This is the spectral triple we will be working with for the rest of the paper, and we refer to it as the gauge spectral triple of the directed graph  $E$  (or algebra  $C^*(E)$ ). We remind the reader that the existence of this spectral triple depends only on the graph  $E$  being locally finite with no sources, and the existence of a faithful, semifinite, gauge invariant, norm lower-semicontinuous trace  $\tau : A \rightarrow \mathbf{C}$ . The latter is a nontrivial condition.

### 3. CONDITIONS FOR LOCALLY COMPACT SEMIFINITE MANIFOLDS

We now review in turn the conditions for noncommutative manifolds as presented in [RV]. We will consider natural generalisations of these conditions to the semifinite and nonunital setting and consider when the gauge spectral triple satisfies these generalisations.

We will present each condition as stated for the type I and unital case, where  $(\mathcal{N}, \tau) = (\mathcal{B}(\mathcal{H}), \text{Trace})$  and  $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}(\mathcal{H})$ , and then present the necessary modification of the condition, if it requires modification.

When dealing with these generalisations we will suppose that  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is a local semifinite spectral triple relative to  $\mathcal{N}, \tilde{\tau}$ . We will not suppose that  $\mathcal{A}$  is unital, but will suppose that  $\mathcal{A}_c \subset \mathcal{A}$  gives us a quasi-local algebra.

When considering the conditions as applied to graph algebras, we will suppose that  $E$  is a locally finite directed graph with no sources and possessing a faithful graph trace  $g$ . We will let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be the gauge spectral triple of  $E$  described in the previous section.

The conditions are somewhat interdependent, and we have found it is difficult to present them in a logical fashion. It seems that this difficulty is greatly eased if we assume at the outset that the Hilbert space  $\mathcal{H}$  carries commuting representations  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  and  $\pi^{op} : \mathcal{A}^{op} \rightarrow \mathcal{B}(\mathcal{H})$ . The former representation actually has  $\pi(\mathcal{A}) \subset \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ , but we do not assume this for the latter representation. We will explicitly state this bimodule requirement again when we look at the first order condition, but it will be apparent that several of our conditions require a bimodule structure for their statement. In all the following, we identify  $a \in \mathcal{A}$  with  $\pi(a) \in \mathcal{N}$  unless stated otherwise.

#### 3.1. The Analytic Conditions.

**Old Condition 1** (Dimension). *The type I unital spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is  $(p, \infty)$ -summable for a fixed positive integer  $p$ , for which  $\text{Trace}_\omega((1 + \mathcal{D}^2)^{-p/2}) > 0$  for all Dixmier limits  $\omega$ .*

To generalise this condition we evidently need to replace the operator trace,  $\text{Trace}$ , by the trace  $\tilde{\tau} : \mathcal{N} \rightarrow \mathbf{C}$  which determines the compactness and summability requirements of our spectral triple. We also need to restate the requirement, since in general for a nonunital spectral triple, even type I, we will not have  $(1 + \mathcal{D}^2)^{-p/2} \in \mathcal{L}^{(1, \infty)}$ , [R2]. So we have a simultaneous generalisation to the nonunital and semifinite case.

**New Condition 1** (Semifinite Nonunital Dimension). *The local semifinite spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is  $(p, \infty)$ -summable for a fixed positive integer  $p$ , for which  $\tilde{\tau}_\omega(a(1 + \mathcal{D}^2)^{-p/2}) > 0$  for all  $\omega$  and all  $0 \neq a \in \mathcal{A}_c$  with  $a \geq 0$ .*

**Remark.** In the type I setting, we also have the condition of *Absolute Continuity* which states: *For all nonzero  $a \in \mathcal{A}$  with  $a \geq 0$ , and for any  $\omega$ -limit, the following Dixmier trace is positive:*

$$\text{Trace}_\omega(a(1 + \mathcal{D}^2)^{-p/2}) > 0.$$

This is half of Connes' finiteness and absolute continuity condition, [C1, C2, GVF], the other half being finiteness discussed in Section 3.3 below; see also [RV]. Here we have demanded positivity only for positive elements of  $\mathcal{A}_c$ , but this extends to positive elements of  $\mathcal{A}$ , provided we allow the value  $+\infty$ . Of course our reformulation of the dimension condition already subsumes a semifinite version of absolute continuity, so the natural generalisation of the absolute continuity condition is already satisfied by our gauge spectral triples. This shows that even in the unital case it makes sense to combine the dimension and absolute continuity conditions, as mentioned in [RV].

Thus our formulation of the conditions has rendered the absolute continuity condition redundant.

This generalisation of the dimension condition is satisfied by the gauge spectral triple of a directed graph with  $p = 1$ . Provided the graph  $E$  has no sinks this follows from Theorem 2.21 since the Dixmier trace of  $a^*a$ ,  $0 \neq a \in \mathcal{A}_c$ , is given by  $\tilde{\tau}_\omega(a^*a(1 + \mathcal{D}^2)^{-1/2}) = 2\tau(a^*a) > 0$ . Even if the graph has sinks, the proof of Theorem 2.21 in [PRen] shows that we still have positivity.

**Old Condition 2** (Regularity). *The spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is  $QC^\infty$ . Without loss of generality, we assume that  $\mathcal{A}$  is complete in the  $\delta$ -topology and so is a Fréchet pre- $C^*$ -algebra.*

It follows from Lemma 2.19 that this condition is satisfied with no need to modify it at all.

**New Condition 2** (Regularity). *The spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is  $QC^\infty$ . Without loss of generality, we assume that  $\mathcal{A}$  is complete in the  $\delta$ -topology and so is a Fréchet pre- $C^*$ -algebra.*

**3.2. The Orientation and Closedness Conditions.** This section examines the orientation and finiteness conditions. The orientability condition for spectral triples with unital algebra  $\mathcal{A}$  is

**Old Condition 3** (Orientability). *Let  $p$  be the metric dimension of  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ . We require that the spectral triple be even, with  $\mathbf{Z}_2$ -grading  $\Gamma$ , if and only if  $p$  is even. For convenience, we take  $\Gamma = Id_{\mathcal{H}}$  when  $p$  is odd. We say the spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is orientable if there exists a Hochschild  $p$ -cycle*

$$(6a) \quad c = \sum_{\alpha=1}^n a_\alpha^0 \otimes b_\alpha^{op} \otimes a_\alpha^1 \otimes \cdots \otimes a_\alpha^p \in Z_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op})$$

whose Hochschild class  $[c] \in HH_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op})$  may be called the “orientation” of  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ , such that

$$(6b) \quad \pi_{\mathcal{D}}(c) := \sum_{\alpha} a_\alpha^0 b_\alpha^{op} [\mathcal{D}, a_\alpha^1] \cdots [\mathcal{D}, a_\alpha^p] = \Gamma.$$

Here  $\mathcal{A} \otimes \mathcal{A}^{op}$  is a bimodule for  $\mathcal{A}$  via

$$a \cdot (x \otimes y^{op}) = ax \otimes y^{op}, \quad (x \otimes y^{op}) \cdot a = xa \otimes y^{op}, \quad a, x, y \in \mathcal{A}.$$

Now, typically, we have a nonunital algebra, and require a different formulation. We adopt the attitude that we should have a locally finite but possibly infinite cycle, as would be the case for a volume form on a noncompact manifold.

**New Condition 3** (Nonunital Orientability). *Let  $p$  be the metric dimension of  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ . We require that the spectral triple be even, with  $\mathbf{Z}_2$ -grading  $\Gamma$ , if and only if  $p$  is even. For convenience, we take  $\Gamma = Id_{\mathcal{H}}$  when  $p$  is odd. We say the spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is orientable if there exists a Hochschild  $p$ -cycle*

$$(7a) \quad c = \sum_{\alpha=1}^{\infty} a_\alpha^0 \otimes b_\alpha^{op} \otimes a_\alpha^1 \otimes \cdots \otimes a_\alpha^p$$

whose Hochschild class  $[c]$  may be called the “orientation” of  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ , such that

$$(7b) \quad \pi_{\mathcal{D}}(c) := \sum_{\alpha} a_{\alpha}^0 b_{\alpha}^{op} [\mathcal{D}, a_{\alpha}^1] \dots [\mathcal{D}, a_{\alpha}^p] = \Gamma$$

where the sum in (7b) converges strongly.

**Remark** We have deliberately omitted any mention of the homology groups that  $c$  should belong to, there being many possibilities and few examples to guide us. We offer one possible candidate, without examining the subject in detail.

Let  $C_n(A_c, A_c \otimes A_c^{op})$  be the linear space of algebraic Hochschild  $n$ -chains for  $A_c$ . Suppose  $\mathcal{A}$  is the completion of  $A_c$  in the topology determined by the seminorms  $q_k$ , let  $\{q_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}^n}$  be a corresponding family of seminorms on  $C_n(A_c, A_c \otimes A_c^{op})$  and let  $\{\phi_j\}$  be a local approximate unit for  $\mathcal{A}$ , [R1]. Define  $C_n^{\infty}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op})$  to be the completion of  $C_n(A_c, A_c \otimes A_c^{op})$  for the topology determined by the family of seminorms

$$q_{\mathbf{k}, j}(a^0 \otimes b^{op} \otimes a^1 \otimes \dots \otimes a^n) := q_{\mathbf{k}}(\phi_j a^0 \otimes (\phi_j b)^{op} \otimes \phi_j a^1 \otimes \dots \otimes \phi_j a^n).$$

This should be viewed as similar to uniform convergence of all derivatives on compacta, and so analogous to a  $C^{\infty}$  topology. Ultimately more nonunital examples are required to clarify this issue; for more comments see [GGISV, R1, R2]. We leave these homological questions for future investigation.

