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Inf-sup stable finite element pairs based on dual meshes and bases for nearly incompressible elasticity

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We consider finite element methods based on simplices to solve the problem of nearly incompressible elasticity. Two different approaches based respectively on dual meshes and dual bases are presented, where in both approaches pressure is discontinuous and can be statically condensed out from the system. These novel approaches lead to displacement-based low order finite element methods for nearly incompressible elasticity based on rigorous mathematical framework. Numerical results are provided to demonstrate the efficiency of the approach.

Keywords: mixed finite elements, nodal average pressure, nearly incompressible elasticity, dual bases, dual meshes.

AMS Subject Classification: 65F30, 65N15, 74B10

1. Introduction

Low order finite elements based on quadrilaterals, hexahedra or simplices exhibit a poor performance when applied to a nearly incompressible elasticity problems. This poor performance is well-known to be the locking effect, that means, they do not converge uniformly with respect to the Lamé parameter λ. There are many approaches to overcome this difficulty. One obvious remedy for such problem is to work with higher order finite elements. For example, in Scott & Vogelius (1985), it is shown that working with the *h*-version finite elements of order higher than three on a class of triangular meshes locking in linear elasticity can completely be avoided. On the other hand, in Babuška & Suri (1992a), it has been shown that the *h*-version can never be fully free of locking in rectangular meshes no matter how higher-order finite elements are used in the sense that suboptimal convergence rates are observed. For the mathematical analysis of the locking effect, we refer to Babuška & Suri (1992a,b). An another approach is related to working with mixed methods. The linear elasticity problem can be formulated as a mixed formulation in many different ways, see Brezzi & Fortin (1991); Braess (1996, 2001); Arnold & Winther (2002). The general approach in these mixed formulations is to introduce extra variables leading to a saddle point problem. The essential point is to prove that the method is robust for the limiting problem, which is the Stokes problem. Treating displacement and the 'mean pressure' as two independent field variables, one of the mixed formulations is given by Herrmann (1965), which is also known as the reduced form of Hellinger-Reissner principle. The Herrmann principle has also been extended to nonlinear hyperelastic problems in Reissner & Atluri (1989); van den Bogert *et al.* (1991); de Souza *et al.* (1996); Piltner & Taylor (1999). Other mixed formulations for nonlinear elasticity are introduced in Gatica & Stephan (2002); Gatica *et al.* (2007), see also Carstensen & Funken (2001); Brink & Stephan (2001); Carstensen *et al.* (2005). The approaches based on the bilinear or trilinear displacement and piecewise constant pressure are often used to overcome the volumetric locking in nearly incompressible elasticity, see Hughes (1987); Brezzi & Fortin (1991); Braess (2001). Separation of volumetric and deviatoric response leads to the so-called B-bar method for linear and nonlinear elasticity, see Hughes (1987); Nagtegaal *et al.* (1974); Simo *et al.* (1985). The approach based on mixed formulations has a close link with reduced integration schemes, see Belytschko & Bachrach (1986); Brenner & Sung (1992). Nonconforming finite element methods have also been analyzed for linear elastic problems leading to the uniform convergence in the nearly incompressible case, see Falk (1991); Brenner & Sung (1992); Lee *et al.* (2003); Brenner (1993, 1994).

Most popular methods with low order finite elements based on four-noded quadrilaterals or eight-noded hexahedra are the methods associated with the enhancement of the strain or stress field. The method of enhanced assumed strain is derived by Simo & Rifai (1990) using Hu-Washizu formulation and has become a popular approach in overcoming these difficulties of standard elements in linear and nonlinear elasticity (Simo & Armero (1992); Simo *et al.* (1993)). The method of mixed enhanced strain approach introduced in Kasper & Taylor (2000a,b) is also based on Hu-Washizu formulation. The rigorous mathematical analysis of these approaches can be found in Braess *et al.* (2004); Lamichhane *et al.* (2006).