For the case of graph algebras, we consider the sum over all edges in the graph

$$(8) \quad c = \sum_{e \in E^1} S_e^* \otimes S_e.$$

Before worrying about the convergence of this sum (in the multiplier algebra), we apply the Hochschild boundary  $b$  to find

$$b(c) = \sum_e (S_e^* S_e - S_e S_e^*) = \sum_e p_{r(e)} - \sum_{v \text{ not sink}} p_v,$$

where we have used the Cuntz-Krieger relation to obtain the second sum on the right-hand side. Thus if there are no sinks, the second sum on the right-hand side converges to the identity (in the multiplier algebra or the ‘one-point’ unitization).

The first sum on the right-hand side contains each vertex projection  $p_v$  with multiplicity equal to the number of edges entering it, which we denote by  $|v|_1$ . Thus

$$b(c) = \sum_{v \in E^0, v \text{ not sink}} (|v|_1 p_v - p_v) + \sum_{v \text{ a sink}} |v|_1 p_v.$$

In particular, if each vertex has precisely one edge entering it, and no vertex is a sink,  $b(c) = 0$ . We say that such a graph  $E$  has no sinks, and satisfies the *single entry condition*.

Observe that the single entry condition (together with the requirement that no loop has an exit) rules out loops except for the case where the (connected) graph comprises a single loop. The  $C^*$ -algebra of a graph consisting of a simple loop on  $n$  vertices is isomorphic to  $M_n(C(S^1))$ . For a one-edge loop, the Hochschild cycle  $c$  is  $z^{-1} \otimes z$ , the usual volume form for the circle. The single entry condition also rules out sources, so unless our (connected) graph comprises a single loop, it is an infinite directed tree with no sources or sinks, in which case the  $C^*$ -algebra is AF [KPR, Theorem 2.4].

If  $E$  satisfies the single entry condition then we claim that  $\sum S_e^* \otimes S_e$  converges to a partial isometry in the multiplier algebra of  $C^*(E) \otimes C^*(E)$ . Let  $X_e = S_e^* \otimes S_e$  then it is clear that  $X_e$  is a partial

isometry in  $C^*(E) \otimes C^*(E)$  with

$$\begin{aligned} X_e X_e^* &= (S_e^* \otimes S_e)(S_e \otimes S_e^*) = P_{r(e)} \otimes S_e S_e^* \\ X_e^* X_e &= (S_e \otimes S_e^*)(S_e^* \otimes S_e) = S_e S_e^* \otimes P_{r(e)}. \end{aligned}$$

By the relations in  $C^*(E)$  the  $S_e S_e^*$  are mutually orthogonal, and then by the single entry hypothesis the  $P_{r(e)}$  are too. Hence the  $X_e$  have mutually orthogonal ranges, and a standard argument (see [PR2, Lemma 1.1] or [BPRS, Lemma 1.1]) finishes off the claim.

Using the single-entry condition, we see that the Hochschild cycle defined in (8) is represented by

$$(9) \quad \pi_{\mathcal{D}}(c) = \sum_e S_e^* [\mathcal{D}, S_e] = \sum_e S_e^* S_e = \sum_e p_{r(e)} = Id_{\mathcal{H}},$$

showing that the new condition of orientation is satisfied for this cycle. The sums in (9) converge in the strict topology as an operator on the  $C^*$ -module  $X$ , and also converge strongly on  $\mathcal{H}$ .

It may well be possible that there is a Hochschild cycle for a more general family of graphs, and we are not claiming that the above conditions are necessary for the orientability condition to hold, only sufficient.

**From now on we suppose that  $E$  has no sinks and satisfies the single entry condition.** As noted above, it follows that the algebra  $C^*(E)$  is then AF unless it is  $M_n(C(S^1))$ . In the AF case,  $E$  is a directed tree. We record the following Lemma describing the fixed-point algebra of the directed tree examples.

**Lemma 3.1.** *Suppose that  $E$  is a directed tree with no sinks satisfying the single entry condition and having finitely many ends. Then  $F = C^*(E)^\gamma$  is an abelian algebra, isomorphic to the continuous functions on the infinite path space  $E^\infty$  of  $E$ . Letting  $N$  denote the number of ends, each  $f \in F_c$  can be represented as*

$$f = \sum_{v \in E^0} \sum_{n=1}^N c_{v,n} S_{v,n} S_{v,n}^*, \quad c_{v,n} \in \mathbf{C}$$

where  $(v, n)$  denotes a path with source  $v$  and range in the  $n$ -th tail. The  $C^*$ -norm of such an  $f$  is  $\|f\|_F^2 = \sup |c_{v,n}|^2$ .

*Proof.* The assertion that  $F \cong C_0(E^\infty)$  follows from an argument along the lines of [KPRR, Lemma 4.3]. To see that it is possible to write  $f \in F_c$  in the above form, consider a path  $\alpha$  with range  $r(\alpha)$  a vertex emitting one edge  $e$ . Then

$$S_{\alpha e} S_{\alpha e}^* = S_\alpha S_e S_e^* S_\alpha^* = S_\alpha p_{s(e)} S_\alpha^* = S_\alpha p_{r(\alpha)} S_\alpha^* = S_\alpha S_\alpha^*.$$

So any  $S_\alpha S_\alpha^*$  is equal to  $S_\beta S_\beta^*$  where  $\beta$  is an extension of  $\alpha$  not passing through a vertex emitting more than one edge. If  $\alpha$  is a path with range a vertex emitting, say,  $k$  edges,  $e_1, \dots, e_k$ , then

$$S_\alpha S_\alpha^* = S_\alpha p_{r(\alpha)} S_\alpha^* = \sum_{i=1}^k S_\alpha S_{e_i} S_{e_i}^* S_\alpha^*,$$

and this can be subsequently extended until the next vertices emitting more than one edge. This process terminates after finitely many steps because there are finitely many ends. The  $S_{v,n} S_{v,n}^*$  are mutually orthogonal, so

$$f^* f = \sum_v \sum_n |c_{v,n}|^2 S_{v,n} S_{v,n}^*,$$

and  $\|f\|_F^2 = \sup |c_{v,n}|^2$ . □

The next condition is closedness, which, in its original form, is basically Stoke's theorem for the Dixmier trace applied to elements of  $\mathcal{A} \otimes \mathcal{A}^{op}$ . The original formulation for  $(p, \infty)$ -summable triples using the operator trace  $\text{Trace}$  is

**Old Condition 4** (Closedness). *The  $(p, \infty)$ -summable spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is closed if for any  $a_1, \dots, a_p \in \mathcal{A} \otimes \mathcal{A}^{op}$ , the operator  $\Gamma[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p](1 + \mathcal{D}^2)^{-p/2}$  has vanishing Dixmier trace; thus, for any Dixmier trace  $\text{Trace}_\omega$ ,*

$$(10) \quad \text{Trace}_\omega(\Gamma[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p](1 + \mathcal{D}^2)^{-p/2}) = 0.$$

**Remark** By setting  $\phi(a_0, \dots, a_p) := \text{Trace}_\omega(\Gamma a_0 [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p](1 + \mathcal{D}^2)^{-p/2})$ , the equation (10) may be rewritten [C, VI.2] as  $B_0\phi = 0$ , where  $B_0$  is defined on  $(k+1)$ -linear functionals by

$$(B_0\phi)(a_1, \dots, a_k) := \phi(1, a_1, \dots, a_k) + (-1)^k \phi(a_1, \dots, a_k, 1).$$

To see the utility of this condition, we introduce some notation so that we can quote Lemma 3 of [C, VI.4.γ]. Let  $\Omega^*(\mathcal{A})$  be the universal differential algebra of  $\mathcal{A}$ , [C, II.1.α]. Then  $\pi_{\mathcal{D}} : \Omega^*(\mathcal{A}) \rightarrow \mathcal{C}_{\mathcal{D}}(\mathcal{A})$  defined by  $\pi_{\mathcal{D}}(a_0 \delta a_1 \cdots \delta a_n) = a_0 [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_n]$  is a  $*$ -algebra representation. Denote by  $\Lambda_{\mathcal{D}}^*(\mathcal{A})$  the graded differential algebra we obtain by quotienting  $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$  by the differential ideal  $\pi_{\mathcal{D}}(\delta(\ker \pi_{\mathcal{D}}))$ , where  $\delta$  is the universal derivation on  $\Omega^*(\mathcal{A})$ . We denote by  $d$  the derivation on  $\Lambda_{\mathcal{D}}^*(\mathcal{A})$ . See [C, Chap VI] for more information. Finally, let  $Z^k(\mathcal{A}, \mathcal{A}^*)$  denote the Hochschild cocycles.

**Lemma 3.2.** *Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be  $(p, \infty)$ -summable and satisfy Old Condition 5 (first order). Then for each  $k = 0, 1, \dots, p$  and  $\eta \in \Omega^k \mathcal{A}$ , a Hochschild cocycle  $C_\eta \in Z^{p-k}(\mathcal{A}^{op}, (\mathcal{A}^{op})^*)$  is defined by*

$$C_\eta(a^0, \dots, a^{p-k}) := \text{Trace}_\omega(\Gamma \pi_{\mathcal{D}}(\eta) a^0 [\mathcal{D}, a^1] \cdots [\mathcal{D}, a^{p-k}](1 + \mathcal{D}^2)^{-p/2}), \quad a^0, \dots, a^{p-k} \in \mathcal{A}^{op}.$$

*Moreover, if Old Condition 4 (closedness) also holds, then  $C_\eta$  depends only on the class of  $\pi_{\mathcal{D}}(\eta)$  in  $\Lambda_{\mathcal{D}}^k \mathcal{A}$ , and*

$$B_0 C_\eta = (-1)^k C_{d\eta}. \quad \square$$

Thus the first order condition together with closedness give us tools to study the Hochschild and cyclic homology of the algebra  $\mathcal{A}$ . More information can be found in [C, VI.4.γ].

The difficulty we face is that we have a Dixmier trace defined on  $\mathcal{N} \supset \mathcal{A}$  which we can not apply to  $\mathcal{A} \otimes \mathcal{A}^{op}$ . As we discuss in the next section, we do not believe having a spectral triple for  $\mathcal{A} \otimes \mathcal{A}^{op}$  is of central importance. Nevertheless, the utility of Lemma 3.2 is greatly reduced by our new formulation.

**New Condition 4** (Semifinite Closedness). *The  $(p, \infty)$ -summable local semifinite spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is closed if for any Dixmier trace  $\tilde{\tau}_\omega$  we have*

$$(11) \quad \tilde{\tau}_\omega(\Gamma[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p](1 + \mathcal{D}^2)^{-p/2}) = 0$$

*for all  $a_1, \dots, a_p \in \mathcal{A}$ .*

It would seem that this formulation does not give us tools to study the Hochschild and cyclic cohomology of  $\mathcal{A}$  as in the type I case described above, [C, VI.4.γ]. More examples are required to understand the proper extension of this condition to the semifinite setting.

For the gauge spectral triple of a graph algebra and generator  $S_\mu S_\nu^* \in \mathcal{A}$ , [PRen, Theorem 5.8],

$$\begin{aligned} \tilde{\tau}_\omega([\mathcal{D}, S_\mu S_\nu^*](1 + \mathcal{D}^2)^{-1/2}) &= (|\mu| - |\nu|) \tilde{\tau}_\omega(S_\mu S_\nu^*(1 + \mathcal{D}^2)^{-1/2}) \\ &= 2(|\mu| - |\nu|) \tau(S_\mu S_\nu^*). \end{aligned}$$

The gauge invariance of the trace says that  $\tau(S_\mu S_\nu^*)$  is non-zero only if  $|\mu| = |\nu|$ , whence the whole expression always vanishes. Hence the new closedness condition holds for the gauge spectral triple.

**3.3. The Bimodule Conditions.** This section is concerned with the relation between the bimodule structure of the Hilbert space and the spectral triple.

First we have the first order condition which specifies the bimodule structure. In the original type I setting we have

**Old Condition 5** (First Order). *There are commuting representations  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  and  $\pi^{op} : \mathcal{A}^{op} \rightarrow \mathcal{B}(\mathcal{H})$  of the opposite algebra  $\mathcal{A}^{op}$  (or equivalently, an antirepresentation of  $\mathcal{A}$ ). Writing  $a$  for  $\pi(a)$ , and  $b^{op}$  for  $\pi^{op}(b)$ , we ask that  $[a, b^{op}] = 0$ . In addition, the bounded operators in  $[\mathcal{D}, \mathcal{A}]$  commute with  $\mathcal{A}^{op}$ ; in other words,*

$$(12) \quad [[\mathcal{D}, a], b^{op}] = 0 \quad \text{for all } a, b \in \mathcal{A}.$$

In the type I setting the first order condition gives us a spectral triple for  $\mathcal{A} \otimes \mathcal{A}^{op}$ , but we believe this is not essential, and just an artefact of the type I setting. Rather we focus on the fact that in the type I setting the algebra  $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$  is contained in the endomorphism algebra of the right  $\mathcal{A}$  module  $\mathcal{H}_{\infty}$ .

The finiteness condition (below) asks that  $\mathcal{H}_{\infty} = \bigcap_{m \geq 1} \text{dom } \mathcal{D}^m$  be a finite projective (right)  $\mathcal{A}$  module. The first order condition then says that  $\mathcal{C}_{\mathcal{D}}(\mathcal{A}) \subseteq \text{End}_{\mathcal{A}}^R(\mathcal{H}_{\infty})$ , where  $R$  is for right. One would expect this finite projective condition to be symmetric in some sense, but this is an extra requirement. If  $\mathcal{H}_{\infty}$  is also a finite projective left  $\mathcal{A}$ -module, then  $\mathcal{C}_{\mathcal{D}}(\mathcal{A}^{op}) \subseteq \text{End}_{\mathcal{A}}^L(\mathcal{H}_{\infty})$ ,  $L$  for left. Typically however, these two algebras of endomorphisms, one left and one right, will *not* commute with each other. They do for the gauge spectral triple of a graph algebra, but this is a one-dimensional phenomenon (see also [GGISV]).

A moment's thought shows that regarding the (sections of the) spinor bundle of a spin manifold  $M$  as a  $C^{\infty}(M)$  bimodule, the two collections of endomorphisms we obtain do not commute, since both algebras of endomorphisms are the same Clifford algebra.

These arguments show that the most important aspect of the first order condition is that the algebra  $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$  acts as endomorphisms of a noncommutative bundle, and that the ‘symbol’ of  $\mathcal{D}$  is such an endomorphism.

Moreover, in the semifinite setting we begin with a representation  $\pi : \mathcal{A} \rightarrow \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ . The von Neumann algebra  $\mathcal{N}$  is thus required to contain  $\mathcal{A}$  and the spectral projections of  $\mathcal{D}$ , and these are the only requirements. So typically,  $\pi^{op}(\mathcal{A}^{op}) \not\subset \mathcal{N}$ , and this is the case for the gauge spectral triple. In particular,  $\mathcal{A}^{op}$  need not lie in the domain of the trace we employ, and even supposing we have a version of the first order condition, we will not obtain a spectral triple for  $\mathcal{A} \otimes \mathcal{A}^{op}$ .

We therefore change the first-order condition only very slightly as follows:

**New Condition 5** (Semifinite First Order). *There are commuting representations  $\pi : \mathcal{A} \rightarrow \mathcal{N}$  and  $\pi^{op} : \mathcal{A}^{op} \rightarrow \mathcal{B}(\mathcal{H})$  of the opposite algebra  $\mathcal{A}^{op}$ . Writing  $a$  for  $\pi(a)$ , and  $b^{op}$  for  $\pi^{op}(b)$ , we ask that  $[a, b^{op}] = 0$ . In addition, the bounded operators in  $[\mathcal{D}, \mathcal{A}]$  commute with  $\mathcal{A}^{op}$ ; in other words,*

$$(13) \quad [[\mathcal{D}, a], b^{op}] = 0 \quad \text{for all } a, b \in \mathcal{A}.$$

For the gauge spectral triple of a directed graph, the Hilbert space naturally carries commuting representations of  $\mathcal{A}$  and  $\mathcal{A}^{op}$ . The first order condition

$$[[\mathcal{D}, \mathcal{A}], \mathcal{A}^{op}] = 0$$

follows since  $[\mathcal{D}, \mathcal{A}] \subset \mathcal{A}$ , and the left and right actions of  $\mathcal{A}$  on the Hilbert space commute.

The condition of finiteness in the unital case is

**Old Condition 6** (Finiteness). *The dense subspace of  $\mathcal{H}$  which is the smooth domain of  $\mathcal{D}$ ,*

$$\mathcal{H}_\infty := \bigcap_{m \geq 1} \text{dom } \mathcal{D}^m$$

*is a finitely generated projective right  $\mathcal{A}$ -module.*

Thus  $\mathcal{H}_\infty \simeq q\mathcal{A}^m$  where  $q \in M_m(\mathcal{A})$  is an idempotent. Without loss of generality, we may suppose  $q = q^*$  also, so that, without further hypotheses,  $\mathcal{H}_\infty$  carries an  $\mathcal{A}$ -valued Hermitian pairing, namely, that given by  $(\xi, \eta)' := \sum_{j,k} a_j^* q_{jk} b_k$  when  $\xi = (\sum_j q_{ij} a_j)_{i=1}^m$ ,  $\eta = (\sum_k q_{ik} b_k)_{i=1}^m$ .

In the nonunital case, this is necessarily more subtle as the elements of  $\mathcal{H}_\infty = \bigcap_m \text{dom } \mathcal{D}^m$  must also satisfy integrability conditions. In [R1], the notion of smooth module was introduced for nonunital algebras which are local. As we are dealing with quasi-local algebras, most of the results on smooth modules in [R1] are not applicable.

We take the attitude that:

Point 1)  $\mathcal{H}_\infty$  should be a continuous  $\mathcal{A}$ -module,

Point 2)  $\mathcal{H}_\infty$  should embed continuously as a dense subspace in the  $C^*$ - $\mathcal{A}$ -module  $X_A = \overline{\mathcal{H}_\infty}$ ,

Point 3)  $X$  should be the completion of  $qA^N$  for some  $N$  and some projection  $q$  in  $M_N(A_b)$  where  $A_b$  is a unitization of  $A$ ,

Point 4) the Hermitian product  $\mathcal{H}_\infty \ni x, y \rightarrow (x|y)$  should have range in  $\mathcal{A}$  (acting on the right).

Point 1) is implied by the condition of regularity.

*Proof.* For  $x \in \mathcal{H}_\infty$  and  $a \in \mathcal{A}$  we have

$$\begin{aligned} \|\mathcal{D}^n(xa)\|_{\mathcal{H}} &= \|\mathcal{D}^n(x)\|_{\mathcal{H}} = \left\| \sum_{j=0}^n \binom{n}{j} \delta^{n-j}(a^{op}) |\mathcal{D}|^j(x) \right\|_{\mathcal{H}} \\ (14) \qquad \qquad \qquad &\leq \sum_{j=0}^n \binom{n}{j} \|\delta^{n-j}(a^{op})\| \|\mathcal{D}|^j(x)\|_{\mathcal{H}}. \end{aligned}$$

The continuity of the action of  $\mathcal{A}$  on  $\mathcal{H}_\infty$  now follows easily. □

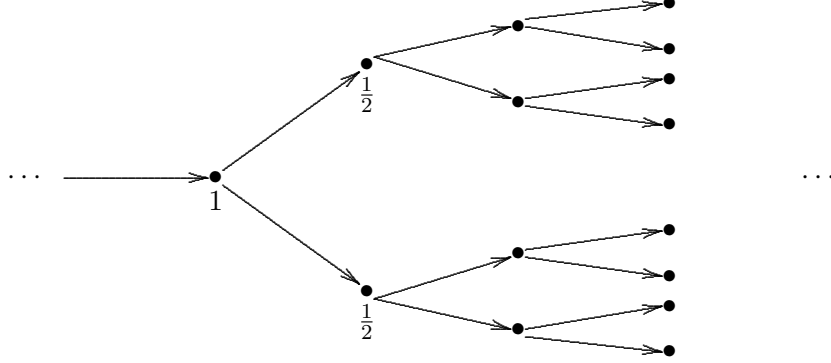
Point 2 above is included to ensure that we can recover the ‘module of continuous sections vanishing at infinity’ from  $\mathcal{H}_\infty$ , and it is a nontrivial condition as we shall see. Once we have a continuous embedding, the image will be dense for our graph algebra examples, since  $A_c \subset \mathcal{H}_\infty$ .