The mixed enhanced formulation is also extended to simplicial meshes in Taylor (2000). However, the formulation in Taylor (2000) is derived by using mini-element requiring that the pressure variable is continuous. Similarly, the finite calculus formulation has also been shown to perform well for simplices for incompressible solids in Onate *et al.* (2004). There are a large number of publications devoted to the analysis of average nodal pressure element based on simplices, see e.g., Bonet & Burton (1998); Guo *et al.* (2000); Bonet *et al.* (2001); de Souza Neto *et al.* (2005). Although there are many approaches proposed to solve a nearly incompressible elasticity problem on simplices, a few approaches are based on rigorous mathematical analysis. For example, the mixed displacement-pressure formulation and the non-conforming approach analyzed in Brenner & Sung (1992) are based on rigorous mathematical analysis. However, the mixed formulation requires an inf-sup stable pair for the displacement and pressure, and working with discontinuous pressure, higher order finite elements should be used. On the other hand, the low order non-conforming approach should be modified to fulfil the discrete Korn's inequality (Falk (1991)). Hence, our approach is motivated to obtain a simple displacement-based low order method with rigorous mathematical analysis.

In this paper, we consider a discretization scheme similar to the approach based on enforcing a constant pressure field over a patch of triangles or tetrahedra (Bonet & Burton (1998); Guo *et al.* (2000); Dohrmann *et al.* (2000); Bonet *et al.* (2001); Pires *et al.* (2004)). This type of scheme is introduced in Bonet & Burton (1998) and is called average nodal pressure formulation. It is extremely easy to implement such scheme in the existing codes and has many nice properties. The idea of using average nodal pressure formulation can be traced back to finite volume schemes, where control volumes are built by using a finite element mesh (Bank & Rose (1987); Ewing *et al.* (2002)). We combine the idea of finite volume element method with the mixed finite element method to replace the continuous pressure with the discontinuous one and analyze the average nodal pressure formulation under the framework of mixed finite elements. In particular, we consider the nearly incompressible linear elastic problem and discuss the stability of the finite element formulations. The analysis is done by taking into account the primal and dual meshes, where the respective variables: the displacement and the pressure are defined. The analysis shows that the average nodal pressure formulation does not satisfy the uniform inf-sup condition and it is necessary to enrich the displacement space with bubble functions as in the mini-element (Arnold *et al.* (1984)) to obtain a stable formulation. Using the piece-wise linear polynomial space enriched with a bubble function per element for the displacement, we show that the inf-sup condition holds uniformly for the discontinuous pressure defined on the dual mesh. The degree of freedom corresponding to the pressure variable can be statically condensed out from the system leading to a simple displacement-based formulation. The implementation is made easy using the linearity of the displacement field and an interesting property of the bubble function on the control volume.

We also present a next approach based on primal and dual bases which is alternative to the average nodal pressure formulation. In this approach, the solution space and the test space for the pressure are different leading to a scheme like Petrov-Galerkin. The solution space for the pressure is discretized by using the standard linear finite element basis whereas we use the dual basis to discretize the test space for the pressure. The dual basis can be locally constructed yielding a diagonal mass matrix. As in the average nodal pressure formulation the pressure can be statically condensed out from the system. The inf-sup condition follows easily by using standard arguments. Furthermore, this idea paves a general way to show that many formulations based on primal and dual meshes can be written as formulations based on primal and dual bases. In particular, this is useful for problems where the construction of dual meshes is complicated.

The structure of the rest of the paper is organized as follows. In the next section, we briefly recall the standard and a mixed formulation of linear elasticity. Section 3 is devoted to the mathematical analysis of the discrete problem. Using the fact that the mini-element introduced in Arnold *et al.* (1984) satisfies the uniform inf-sup condition, we show that the uniform inf-sup condition also holds for the discontinuous pressure as far as we work with the dual mesh for the pressure. In Section 4, we introduce the alternative approach based on dual bases, and numerical results are presented in Section 5. Finally, we draw conclusion in the last section.

2. The boundary value problem of linear elasticity

This section is devoted to the introduction of the boundary value problem of linear elasticity. We consider a homogeneous isotropic linear elastic material body occupying a bounded domain Ω in $\mathbb{R}^d$, $d = \{2,3\}$ with Lipschitz boundary Γ. For a prescribed body force $\mathbf{f} \in [L^2(\Omega)]^d$, the governing equilibrium equation in Ω reads

$$-\operatorname{div} \sigma = \mathbf{f}, \tag{2.1}$$

where σ is the symmetric Cauchy stress tensor. The stress tensor σ is defined as a function of the displacement $\mathbf{u}$ by the Saint-Venant Kirchhoff constitutive law

$$\sigma = \frac{1}{2}\mathscr{C}(\nabla \mathbf{u} + [\nabla \mathbf{u}]^T), \tag{2.2}$$

where $\mathscr{C}$ is the fourth-order elasticity tensor. The action of the elasticity tensor $\mathscr{C}$ on a tensor $\mathbf{d}$ is defined as

$$\sigma = \mathscr{C}\mathbf{d} := \lambda(\operatorname{tr}\mathbf{d})\mathbf{1} + 2\mu\,\mathbf{d}. \tag{2.3}$$

Here, $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and which are assumed positive. The displacement is assumed to satisfy homogeneous Dirichlet boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma. \tag{2.4}$$

We are interested in the nearly incompressible case, which corresponds to λ being very large.

Standard weak formulation.

We will make use of the space $L^2(\Omega)$ of square-integrable functions defined on Ω with the inner product and norm being denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, respectively. The space $H^1_0(\Omega)$ consists of functions in

$H^1(\Omega)$ which vanish on the boundary in the sense of traces. To write the weak or variational formulation of the boundary value problem, we introduce the space $\mathbf{V} := [H_0^1(\Omega)]^d$ of displacements with inner product $(\cdot,\cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way; that is, $(\mathbf{u},\mathbf{v})_1 := \sum_{i=1}^d (u_i, v_i)_1$, with the norm being induced by this inner product.

We define the bilinear form $A(\cdot,\cdot)$ and the linear functional $\ell(\cdot)$ by

$$\begin{array}{llll} A : \mathbf{V}\times\mathbf{V} \to \mathbb{R}, & A(\mathbf{u},\mathbf{v}) & := & \int_\Omega \mathscr{C}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\,dx, \\ \ell : V \to \mathbb{R}, & \ell(\mathbf{v}) & := & \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx. \end{array}$$

Then the standard weak form of linear elasticity problem is as follows: given $\ell \in \mathbf{V}'$, find $\mathbf{u} \in \mathbf{V}$ that satisfies

$$A(\mathbf{u},\mathbf{v}) = \ell(\mathbf{v})\,, \quad \mathbf{v} \in \mathbf{V}. \tag{2.5}$$

The assumptions on $\mathscr{C}$ guarantee that $A(\cdot,\cdot)$ is symmetric, continuous, and V-elliptic. Hence by using standard arguments it can be shown that (2.5) has a unique solution $\mathbf{u} \in \mathbf{V}$. Furthermore, we assume that the domain Ω is convex and polygonal or polyhedral such that $\mathbf{u} \in [H^2(\Omega)]^d \cap \mathbf{V}$, and there exists a constant C independent of λ such that

$$\|\mathbf{u}\|_2 + \lambda \|\operatorname{div}\mathbf{u}\|_1 \leqslant C\,\|\mathbf{f}\|_0. \tag{2.6}$$

The a priori estimate (2.6) has been shown in Brenner & Sung (1992) for the two-dimensional linear elasticity posed in a convex domain with polygonal boundary, see Kozlov *et al.* (2001) for the three-dimensional case with convex domain and polyhedral boundary.

Mixed formulation. As pointed out in the introduction, the linear elasticity problem can be recast into different mixed formulations. The easiest mixed formulation is given by introducing pressure as an extra variable, which leads to penalized Stokes equations. Defining $p := \lambda \operatorname{div}\mathbf{u}$ and $L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}$, a mixed variational formulation of linear elastic problem (2.5) is given by: find $(\mathbf{u}, p) \in \mathbf{V}\times L_0^2(\Omega)$ such that

$$\begin{array}{lllllll} a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & \ell(\mathbf{v}), & \mathbf{v} \in & \mathbf{V}, \\ b(\mathbf{u},q) & - & \frac{1}{\lambda}c(p,q) & = & 0, & q \in & L_0^2(\Omega), \end{array} \tag{2.7}$$

where

$$a(\mathbf{u},\mathbf{v}) := 2\mu\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\,dx,\; b(\mathbf{v},q) := \int_\Omega \operatorname{div}\mathbf{v}\,q\,dx,\; \text{and } c(p,q) := \int_\Omega p\,q\,dx.$$

We note that $p \in L_0^2(\Omega)$ because of the homogeneous Dirichlet boundary condition. The bilinear forms $a(\cdot,\cdot) : \mathbf{V}\times\mathbf{V} \to \mathbf{R}$, $b(\cdot,\cdot) : \mathbf{V}\times L_0^2(\Omega) \to \mathbf{R}$ and $c(\cdot,\cdot) : L_0^2(\Omega)\times L_0^2(\Omega) \to \mathbf{R}$ satisfy the following **three conditions of well-posedness**, see Brezzi & Fortin (1991); Braess (2001).