Once we can recover the module  $X$ , we demand that it be ‘finitely generated and projective’ in the sense of 3): see also [R1, Theorem 8]. The examples arising from graph algebras have  $A$  dense in  $X$ , so taking  $N = 1$  and  $q = id_{A_b}$  in any unitization  $A_b$  of  $A$  shows that 3) is always satisfied for the gauge spectral triple of a graph algebra.

All four points are satisfied in the unital case, so we will ignore the case of a single loop in the following, focussing attention instead on the directed trees.

Roughly speaking, without points 2) and 4),  $\mathcal{H}$  can contain many ‘functions’ on the graph which are unbounded, and so are not in the algebra  $A$  or the module  $X$ . Modules of unbounded ‘functions’ are not terrible per se, but we prefer to remain close to the  $C^*$ -theory.

**Example** Let  $E$  be the ‘dyadic directed tree’



Define a faithful trace as follows. If  $v$  is a vertex before the first split, let  $\tau(p_v) = 1$ . If  $v$  occurs after  $n$  splits and before  $n + 1$  splits, define  $\tau(p_v) = 2^{-n}$ . Finally define  $\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu} \tau(p_{r(\mu)})$ . Then the Hilbert space  $\mathcal{H} = L^2(X, \tau)$  contains

$$(15) \quad a = \lim_{N \rightarrow \infty} \sum_{i=1}^N 2^{i/4} p_i$$

where  $\tau(p_i) = 2^{-i}$ , and the  $p_i$  are mutually orthogonal. The element  $a \in \mathcal{H}$  in equation (15) does not lie in the  $C^*$ -module  $X$ , as the limit does not exist in the norm  $\|\cdot\|_X$ .

**New Condition 6** (Nonunital Finiteness). *The dense subspace of  $\mathcal{H}$  which is the smooth domain of  $\mathcal{D}$ ,*

$$\mathcal{H}_\infty := \bigcap_{m \geq 1} \text{dom } \mathcal{D}^m$$

*has a right inner product  $\mathcal{A}$ -module structure. Moreover,  $\mathcal{H}_\infty$  embeds as a dense subspace of a  $C^*$ - $\mathcal{A}$ -module which is finitely generated and projective over some unitization  $A_b$  of  $A$ .*

Having identified a working generalisation of the finiteness condition, we identify the restrictions it places on a graph  $C^*$ -algebra. So to check that New Condition 6 holds, we must verify points 2) and 4).

**Proposition 3.3.** *Suppose that the locally finite directed graph  $E$  has no sinks, no loops and satisfies the single entry condition. The  $\mathcal{A}$ -module  $\mathcal{H}_\infty$  satisfies 2) if and only if the  $K$ -theory of  $A$  is finitely generated. In this case the Hilbert space  $\mathcal{H}$  also satisfies point 2). If the  $K$ -theory of  $A$  is finitely generated then point 4) holds.*

**Remark** Thus for the directed tree examples, the finiteness condition is satisfied if and only if the  $K$ -theory of  $A$  is finitely generated.

*Proof.* We begin with condition 2) for our directed trees. First of all we must have  $\ker \mathcal{D} \cap \mathcal{H}_\infty = L^2(F_c, \tau) \subset X$ . Thus we require a  $C > 0$  such that

$$\|f\|_X^2 = \|f^* f\|_F^{1/2} \leq C \tau(f^* f)^{1/2} = C \|f\|_{\mathcal{H}}^2,$$

for all  $f \in L^2(F_c, \tau)$ . In particular, we require for all  $v \in E^0$  that

$$1 = \|p_v\|_F \leq C\tau(p_v)^{1/2}.$$

Hence  $\tau(p_v)$  must be bounded below, which implies, by the definition of a graph trace and the faithfulness of  $\tau$ , that there exist at most finitely many ends, and so  $K_0(A)$  is finitely generated. Thus the condition is necessary.

Conversely, suppose that  $K_0(A)$  is finitely generated, and let  $\text{rank}(K_0(A)) = N < \infty$  be the number of ends. Observe that having finitely many ends implies that any faithful graph trace is bounded from below. Then if  $f \in F_c$ , Lemma 3.1 allows us to write

$$f = \sum_{v \in E^0} \sum_{n=1}^N c_{v,n} S_{v,n} S_{v,n}^*, \quad c_{v,n} \in \mathbf{C}$$

where  $(v, n)$  denotes a path with source  $v$  and range in the  $n$ -th end. We have  $\|f\|_F^2 = \sup |c_{v,n}|^2$ .

Now suppose that  $f \in \mathcal{H}$ , so that

$$\|f\|_{\mathcal{H}}^2 = \tau(f^*f) = \sum_v \sum_n |c_{v,n}|^2 \tau(p_n) < \infty$$

where  $p_n$  is any projection in the  $n$ -th end. Then

$$\begin{aligned} \|f\|_{\mathcal{H}}^2 &= \sum_v \sum_n |c_{v,n}|^2 \tau(p_n) \geq \min\{\tau(p_n)\} \sum_v \sum_n |c_{v,n}|^2 \\ &\geq \min\{\tau(p_n)\} \sup_{v,n} |c_{v,n}|^2 = \min\{\tau(p_n)\} \|f\|_F^2 = \min\{\tau(p_n)\} \|f\|_X^2. \end{aligned}$$

Hence  $f \in X$ . Finally, suppose that  $x \in \mathcal{H}$ , so  $x = \sum_{k \in \mathbf{Z}} x_k$  and  $\sum_{k \in \mathbf{Z}} \tau(x_k^* x_k) < \infty$ . As  $f_k := x_k^* x_k \in F$  and is positive, we have

$$\begin{aligned} \|x\|_{\mathcal{H}}^2 &= \sum_k \tau(x_k^* x_k) = \sum_k \|f_k\|_{\mathcal{H}} \geq (\min\{\tau(p_n)\})^{1/2} \sum_k \|f_k\|_X \\ &= (\min\{\tau(p_n)\})^{1/2} \sum_k \|x_k^* x_k\|_X = (\min\{\tau(p_n)\})^{1/2} \sum_k \|(x_k^* x_k)^2\|_F^{1/2} \\ &= (\min\{\tau(p_n)\})^{1/2} \sum_k \|x_k^* x_k\|_F = (\min\{\tau(p_n)\})^{1/2} \sup_k \|x_k^* x_k\|_F \\ &= (\min\{\tau(p_n)\})^{1/2} \|x\|_X^2. \end{aligned}$$

This proves that the finite generation of  $K_0(A)$  is necessary and sufficient for the second point.

For point 4), we assume that  $K_0(A)$  is finitely generated. We observe that if  $x, y \in X_c = A_c \subset \mathcal{H}_\infty$  we have  $x^* y \in A_c \subset \mathcal{A}$ . In particular,  $x^* y$  is in the smooth domain of the derivation  $\delta = [|\mathcal{D}|, \cdot]$ . Thus for  $x, y \in X_c$  we have, by Lemma 2.19,

$$\|\delta^m(x^* y)\|^2 \leq \sum_{k,l \in \mathbf{Z}} |k-l|^{2m} \|x_k^* y_l\|_A^2$$

where the sum over  $k, l$  is finite and we have used  $\|\delta^m((x^* y)^{*op})\| = \|\delta^m(x^* y)\|$  for  $a \in A_c$  to avoid writing  $op$  throughout the following calculation. Now

$$\|x_k^* y_l\|_A^2 = \|y_l^* x_k x_k^* y_l\|_A \leq \|y_l^* y_l\|_A \|x_k x_k^*\|_A \leq C^2 \tau(y_l^* y_l) \tau(x_k^* x_k),$$

the last inequality using the finite generation of  $K_0(A)$ . So we have the inequality

$$\begin{aligned}
\|\delta^m(x^*y)\| &\leq C^2 \sum_{k,l \in \mathbf{Z}} |k-l|^{2m} \tau(y_l^* y_l) \tau(x_k^* x_k) \\
&\leq C^2 \sum_{k,l \in \mathbf{Z}} (|k|+|l|)^{2m} \tau(y_l^* y_l) \tau(x_k^* x_k) \\
&= C^2 \sum_{k,l \in \mathbf{Z}} \sum_{j=0}^{2m} \binom{2m}{j} |k|^{2m-j} |l|^j \tau(y_l^* y_l) \tau(x_k^* x_k) \\
&= C^2 \sum_{k,l \in \mathbf{Z}} \sum_{j=0}^{2m} \binom{2m}{j} \tau((|\mathcal{D}|^{j/2} y_l)^* (|\mathcal{D}|^{j/2} y_l)) \tau((|\mathcal{D}|^{(2m-j)/2} x_k)^* (|\mathcal{D}|^{(2m-j)/2} x_k)) \\
(16) \quad &= C^2 \sum_{j=0}^{2m} \binom{2m}{j} \|\mathcal{D}^{j/2} y\|_{\mathcal{H}}^2 \|\mathcal{D}^{(2m-j)/2} x\|_{\mathcal{H}}^2.
\end{aligned}$$

So suppose that  $\{x^i\} \subset X_c$  is a sequence converging to  $x \in \mathcal{H}_\infty$  in the topology determined by the seminorms  $x \rightarrow \|\mathcal{D}^m x\|_{\mathcal{H}}$ ,  $m \geq 0$ , and similarly  $y^i \rightarrow y$ .