1. The bilinear form $a(\cdot,\cdot)$ is continuous, symmetric and elliptic on $\mathbf{V}_0 := \{\mathbf{v} \in \mathbf{V} : b(\mathbf{v},q) = 0, q \in L_0^2(\Omega)\}$, i.e., there exists $\alpha_0 > 0$ such that
$$a(\mathbf{v},\mathbf{v}) \geqslant \alpha_0\|\mathbf{v}\|_1,\, \mathbf{v} \in \mathbf{V}_0.$$

2. The bilinear form $b(\cdot,\cdot)$ is continuous and satisfies the inf-sup condition, i.e., there exists $\beta_0 > 0$ such that
$$\inf_{q\in L_0^2(\Omega)} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1\|q\|_0} \geqslant \beta_0,\; \|\mathbf{v}\|_1 \neq 0,\, \|q\|_0 \neq 0.$$

3. The bilinear form $c(\cdot,\cdot)$ is continuous, symmetric and positive semi-definite, i.e.,

$$c(q,q) \geqslant 0,\ q \in L_0^2(\Omega).$$

Hence, the problem (2.7) has a unique solution which depends continuously on the right hand side, see (Brezzi & Fortin, 1991, Section II.1.2) and Braess (1996). We note that the bilinear form $a(\cdot,\cdot)$ is elliptic not only on $\mathbf{V}_0$ but also on the whole space $\mathbf{V}$.

3. Finite element discretizations

We consider a locally quasi-uniform triangulation $\mathscr{T}_h$ of the polygonal domain Ω, where $\mathscr{T}_h$ consists of simplices, either triangles or tetrahedra. We will use $\mathscr{N}_h$ to denote the set of all nodes or vertices of $\mathscr{T}_h$,

$$\mathscr{N}_h := \{i : i \text{ is a vertex of element } T \in \mathscr{T}_h\}, \text{ and } N := \#\mathscr{N}_h.$$

For $i \in \mathscr{N}_h$, x_i will denote the geometrical coordinates of the vertex i, and the vertex i will also be identified with its geometrical coordinates x_i. A dual mesh $\mathscr{T}_h^*$ is introduced based on the primal mesh $\mathscr{T}_h$ so that the elements of $\mathscr{T}_h^*$ are called control volumes. We introduce the dual mesh by a general scheme for a triangle. For each triangular element $T \in \mathscr{T}_h$ with vertices x_{T^1}, x_{T^2} and x_{T^3}, we select a point c_T inside T, and select three points $x_{T^{12}}$, $x_{T^{23}}$ and $x_{T^{31}}$ in each of the three edges of T. Connecting c_T to these points on edges by straight lines, a triangle is divided into three quadrilaterals Q_{T^i}, $1 \leqslant i \leqslant 3$, see the right picture of Figure 1. Each quadrilateral Q_{T^i} shares a vertex x_{T^i} of a triangle T, $1 \leqslant i \leqslant 3$, and hence corresponds to the vertex x_{T^i} of the triangle T. For each vertex $i \in \mathscr{N}_h$, we select a set of triangles $\mathscr{T}_i := \{T \in \mathscr{T}_h : T \text{ shares the vertex } i\}$ and form a set of quadrilaterals $\mathscr{Q}_i := \{Q : Q \text{ corresponds to the vertex } i \text{ of the triangle } T, T \in \mathscr{T}_i\}$. The control volume V_i corresponding to the vertex i is defined as $\overline{V}_i := \cup_{Q \in \mathscr{Q}_i} \overline{Q}$, and we call $\overline{\gamma}_{ij} = \overline{V}_i \cap \overline{V}_j$. We also use V_{T^i} to denote the control volume corresponding to the vertex x_{T^i} of the triangle T. The collection of these control volumes then defines a dual mesh, see Figure 1. In Figure 1, the empty circles denote the midpoints of the edges, whereas the filled circles denote the centroids. We point out that a similar construction can be followed for a tetrahedron T selecting four points each on four faces, four points on four edges and a point c_T inside T.

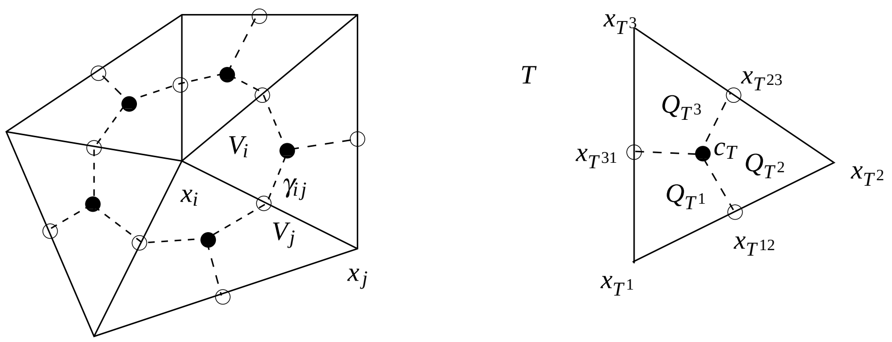

FIG. 1. Primal and dual meshes with vertices x_i and x_j, the interface γ_{ij} of V_i and V_j and the triangle T divided into three quadrilaterals in 2D case

We call the control volume mesh $\mathscr{T}_h^*$ locally regular or locally quasi-uniform if there exists a positive constant $C>0$ such that

$$Ch_i^2 \leqslant |V_i| \leqslant h_i^2,\, V_i \in \mathscr{T}_h^*,$$

where h_i is the maximum diameter of all elements $T \in \mathscr{T}_i$. The quantity h_i also denotes the local mesh-size at the i-th node of $\mathscr{T}_h$. Among many ways to introduce a regular dual mesh $\mathscr{T}_h^*$, depending on the choices of the point c_T in an element $T \in \mathscr{T}_h$ and the points on its edges or faces, we use a popular configuration in which c_T is chosen to be the barycenter or centroid of an element $T \in \mathscr{T}_h$, and the points on the edges or faces are chosen to be the midpoints of the edges or faces of T. It can be shown that, if $\mathscr{T}_h$ is locally regular, i.e., there is a constant C such that

$$Ch_T^2 \leqslant |T| \leqslant h_T^2,\, T \in \mathscr{T}_h$$

with $h_T = \mathrm{diam}(T)$ for all elements $T \in \mathscr{T}_h$, then this dual mesh $\mathscr{T}_h^*$ is also locally regular. Let S_h be the standard linear finite element space defined on the triangulation $\mathscr{T}_h$,

$$S_h := \{v \in H^1(\Omega) : v_{|_T} \in \mathscr{P}_1(T),\, T \in \mathscr{T}_h\},$$

and its dual volume element space S_h^*,

$$S_h^* := \{p \in L_0^2(\Omega) : p_{|_V} \in \mathscr{P}_0(V),\, V \in \mathscr{T}_h^*\}.$$

Let p_i be the nodal value of p_h at the vertex $i \in \mathscr{N}_h$, $1 \leqslant i \leqslant N$. Then, $p_h = \sum_{i=1}^N u_i \phi_i$ if $p_h \in S_h$ and $p_h = \sum_{i=1}^N p_i \chi_i$ if $p_h \in S_h^*$, where ϕ_i are the standard nodal basis functions associated with the node i, and χ_i are the characteristic functions of the volume V_i.

Defining the space of bubble functions

$$B_h := \{b_T \in H^1(T) : b_{T}|_{\partial T} = 0, \text{ and } \int_T b_T\, dx > 0,\, T \in \mathscr{T}_h\},$$

we introduce our finite element space for the displacement as $\mathbf{V}_h^B = [S_h \oplus B_h]^d$. In particular, we consider two types of bubble functions. The first one is the cubic bubble function defined on an element T with centroid c_T as $b_T(x) = c_b \Pi_{i=1}^{d+1} \lambda_{T^i}(x)$, where $\lambda_{T^i}(x)$ are the barycentric coordinates associated with the vertex x_{T^i}, and the constant c_b is defined in such a way that $b_T(c_T) = 1$. The second type of bubble function can be defined by partitioning T into $d+1$ elements by joining the centroid of the triangle or tetrahedron T with its vertices and defining a piecewise linear functions having zero boundary condition on ∂T, and satisfying $b_T(c_T) = 1$, see Girault & Raviart (1986); Brezzi & Fortin (1991); Quarteroni & Valli (1994).