The estimate (16) shows that

$$\begin{aligned}
\|\delta^m(x_j^* y_j - x_i^* y_i)\|_A^2 &= \|\delta^m(x_j^* y_j - x_j^* y_i + x_j^* y_i - x_i^* y_i)\|_A^2 \\
&\leq C^2 \sum_{j=0}^{2m} \binom{2m}{j} \|\mathcal{D}^{j/2} (y_j - y_i)\|_{\mathcal{H}}^2 \|\mathcal{D}^{(2m-j)/2} (x_j)\|_{\mathcal{H}}^2 \\
&\quad + C^2 \sum_{j=0}^{2m} \binom{2m}{j} \|\mathcal{D}^{j/2} (y_i)\|_{\mathcal{H}}^2 \|\mathcal{D}^{(2m-j)/2} (x_j - x_i)\|_{\mathcal{H}}^2,
\end{aligned}$$

and this goes to zero. Hence the sequence  $x_j^* y_j$  is Cauchy in  $\mathcal{A}$ , and so for the limits  $x, y \in \mathcal{H}_\infty$ , the inner product  $(x|y)_A = x^* y$  is in the completion of  $A_c$  for the  $\delta$ -topology, and so  $x^* y \in \mathcal{A}$ .  $\square$

**Remark** We also note that Connes stipulates that when we restrict the Hilbert space inner product to  $\mathcal{H}_\infty$  we should have

$$\langle x, y \rangle = \int (x|y)_{\mathcal{A}} (1 + \mathcal{D}^2)^{-1/2}$$

where the Hermitian product is the  $\mathcal{A}$ -valued one:  $(x|y)_{\mathcal{A}} = x^* y$ . However, the trace satisfies  $\tau = \tau \circ \Phi$ , so

$$\tau((x|y)_{\mathcal{A}}) = \tau(x^* y) = \tau(\Phi(x^* y)) = \tau((x|y)_R),$$

and the inner product does indeed satisfy this formula, up to a factor of 2; see Equation (3). The factor of 2 also occurs in the type I case, and is simply a matter of normalisation of the inner product, and does not affect the Hilbert space; see [R2, Section 5] for the constants in the commutative case.

The following condition describes a  $\text{spin}^c$  structure for the noncommutative manifold, [P].

**Old Condition 7** ( $\text{Spin}^c$ ). *The  $C^*$ - $A$ -module completion of  $\mathcal{H}_\infty$  is a Morita equivalence bimodule between  $A$  and the norm completion of the algebra  $\mathcal{C}_{\mathcal{D}}(A)$  generated by  $\mathcal{A}$  and  $[\mathcal{D}, \mathcal{A}]$ .*

Since for graph algebras the  $A$ -bimodule  $A$  is contained in  $X$ , we have a natural Morita equivalence bimodule between  $A$  and  $A$ . As the norm closed algebra generated by  $\mathcal{A}$  and  $[\mathcal{D}, \mathcal{A}]$  is just  $A$  in the

case of the gauge spectral triple, the Morita equivalence follows. Thus there is no need to alter the  $\text{spin}^c$  condition to deal with semifiniteness or lack of a unit (at least for graph algebras).

**New Condition 7** ( $\text{Spin}^c$ ). *The  $C^*$ - $\mathcal{A}$ -module completion of  $\mathcal{H}_\infty$  is a Morita equivalence bimodule between  $\mathcal{A}$  and the norm completion of the algebra  $\mathcal{C}_{\mathcal{D}}(\mathcal{A})$  generated by  $\mathcal{A}$  and  $[\mathcal{D}, \mathcal{A}]$ .*

In the case where  $\mathcal{A} = C^\infty(M)$ ,  $M$  a manifold, the  $\text{spin}^c$  condition (together with orientability) provides a  $\text{spin}^c$  structure for  $M$ , [P]. Given a  $\text{spin}^c$  manifold  $M$ ,  $M$  is spin if and only if at least one (oriented) Morita equivalence bimodule admits a bijective antilinear map satisfying the requirements of the reality condition, [GVF, Theorem 9.6]. Thus the reality condition below, in conjunction with the  $\text{spin}^c$  condition, may be regarded as a noncommutative spin structure.

**Old Condition 8** (Reality). *There is an antiunitary operator  $J : \mathcal{H} \rightarrow \mathcal{H}$  such that  $Ja^*J^{-1} = a^{op}$  for all  $a \in \mathcal{A}$ ; and moreover,  $J^2 = \pm 1$ ,  $J\mathcal{D}J^{-1} = \pm\mathcal{D}$  and also  $J\Gamma J^{-1} = \pm\Gamma$  in the even case, according to the following table of signs depending only on  $p \bmod 8$ :*

(17)

$p \bmod 8$	0	2	4	6
$J^2 = \pm 1$	+	-	-	+
$J\mathcal{D}J^{-1} = \pm\mathcal{D}$	+	+	+	+
$J\Gamma J^{-1} = \pm\Gamma$	+	-	+	-

$p \bmod 8$	1	3	5	7
$J^2 = \pm 1$	+	-	-	+
$J\mathcal{D}J^{-1} = \pm\mathcal{D}$	-	+	-	+

For the origin of this sign table in  $KR$ -homology, we refer to [GVF, §9.5].

For the gauge spectral triple, the operator  $J : L^2(X, \tau) \rightarrow L^2(X, \tau)$ ,  $J(x) = x^*$  satisfies the reality condition for  $p = 1$ , namely,  $J^2 = 1$ ,  $Ja^*J = a^{op}$  and  $J\mathcal{D}J = -\mathcal{D}$ , so the bimodule and spectral triple are real. This can be directly verified with ease.

For this reason we retain the reality condition in its original form.

**New Condition 8** (Reality). *There is an antiunitary operator  $J : \mathcal{H} \rightarrow \mathcal{H}$  such that  $Ja^*J^{-1} = a^{op}$  for all  $a \in \mathcal{A}$ ; and moreover,  $J^2 = \pm 1$ ,  $J\mathcal{D}J^{-1} = \pm\mathcal{D}$  and also  $J\Gamma J^{-1} = \pm\Gamma$  in the even case, according to the table (17) of signs.*

For the type I case connectedness of the underlying noncommutative space is formulated in the following condition.

**Old Condition 9** (Irreducibility). *The spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is irreducible: that is, the only operators in  $\mathcal{B}(\mathcal{H})$  commuting with  $\mathcal{D}$  and all  $a \in \mathcal{A}$  are the scalars.*

In a von Neumann algebra context it is clear what we should replace this condition with.

**New Condition 9** (Semifinite Irreducibility). *The semifinite spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is irreducible: that is, the only operators in  $\mathcal{N}$  commuting with  $\mathcal{D}$  and all  $a \in \mathcal{A}$  are the scalars.*

For our algebra  $\mathcal{A}$ , only the fixed-point subalgebra  $F$  commutes with  $\mathcal{D}$ . For graphs satisfying the single entry condition,  $F$  is abelian. A graph-theoretic argument shows that if  $E$  is connected, then no nontrivial element of  $F$  can commute with all of  $\mathcal{A}$ .

We summarise our results for graph algebras.

**Theorem 3.4.** *Let  $E$  be a connected locally finite graph with no sinks, admitting a faithful graph trace, satisfying the single entry condition and having finitely generated  $K$ -theory. Then the gauge spectral*

triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  of  $E$  satisfies the new (semifinite, nonunital) conditions 1 to 9. If  $E$  is not a single loop, the gauge spectral triple is both nonunital (noncompact) and semifinite.

#### 4. $k$ -GRAPH MANIFOLDS

In [PRS] we adapted the construction of [PRen] described earlier to construct a Kasparov module and semifinite spectral triple for suitable  $k$ -graph algebras. This was accomplished by ‘pushing forward’ the Dirac operator (of the simplest spin structure) on the  $k$ -torus, using the canonical  $\mathbb{T}^k$  action on a  $k$ -graph algebra.

We will not go into the details of these constructions as we did for graph algebras, noting only that they are essentially analogous to the graph case. We also omit a general discussion of  $k$ -graph algebras, as this is lengthy. We will adopt the definitions, notations and conventions of [PRS], and refer the reader to this work for an introduction to  $k$ -graph algebras adapted to this context.

We do require several notational reminders so that we can state our results here with the minimum of ambiguity. In particular:

**Warning** In this section we reverse our conventions regarding range and source of edges. This means that sinks and sources play opposite roles, the single entry condition becomes the single exit condition, and so on. This is in keeping with the notation employed in [PRS].

Briefly, a  $k$ -graph is a set  $\Lambda$  of paths with a degree map  $d : \Lambda \rightarrow \mathbb{N}^k$ . For  $n \in \mathbb{N}^k$ , we write  $\Lambda^n$  for  $d^{-1}(n)$ , and regard  $\Lambda^0$  as the set of vertices. Paths have the unique factorisation property: if  $d(\lambda) = m + n$  then there are unique paths  $\mu \in \Lambda^m$  and  $\nu \in \Lambda^n$  such that  $\lambda = \mu\nu$ . In particular, if  $m \leq n \leq l = d(\lambda)$ , then there is a unique factorisation  $\lambda = \lambda(0, m)\lambda(m, n)\lambda(n, l)$  where  $d(\lambda(0, m)) = m$  and so forth. It also follows that each path  $\lambda$  has a unique range  $r(\lambda) \in \Lambda^0$  such that  $r(\lambda)\lambda = \lambda$ ; likewise for sources. With this in mind, we write  $v\Lambda^n$  for  $r^{-1}(v) \cap \Lambda^n$  and  $\Lambda^n v$  for  $s^{-1}(v) \cap \Lambda^n$  for  $n \in \mathbb{N}^k$  and  $v \in \Lambda^0$ .

The  $C^*$ -algebra  $C^*(\Lambda)$  of a  $k$ -graph  $\Lambda$  is the universal  $C^*$ -algebra generated by a set  $\{S_\lambda : \lambda \in \Lambda\}$  of partial isometries satisfying Cuntz-Krieger type relations [KP].

For the remainder of this section, ‘ $k$ -graph’ shall be an abbreviation for ‘locally convex, locally finite  $k$ -graph without sinks, which possesses a faithful  $k$ -graph trace’. All the conditions below refer to the general semifinite nonunital versions discussed for graph algebras (with appropriate changes to the dimensions involved where necessary).

The gauge spectral triples  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  for  $k$ -graph algebras satisfy the new dimension, regularity (smoothness) and absolute continuity conditions, with dimension  $k$ . All this is proved in [PRS].

The new first order condition is satisfied just as in the graph case, and the new irreducibility condition is also satisfied if the  $k$ -graph is connected.

The remaining conditions which we need to verify are the new finiteness, orientability, closedness, Morita equivalence (spin<sup>c</sup>) and reality conditions.

In order to do this, we will need to assume that our  $k$ -graphs are row-finite with no sources ( $0 < |v\Lambda^n| < \infty$  for  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ ), and satisfy the *single exit condition* ( $|\Lambda^n v| = 1$  for each  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ ).

#### Finiteness and Morita Equivalence

The proof of our rather strict finiteness condition for  $k$ -graphs is almost identical to the proof for the 1-graph case. In fact, once we have the following result it is virtually identical.

Suppose that  $\Lambda$  is a row-finite  $k$ -graph with no sources satisfying the single-exit condition. We claim there is an isomorphism of the fixed point algebra  $C^*(\Lambda)^\gamma$  onto  $C_0(\Lambda^\infty)$  (where the infinite path space  $\Lambda^\infty$  is endowed with the topology generated by the cylinder sets  $\lambda\Lambda^\infty$ ,  $\lambda \in \Lambda$ ). The isomorphism takes  $S_\lambda S_\lambda^*$  to the characteristic function  $\chi_{\lambda\Lambda^\infty}$  for each  $\lambda \in \Lambda$ . To see this, first recall from [FPS] (see also [KP]) that for an arbitrary row-finite  $k$ -graph  $\Lambda$  with no sources, there is an isomorphism of the diagonal  $D(\Lambda) := \overline{\text{span}}\{S_\lambda S_\lambda^* : \lambda \in \Lambda\}$  onto  $C_0(\Lambda^\infty)$  which takes  $S_\lambda S_\lambda^*$  to  $\chi_{\lambda\Lambda^\infty}$ . We also know that  $C^*(\Lambda)^\gamma = \overline{\text{span}}\{S_\mu S_\nu^* : d(\mu) = d(\nu)\}$ , but the single-exit condition ensures that whenever  $S_\mu S_\nu^* \neq 0$  and  $d(\mu) = d(\nu)$ , we have  $\mu = \nu$ . Hence  $C^*(\Lambda)^\gamma = D(\Lambda)$  when  $\Lambda$  satisfies the single-exit condition, and this establishes the claim.

In particular, it is not hard to deduce from this an exact analogue of Lemma 3.1: if  $\Lambda$  is row-finite, satisfies the single exit condition, and has finitely many (say  $N$ ) ends, then each element  $a$  of  $F_c$  can be expressed as

$$(18) \quad a := \sum_{v \in \Lambda^0} \sum_{i=1}^N b_{(v,i)} S_{(v,i)} S_{(v,i)}^*$$

where  $(v, i)$  is a path from the  $i^{\text{th}}$  end to  $v$ .

As in the graph case, there is almost nothing to prove when the algebra is unital. This follows since then the trace of the identity is finite, and we can compare the Hilbert space and  $C^*$ -module norms easily. For the nonunital case we have the following.

**Proposition 4.1.** *Suppose that the locally finite, locally convex  $k$ -graph  $(\Lambda, d)$  has no sources and satisfies the single exit condition. The  $\mathcal{A}$ -module  $\mathcal{H}_\infty$  embeds continuously in the  $C^*$ - $\mathcal{A}$ -module completion if and only if the  $K$ -theory of  $\mathcal{A}$  is finitely generated; in this case the Hilbert space  $\mathcal{H}$  does also. If  $K_*(\mathcal{A})$  is finitely generated then the  $C^*$ -inner product restricted to  $\mathcal{H}_\infty$  takes values in  $\mathcal{A}$ .*

Apart from the above result describing the fixed-point algebra of  $C^*(\Lambda)$ , we also require the  $K$ -theory computations of [PRS] which show that in the situation of Proposition 4.1 the  $K$ -theory is finitely generated if and only if there are finitely many ends. With these results in hand, our corresponding proof for 1-graphs can be applied with minor modifications.

The Morita equivalence condition is now simple, since  $\mathcal{C}_{\mathcal{D}}(\mathcal{A}) \cong \mathbf{Cliff}(\mathbf{R}^k) \otimes \mathcal{A}$  (or  $\mathbf{Cliff}^+(\mathbf{R}^k)$  in odd dimensions) and  $\mathcal{H}_\infty = \mathcal{A}^{2^{\lfloor k/2 \rfloor}}$ . So the gauge spectral triple of a  $k$ -graph is  $\text{spin}^c$ .

### Orientation

As noted above,  $\mathcal{H}_\infty = \mathcal{A}^{2^{\lfloor k/2 \rfloor}}$ , and the operator  $\mathcal{D}$  acts on generators  $x \in X_c \subset \mathcal{H}$  with degree  $d(x) = n \in \mathbb{N}^k$  by  $\mathcal{D}x = \left( \sum_{j=1}^k \gamma^j (in_j) \right) x$  where the  $\gamma^j$  are constant matrices generating the complex Clifford algebra of  $\mathbf{R}^k$ .

In what follows we write  $\mathbf{1}_k$  for  $(1, \dots, 1) \in \mathbb{N}^k$ . We let  $\Sigma_k$  be the group of permutations of  $\{1, \dots, k\}$ .

Fix a  $k$ -graph  $\Lambda$  and a path  $\mu \in \Lambda^{\mathbf{1}_k}$ . Given a permutation  $\sigma \in \Sigma_k$ , the factorisation property guarantees that there is a unique factorisation  $\mu = \mu_1^\sigma \mu_2^\sigma \dots \mu_k^\sigma$  such that  $\mu_i^\sigma \in \Lambda^{e_{\sigma(i)}}$  for  $1 \leq i \leq k$ .

For example, let  $k = 2$  and let  $\mu = ef = ab$  be a commuting square, so that  $d(e) = d(b) = e_1$  and  $d(f) = d(a) = e_2$ . There are two elements of  $\Sigma_2$ , namely the flip  $(1, 2)$  and the identity  $\text{id}$ . We have  $\mu_1^{(1,2)} = a$  and  $\mu_2^{(1,2)} = b$  whilst  $\mu_1^{\text{id}} = e$  and  $\mu_2^{\text{id}} = f$ .

We use the notation  $(-1)^\sigma$  for the canonical homomorphism  $\sigma \mapsto (-1)^\sigma$  from  $\Sigma_k$  to  $\{-1, 1\}$  which takes the 2-cycles  $(i, j)$  to  $-1$ .

**Proposition 4.2.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, and suppose that for every  $v \in \Lambda^0$  and  $1 \leq i \leq k$  we have  $|\Lambda^{e_i v}| = 1$  (single exit). Define*

$$(19) \quad c_k := i^{\lceil \frac{k+1}{2} \rceil} \sum_{\mu \in \Lambda^{1_k}} \frac{1}{k!} \sum_{\sigma \in \Sigma_k} (-1)^\sigma S_\mu^* \otimes S_{\mu_1^\sigma} \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma}.$$

*Then  $b(c_k) = 0$ , where  $b$  is the Hochschild boundary operator, and  $\pi_{\mathcal{D}}(c_k) = \Gamma$  where  $\Gamma$  is the grading for  $k$  even, and the identity for  $k$  odd.*

*Proof.* We begin by establishing that  $\pi_{\mathcal{D}}(c_k) = \Gamma$  because this is the easier of the two calculations. To see this, we just calculate (here, the  $\gamma^j$  are the generators of  $\mathbf{Cliff}(\mathbf{R}^k)$ ):

$$\begin{aligned} \pi_{\mathcal{D}}(c_k) &= i^{\lceil \frac{k+1}{2} \rceil} \sum_{\mu \in \Lambda^{1_k}} \frac{1}{k!} \sum_{\sigma \in \Sigma_k} (-1)^\sigma S_\mu^* [\mathcal{D}, S_{\mu_1^\sigma}] [\mathcal{D}, S_{\mu_2^\sigma}] \cdots [\mathcal{D}, S_{\mu_k^\sigma}] \\ &= i^{\lceil \frac{k+1}{2} \rceil} \sum_{\mu \in \Lambda^{1_k}} \frac{1}{k!} \sum_{\sigma \in \Sigma_k} (-1)^\sigma S_\mu^* S_{\mu_1^\sigma} \gamma^{\sigma(1)} S_{\mu_2^\sigma} \gamma^{\sigma(2)} \cdots S_{\mu_k^\sigma} \gamma^{\sigma(k)} \\ &= i^{\lceil \frac{k+1}{2} \rceil} \gamma^1 \cdots \gamma^k \sum_{\mu \in \Lambda^{1_k}} \frac{1}{k!} \sum_{\sigma \in \Sigma_k} S_\mu^* S_{\mu_1^\sigma} S_{\mu_2^\sigma} \cdots S_{\mu_k^\sigma} = \omega_{\mathbf{C}} \sum_{\mu \in \Lambda^{1_k}} p_{s(\mu)}, \end{aligned}$$

where  $\omega_{\mathbf{C}}$  is the complex volume form in  $\mathbf{Cliff}(\mathbf{R}^k)$ . The single exit assumption ensures that the sum of vertex projections in the last line has exactly one term for each vertex of  $\Lambda$ , and hence converges to the identity in the multiplier algebra of  $C^*(\Lambda)$ , establishing that  $\pi_{\mathcal{D}}(c_k) = \Gamma$ .