Then, the finite element approximation of (2.7) is defined as a solution to the following problem: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^B \times S_h^*$ such that

$$\begin{array}{rclcrcl} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) & = & \ell(\mathbf{v}_h), & & \mathbf{v}_h & \in & \mathbf{V}_h^B, \\ b(\mathbf{u}_h, q_h) - \frac{1}{\lambda} c(p_h, q_h) & = & 0, & & q_h & \in & S_h^*. \end{array} \tag{3.1}$$

To establish a priori estimates for the discretization errors, we consider the saddle point formulation (3.1) of the elasticity problem and apply the theory of mixed finite elements. In particular, we have to check the **three conditions of well-posedness** given in the previous section. The continuity of the bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}_h^B \times \mathbf{V}_h^B$, of $b(\cdot,\cdot)$ on $\mathbf{V}_h^B \times S_h^*$ and of $c(\cdot,\cdot)$ on $S_h^* \times S_h^*$ is straightforward. By using the Korn's inequality, it is standard that the ellipticity of the bilinear form $a(\cdot,\cdot)$ holds on $\mathbf{V}_h^B \times \mathbf{V}_h^B$. It is also trivial to see that $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are symmetric, and $c(\cdot,\cdot)$ is positive semi-definite. It remains to show that the uniform inf-sup condition holds for the bilinear form $b(\cdot,\cdot)$ on $\mathbf{V}_h^B \times S_h^*$. That means, there exists a constant $\beta > 0$ independent of the mesh-size such that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h^B} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geqslant \beta \|q_h\|_0, \; q_h \in S_h^*. \tag{3.2}$$

In the first step, we show that the average nodal pressure formulation presented in Bonet & Burton (1998) does not satisfy the inf-sup condition making it essential to enrich the space of linear displacement $\mathbf{V}_h$ with bubble functions. The average nodal pressure formulation is obtained from (3.1) by replacing the space of displacement $\mathbf{V}_h^B$ with $\mathbf{V}_h := [S_h]^d$. We start with the P_1-P_1 finite element pair, which is well-known to be unstable (Girault & Raviart (1986); Brezzi & Fortin (1991); Quarteroni & Valli (1994)). Assume that $p_h \in S_h \cap L_0^2(\Omega)$ with $p_h = \sum_{i=1}^N p_i \phi_i$. We introduce an operator

$$I_h : S_h \cap L_0^2(\Omega) \to S_h^*$$

with

$$\int_T (p_h - I_h p_h)\, dx = 0, \; T \in \mathscr{T}_h.$$

If the dual mesh $\mathscr{T}_h^*$ is constructed by using the barycenters of triangles or tetrahedra and the mid-points on the edges or faces, the operator I_h is a standard interpolation operator with

$$I_h p_h = \sum_{i=1}^N p_i \chi_i,$$

see Ewing *et al.* (2002). Then, it is easy to see that there exist positive constants C_1 and C_2 with

$$C_1 \|p_h\|_0 \leqslant \|I_h p_h\|_0 \leqslant C_2 \|p_h\|_0.$$

Furthermore, we have

$$b(\mathbf{v}_h, p_h - I_h p_h) = \int_\Omega \operatorname{div} \mathbf{v}_h \, (p_h - I_h p_h)\, dx = \sum_{T \in \mathscr{T}_h} \int_T \operatorname{div} \mathbf{v}_h \, (p_h - I_h p_h)\, dx.$$

Using the fact that $\operatorname{div} \mathbf{v}_h$ is constant in each element $T \in \mathscr{T}_h$, we obtain

$$b(\mathbf{v}_h, p_h - I_h p_h) = \sum_{T \in \mathscr{T}_h} \operatorname{div} \mathbf{v}_{h|T} \int_T (p_h - I_h p_h)\, dx = 0.$$

Thus for any $p_h \in S_h \cap L_0^2(\Omega)$, we obtain $I_h p_h \in S_h^*$ so that $b(\mathbf{v}_h, p_h) = b(\mathbf{v}_h, I_h p_h)$. Hence, without enriching the displacement space, no uniform inf-sup condition can be shown.