Now we need to establish that  $b(c_k) = 0$ . To begin with, fix  $\mu \in \Lambda^{1_k}$ . We claim that

$$(20) \quad \begin{aligned} &b\left(\sum_{\sigma \in \Sigma_k} (-1)^\sigma S_\mu^* \otimes S_{\mu_1^\sigma} \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma}\right) \\ &= \sum_{\sigma \in \Sigma_k} (-1)^\sigma \left( S_{\mu_2^\sigma \dots \mu_k^\sigma}^* \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} \right. \\ &\quad \left. + (-1)^k S_{\mu_k^\sigma} S_{\mu_k^\sigma}^* S_{\mu_1^\sigma \dots \mu_{k-1}^\sigma}^* \otimes S_{\mu_1^\sigma} \otimes \cdots \otimes S_{\mu_{k-1}^\sigma} \right). \end{aligned}$$

To see this, we apply the definition of the Hochschild boundary  $b$  to obtain

$$\begin{aligned} &b\left(\sum_{\sigma \in \Sigma_k} (-1)^\sigma S_\mu^* \otimes S_{\mu_1^\sigma} \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma}\right) \\ &= \sum_{\sigma \in \Sigma_k} (-1)^\sigma \left( S_{\mu_2^\sigma \dots \mu_k^\sigma}^* \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} \right. \\ &\quad \left. + \sum_{j=1}^{k-1} (-1)^j S_\mu^* \otimes S_{\mu_1^\sigma} \otimes \cdots \otimes S_{\mu_j^\sigma} S_{\mu_{j+1}^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} \right. \\ &\quad \left. + (-1)^k S_{\mu_k^\sigma} S_{\mu_k^\sigma}^* S_{\mu_1^\sigma \dots \mu_{k-1}^\sigma}^* \otimes S_{\mu_1^\sigma} \otimes \cdots \otimes S_{\mu_{k-1}^\sigma} \right). \end{aligned}$$

To establish (20), it therefore suffices to show that for  $1 \leq j \leq k-1$ , we have

$$\sum_{\sigma \in \Sigma_k} (-1)^\sigma S_\mu^* \otimes S_{\mu_1^\sigma} \otimes \cdots \otimes S_{\mu_j^\sigma} S_{\mu_{j+1}^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} = 0.$$

To see this, we fix  $1 \leq j \leq k-1$ , and note that we may partition  $\Sigma_k$  as  $\Sigma_k = A_j \sqcup B_j$  where  $A_j := \{\sigma \in \Sigma_k : \sigma(j) < \sigma(j+1)\}$  and  $B_j := \{\sigma \in \Sigma_k : \sigma(j) > \sigma(j+1)\}$ . Let  $t_j \in \Sigma_k$  be the transposition  $(j, j+1)$ . Then  $\sigma \mapsto \sigma \circ t_j$  is a bijection from  $A_j$  to  $B_j$ .

Hence

$$\begin{aligned} & \sum_{\sigma \in \Sigma_k} (-1)^\sigma S_\mu^* \otimes S_{\mu_1^\sigma} \otimes \cdots \otimes S_{\mu_j^\sigma} S_{\mu_{j+1}^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} \\ &= \sum_{\sigma \in A_j} \left( (-1)^\sigma S_\mu^* \otimes S_{\mu_1^\sigma} \otimes \cdots \otimes S_{\mu_j^\sigma} S_{\mu_{j+1}^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} \right. \\ & \quad \left. + (-1)^{\sigma \circ t_j} S_\mu^* \otimes S_{\mu_1^{\sigma \circ t_j}} \otimes \cdots \otimes S_{\mu_j^{\sigma \circ t_j}} S_{\mu_{j+1}^{\sigma \circ t_j}} \otimes \cdots \otimes S_{\mu_k^{\sigma \circ t_j}} \right). \end{aligned}$$

The definition of  $t_j$  guarantees that  $(-1)^\sigma + (-1)^{\sigma \circ t_j} = 0$  for all  $\sigma \in A_j$ , and we will therefore have established (20) if we can show that for fixed  $1 \leq j \leq k-1$  and fixed  $\sigma \in A_j$ , we have

$$(21) \quad S_\mu^* \otimes S_{\mu_1^\sigma} \otimes \cdots \otimes S_{\mu_j^\sigma} S_{\mu_{j+1}^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} = S_\mu^* \otimes S_{\mu_1^{\sigma \circ t_j}} \otimes \cdots \otimes S_{\mu_j^{\sigma \circ t_j}} S_{\mu_{j+1}^{\sigma \circ t_j}} \otimes \cdots \otimes S_{\mu_k^{\sigma \circ t_j}}$$

By definition of  $t_j$  we have  $\mu_i^\sigma = \mu_i^{\sigma \circ t_j}$  whenever  $i \neq j, j+1$ . If we set  $m := \sum_{i=1}^{j-1} e_{\sigma(i)} \in \mathbb{N}^k$ , then the factorisation property in  $\Lambda$  ensures that

$$\mu_j^\sigma \mu_{j+1}^\sigma = \mu(m, m + e_{\sigma(j)} + e_{\sigma(j+1)}) = \mu(m, m + e_{\sigma \circ t_j(j)} + e_{\sigma \circ t_j(j+1)}) = \mu_j^{\sigma \circ t_j} \mu_{j+1}^{\sigma \circ t_j}.$$

It follows that corresponding terms in the elementary tensors on either side of (21) are identical. This establishes (20).

We must now show that if we sum the right-hand side of (20) over all  $\mu \in \Lambda^{1^k}$ , we obtain zero.

Fix, for the time being,  $\mu \in \Lambda^{1^k}$  and  $\sigma \in \Sigma_k$ . Consider the expression

$$(22) \quad (-1)^\sigma S_{\mu_2^\sigma \dots \mu_k^\sigma}^* \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma}$$

appearing as a summand in the first term on the right-hand side of (20). Let  $\lambda := \mu_2^\sigma \mu_3^\sigma \dots \mu_k^\sigma$ , so that  $\mu = \mu_1^\sigma \lambda$ . Let  $\psi_k \in \Sigma_k$  be the permutation defined by  $\psi_k(i) = i+1$  for  $i \leq k-1$  and  $\psi_k(k) = 1$ . Fix  $\alpha \in s(\lambda) \Lambda^{e_{\sigma(1)}}$ . Then  $\lambda \alpha \in \Lambda^{1^k}$ . Consider the expression

$$x(\lambda, \sigma \circ \psi_k, \alpha) := (-1)^{\sigma \circ \psi_k} (-1)^k S_{(\lambda \alpha)_k^{\sigma \circ \psi_k}}^* S_{(\lambda \alpha)_k^{\sigma \circ \psi_k}}^* S_{(\lambda \alpha)_1^{\sigma \circ \psi_k} \dots (\lambda \alpha)_{k-1}^{\sigma \circ \psi_k}}^* \otimes S_{(\lambda \alpha)_1^{\sigma \circ \psi_k}} \otimes \cdots \otimes S_{(\lambda \alpha)_{k-1}^{\sigma \circ \psi_k}}$$

which appears in the second term on the right-hand side of (20) for  $\lambda \alpha \in \Lambda^{1^k}$  and  $\sigma \circ \psi_k \in \Sigma_k$ . We have  $(-1)^{\psi_k} = (-1)^{k-1}$ , and hence  $(-1)^{\sigma \circ \psi_k} (-1)^k = -(-1)^\sigma$ . By definition of  $\psi_k$ , we have  $(\lambda \alpha)_k^{\sigma \circ \psi_k} = \alpha$ , and  $(\lambda \alpha)_j^{\sigma \circ \psi_k} = \mu_{j+1}^\sigma$  for  $1 \leq j \leq k-1$ . Hence, we may rewrite

$$x(\lambda, \sigma \circ \psi_k, \alpha) = -(-1)^\sigma S_\alpha S_\alpha^* S_{\mu_2^\sigma \dots \mu_k^\sigma}^* \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma}.$$

By the Cuntz-Krieger relation, we have

$$\sum_{\alpha \in s(\lambda) \Lambda^{e_{\sigma(1)}}} S_\alpha S_\alpha^* = p_{s(\lambda)} = p_{s(\mu_k^\sigma)},$$

and hence

$$(23) \quad S_{\mu_2^\sigma \dots \mu_k^\sigma}^* \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} - \sum_{\alpha \in s(\lambda) \Lambda^{e_{\sigma(1)}}} x(\mu_2^\sigma \dots \mu_k^\sigma, \sigma \circ \psi_k, \alpha) = 0$$

The single-exit condition and the unique factorisation property guarantee that each

$$S_{\mu_2^\sigma \dots \mu_k^\sigma}^* \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma}$$

occurs exactly once in the first summand of the right-hand side of (20) as  $\mu$  ranges over  $\Lambda^{1_k}$  and  $\sigma$  ranges over  $\Sigma_k$ . The factorisation property shows that for fixed  $\mu$  and  $\sigma$ , a term  $x(\lambda, \sigma', \alpha)$  is of the form

$$x \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma}$$

for some  $x \in C^*(\Lambda)$  only if  $\sigma' = \sigma \circ \psi_k$ ,  $\lambda = \mu_2^\sigma \dots \mu_k^\sigma$  and  $\alpha \in s(\lambda)\Lambda^{e_{\sigma(1)}}$ .

Hence we may formally rewrite

$$b(c_k) = \sum_{\mu \in \Lambda^{1_k}} \sum_{\sigma \in \Sigma_k} (-1)^\sigma \left( S_{\mu_2^\sigma \dots \mu_k^\sigma}^* \otimes S_{\mu_2^\sigma} \otimes \cdots \otimes S_{\mu_k^\sigma} - \sum_{\alpha \in s(\mu_k^\sigma)\Lambda^{e_{\sigma(1)}}} x(\mu_2^\sigma \dots \mu_k^\sigma, \sigma \circ \psi_k, \alpha) \right),$$

which formally collapses to zero by (23).