In the next step, starting with the stable pair $(\mathbf{V}_h^B, S_h \cap L_0^2(\Omega))$, see Arnold *et al.* (1984), we show that the pair $(\mathbf{V}_h^B, S_h^*)$ also satisfies the uniform inf-sup condition. The crucial point of the analysis is the possibility to define an interpolation operator (Ewing *et al.* (2002))

$$I_h^* : S_h^* \to S_h \cap L_0^2(\Omega)$$

satisfying the following properties: given $\mathbf{v}_h \in \mathbf{V}_h^B$, we can construct an element $\mathbf{v}_h^* \in \mathbf{V}_h^B$ so that there exists positive constants C_1 and C_2 with

$$\|\mathbf{v}_h^*\|_1 \leqslant C_1\|\mathbf{v}_h\|_1, \quad b(\mathbf{v}_h^*, q_h) \geqslant C_2 b(\mathbf{v}_h, I_h^* q_h),\ q_h \in S_h^*, \tag{3.3}$$

and there exist positive constants c_1 and c_2 with

$$c_1\|q_h\|_0 \leqslant \|I_h^* q_h\|_0 \leqslant c_2\|q_h\|_0. \tag{3.4}$$

In the following, we use generic constants c and C. They do not depend on the mesh-size and do not degenerate when $\lambda \to \infty$.

THEOREM 3.1 Assume that the interpolation operator I_h^* satisfies the equations (3.3) and (3.4). Then the finite element pair $(\mathbf{V}_h^B, S_h^*)$ satisfies the inf-sup condition (3.2).

Proof. Let $p_h \in S_h^*$. With the definition of the interpolation operator, we have $I_h^* q_h \in S_h \cap L_0^2(\Omega)$. Since the pair $(\mathbf{V}_h^B, S_h \cap L_0^2(\Omega))$ satisfies the inf-sup condition, we can find an element $\mathbf{v}_h \in \mathbf{V}_h^B$ satisfying

$$b(\mathbf{v}_h, I_h^* q_h) \geqslant C\|I_h^* q_h\|_0^2, \text{ and } \|\mathbf{v}_h\|_1 \leqslant C\|I_h^* q_h\|_0.$$

Hence, using the properties (3.3) and (3.4) of the interpolation operator I_h^*, we can find an element $\mathbf{v}_h^* \in \mathbf{V}_h^B$ with

$$b(\mathbf{v}_h^*, q_h) \geqslant C b(\mathbf{v}_h, I_h^* q_h) \geqslant C\|I_h^* q_h\|_0^2 \geqslant C\|q_h\|_0^2,$$

and

$$\|\mathbf{v}_h^*\|_1 \leqslant C\|\mathbf{v}_h\|_1 \leqslant C\|I_h^* q_h\|_0 \leqslant C\|q_h\|_0.$$

Hence, the proof follows. □

Assume that $\hat{T}$ is either the reference triangle or reference tetrahedron, and $\hat{\theta}$ is a positive constant with $\hat{\theta} := \int_{\hat{T}} b_{\hat{T}}\, dx$. Suppose that $T \in \mathscr{T}_h$ is a triangle or tetrahedron with vertices $x_{T^i}, 1 \leqslant i \leqslant d+1$, where $T^i \in \mathscr{N}_h$, and V_{T^i} are the control volumes associated with the vertices x_{T^i}. It is easy to see that $\partial(T \cap V_{T^i})$ can be decomposed into $2d$-line segments in 2D and $2d$-polygonal domains in 3D, where d-line segments or d-polygonal domains lie inside the element T, and the rest lies on the boundary of T. We denote these d-line segments or d-polygonal domains lying inside the element T by $\gamma_{T^i}^j, 1 \leqslant j \leqslant d$.