One can check relatively easily, using the approximate identity  $\sum_{\mu \in \Lambda^{1_k}} S_\mu S_\mu^*$  for  $C^*(\Lambda)$ , that the infinite sums involved in the definition of  $c_k$  and the formal calculations in this proof make sense in the multiplier algebra of the  $k + 1$ -fold tensor power of  $C^*(\Lambda)$ .  $\square$

### Closedness

To show that for all  $a_1, \dots, a_k \in \mathcal{A}$  we have

$$\tilde{\tau}_\omega(\Gamma[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k](1 + \mathcal{D}^2)^{-k/2}) = 0$$

it suffices to prove the result for generators of the algebra. So let  $T_{\mu_j, \nu_j} = S_{\mu_j} S_{\nu_j}^*$ ,  $j = 1, \dots, k$ , be generators. Then

$$[\mathcal{D}, T_{\mu_j, \nu_j}] = \gamma(id_j) = i \sum_{m=1}^k \gamma^m n_{m,j} T_{\mu_j, \nu_j}$$

where  $d_j = (n_{1,j}, \dots, n_{k,j})$  is the degree of  $T_{\mu_j, \nu_j}$ . With this notation we have

$$(24) \quad \begin{aligned} & \tilde{\tau}_\omega(\Gamma[\mathcal{D}, T_{\mu_1, \nu_1}] \cdots [\mathcal{D}, T_{\mu_k, \nu_k}](1 + \mathcal{D}^2)^{-k/2}) \\ &= i^p \tilde{\tau}_\omega \left( \Gamma \left( \sum_{j_1} \gamma^{j_1} n_{j_1,1} \right) \cdots \left( \sum_{j_k} \gamma^{j_k} n_{j_k,k} \right) T_{\mu_1, \nu_1} \cdots T_{\mu_k, \nu_k} (1 + \mathcal{D}^2)^{-k/2} \right). \end{aligned}$$

Now  $\Gamma = \omega_{\mathbf{C}} \otimes 1$  where  $\omega_{\mathbf{C}}$  is the (representation of) the complex volume form in the Clifford algebra. Since the only products of generators of the Clifford algebra with non-zero trace are multiples of the identity, the only surviving terms on the right hand side of Equation (24) when we expand the products are those with precisely one of each generator  $\gamma^j$ . Thus

$$\begin{aligned}
& \tilde{\tau}_\omega(\Gamma[\mathcal{D}, T_{\mu_1, \nu_1}] \cdots [\mathcal{D}, T_{\mu_k, \nu_k}](1 + \mathcal{D}^2)^{-k/2}) \\
&= i^p \tilde{\tau}_\omega \left( \Gamma \sum_{\sigma \in S_k} \gamma^{\sigma(1)} n_{\sigma(1), 1} \cdots \gamma^{\sigma(k)} n_{\sigma(k), k} T_{\mu_1, \nu_1} \cdots T_{\mu_k, \nu_k} (1 + \mathcal{D}^2)^{-k/2} \right) \\
&= i^{p - [(p+1)/2]} \tilde{\tau}_\omega \left( \sum_{\sigma \in S_k} (-1)^\sigma n_{\sigma(1), 1} \cdots n_{\sigma(k), k} T_{\mu_1, \nu_1} \cdots T_{\mu_k, \nu_k} (1 + \mathcal{D}^2)^{-k/2} \right) \\
&= i^{p - [(p+1)/2]} \det(n_{j,k}) \tilde{\tau}_\omega \left( T_{\mu_1, \nu_1} \cdots T_{\mu_k, \nu_k} (1 + \mathcal{D}^2)^{-k/2} \right)
\end{aligned}$$

Now, the trace  $\tilde{\tau}_\omega(T_{\mu_1, \nu_1} \cdots T_{\mu_k, \nu_k} (1 + \mathcal{D}^2)^{-k/2}) = \tau(T_{\mu_1, \nu_1} \cdots T_{\mu_k, \nu_k})$  is zero unless  $T_{\mu_1, \nu_1} \cdots T_{\mu_k, \nu_k} \in F$ , since  $\tau$  is gauge invariant. This is equivalent to

$$\sum_{j=1}^k d_j = 0 \Leftrightarrow \forall l \sum_{m=1}^k n_{l,m} = 0.$$

Hence the first, say, column of the matrix  $(n_{j,k})$  is a linear combination of the other columns, and  $\det(n_{j,k}) = 0$ . Hence for any generators  $T_{\mu_j, \nu_j} = S_{\mu_j} S_{\nu_j}^*$ , we have

$$\tilde{\tau}_\omega(\Gamma[\mathcal{D}, T_{\mu_1, \nu_1}] \cdots [\mathcal{D}, T_{\mu_k, \nu_k}](1 + \mathcal{D}^2)^{-k/2}) = 0.$$

### Reality

We take the complex Clifford algebra  $\mathbf{Cliff}_k$  to be generated by  $k$  elements  $\gamma^j$ ,  $j = 1, \dots, k$  such that  $(\gamma^j)^* = -\gamma^j$  and

$$\gamma^j \gamma^l + \gamma^l \gamma^j = -2\delta_{l,j} Id.$$

We make some further specifications on the generators consistent with these conventions. Denote by  $\mathbf{j}$  the antilinear operator on  $X$  such that

$$\mathbf{j}x = \mathbf{j} \begin{pmatrix} x_1 \\ \vdots \\ x_{2^{\lfloor k/2 \rfloor}} \end{pmatrix} = \begin{pmatrix} x_1^* \\ \vdots \\ x_{2^{\lfloor k/2 \rfloor}}^* \end{pmatrix}.$$

Let  $s(k) = \lfloor \frac{k}{2} \rfloor (k+1) - k$  and label the generators of the Clifford algebra so that

$$\overline{\gamma^j} = \mathbf{j} \gamma^j \mathbf{j} = \begin{cases} (-1)^{s(k)} \gamma^j & j \text{ odd} \\ (-1)^{s(k)+1} \gamma^j & j \text{ even} \end{cases}$$

Observe that  $s(k)$  is even only when  $k = 4n$ , so except for these dimensions the odd generators have complex entries and are invariant under transpose, while the even generators have real entries and are antisymmetric. In dimensions  $4n$  the situation is of course reversed.

Let  $\chi = \gamma^2 \gamma^4 \cdots \gamma^{2^{\lfloor k/2 \rfloor}}$  be the product of the even generators (take  $\chi = 1$  when  $k = 1$ ). Since the entries of  $\chi$  are real for all  $k$  (if  $k = 4n$  there are  $2n$  factors in  $\chi$  and so the entries of  $\chi$  are real) we have

$$\bar{\chi} = \chi.$$

Using  $(\gamma^j)^* = -\gamma^j$  we find  $\chi^* = (-1)^{\lfloor k/2 \rfloor (\lfloor k/2 \rfloor + 1)/2} \chi$ . We then define

$$J := \chi \circ \mathbf{j} = \mathbf{j} \circ \chi.$$

**Lemma 4.3.** *The operator  $J$  satisfies  $J^2 = \epsilon$ ,  $J\mathcal{D} = \epsilon'\mathcal{D}J$  and for  $k$  even  $J\Gamma = \epsilon''\Gamma J$ , where  $\epsilon, \epsilon', \epsilon''$  are given in the table in Equation (17).*

*Proof.* To check the sign  $\epsilon$ , one needs only  $J^*J = 1$  (which is straightforward) and

$$J^* = \mathbf{j}^* \circ \chi^* = (-1)^{[k/2]([k/2]+1)/2} \mathbf{j} \circ \chi = (-1)^{[k/2]([k/2]+1)/2} J.$$

The sign can now be easily checked. The sign  $\epsilon''$ , in even dimensions, arises because  $\mathbf{j}$  preserves the  $\pm 1$  eigenspace decomposition of  $\omega_{\mathbf{C}}$ , and so commutes with  $\omega_{\mathbf{C}}$ , while  $\omega_{\mathbf{C}}\chi = (-1)^{k/2}\chi\omega_{\mathbf{C}}$ .

For  $\epsilon'$  this is more subtle. We require the straightforward identity  $\mathbf{j}\Phi_n\mathbf{j} = \Phi_{-n}$  which may be checked on generators. Then we compute

$$\begin{aligned} J\mathcal{D}\Phi_n J^* &= J\left(\sum_j i\gamma^j n_j\right)\Phi_n J^* = \chi\left(-i\sum_j \mathbf{j}\gamma^j \mathbf{j}n_j\right)\mathbf{j}\Phi_n J^* \\ &= -i\chi\left(\sum_{j \text{ odd}} (-1)^{s(k)}\gamma^j n_j + \sum_{j \text{ even}} (-1)^{s(k)+1}\gamma^j n_j\right)\mathbf{j}\Phi_n J^* \\ (25) \quad &= -i(-1)^{[(k+1)/2](k+2)}\sum_j \gamma^j n_j J\Phi_n J^* = (-1)^{[(k+1)/2](k+2)}\mathcal{D}\Phi_{-n}. \end{aligned}$$

Using the orthogonality of the  $\Phi_n$ , for any  $x \in \text{Dom}\mathcal{D}$  we have

$$J\mathcal{D}J^*x = \sum_{n \in \mathbf{Z}^k} J\mathcal{D}\Phi_n J^*x = (-1)^{[(k+1)/2](k+2)}\sum_{n \in \mathbf{Z}^k} \mathcal{D}\Phi_{-n}x = (-1)^{[(k+1)/2](k+2)}\mathcal{D}x.$$

The reader will check that the sign appearing here agrees with the values of  $\epsilon'$  in the table above.  $\square$

**Theorem 4.4.** *Let  $(\Lambda, d)$  be a connected, locally convex, locally finite  $k$ -graph with no sources, a faithful  $k$ -graph trace, satisfying the single exit condition and having finitely generated  $K$ -theory. Then the gauge spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  of  $\Lambda$  satisfies the (semifinite, nonunital) Conditions 1 to 9.*

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