LEMMA 3.1 If M_T is the Jacobian of transformation from a physical element T to the reference element $\hat{T}$, $J_T := \det(M_T)$ and $\hat{\theta}_{i,j} := \int_{\gamma_{\hat{T}^i}^j} b_{\hat{T}}\, d\sigma$, then we have for $1 \leqslant i \leqslant d+1$

$$\int_T \nabla b_T\, \phi_i\, dx = -\hat{\theta} J_T \nabla\hat{\phi}_i M_T^{-1}, \text{ and } \int_{T\cap V_{T^i}} \nabla b_T\, dx = J_T \sum_{j=1}^{d} \hat{\theta}_{i,j}\hat{\mathbf{n}}_j M_T^{-1},$$

where $\hat{\mathbf{n}}_j$ are the normals to the line segments or polygonal domains $\gamma_{\hat{T}^i}^j, 1 \leqslant j \leqslant d$, written as row vectors in $\mathbb{R}^d$, and $\hat{\phi}_i, 1 \leqslant i \leqslant d+1$, are the linear finite element basis functions corresponding to the reference element $\hat{T}$.

Proof. Observing

$$\int_T \nabla b_T\, \phi_i\, dx = \int_T \nabla(b_T\, \phi_i)\, dx - \int_T b_T\, \nabla\phi_i\, dx,$$

we use Gauss theorem and the fact that $b_T = 0$ on ∂T to obtain

$$\int_T \nabla b_T \, \phi_i \, dx = -\int_T b_T \, \nabla \phi_i \, dx.$$

Since $\nabla \phi_i$ is constant on T, we have

$$\int_T \nabla b_T \, \phi_i \, dx = -\nabla \phi_i \int_T b_T \; dx.$$

The first result follows by transforming the element T to the reference element $\hat{T}$. For the second result, we start with the transformation of T into $\hat{T}$, and apply the Gauss theorem to get

$$\int_{T \cap V_{T^i}} \nabla b_T \, dx = J_T \sum_{j=1}^{d} \int_{\gamma^j_{\hat{T}^i}} b_{\hat{T}} \, d\sigma \, \hat{\mathbf{n}}_j M_T^{-1}.$$

Finally, the second result follows by using the definition of $\hat{\theta}_{i,j}$. □

By directly computing the relevant quantities on the reference element, the following identity can easily be verified for both types of bubble functions.

$$-\nabla \hat{\phi}_i = \alpha \sum_{j=1}^{d} \hat{\theta}_{i,j} \hat{\mathbf{n}}_j, \tag{3.5}$$

where α is a constant depending on the bubble function.

LEMMA 3.2 Using Lemma 3.1 in combination with equation 3.5, we have

$$\int_T \nabla b_T \, \phi_i \, dx = \alpha \hat{\theta} \int_{T \cap V_{T^i}} \nabla b_T \, dx.$$

In the following, we have assumed that our bubble function satisfies the relation (3.5), which is verified for two types of bubble functions. We note that the result of Lemma 3.2 can be extended to any symmetric bubble function.

LEMMA 3.3 Assume that the dual mesh $\mathscr{T}_h^*$ is constructed by using the barycenters of triangles or tetrahedra and the mid-points on the edges or faces, and the bubble functions satisfy 3.5. Then, the interpolation operator $I_h^* : S_h^* \to S_h \cap L_0^2(\Omega)$ defined by

$$I_h^* q_h = \sum_{i=1}^{N} q_i \phi_i$$

satisfies the conditions (3.3) and (3.4), where $q_h = \sum_{i=1}^{N} q_i \chi_i \in S_h^*$.

Proof.

Here, it is easy to check that for any $q_h \in S_h^*$, we have

$$\int_T (q_h - I_h^* q_h) \, dx = 0, \; T \in \mathscr{T}_h$$

so that $\int_\Omega (q_h - I_h^* q_h) \, dx = 0$, and hence $I_h^* q_h \in L_0^2(\Omega)$. We start with the definition of the bilinear form $b(\cdot,\cdot)$ so that for $\mathbf{v}_h \in \mathbf{V}_h^B$ and $q_h \in S_h^*$, we have

$$b(\mathbf{v}_h, q_h - I_h^* q_h) = \int_\Omega \operatorname{div} \mathbf{v}_h \, (q_h - I_h^* q_h) \, dx = \sum_{T \in \mathscr{T}_h} \int_T \operatorname{div} \mathbf{v}_h \, (q_h - I_h^* q_h) \, dx.$$

Now, $\mathbf{v}_h = \mathbf{s}_h + \mathbf{b}_h$ with $\mathbf{s}_h \in S_h^d$, and $\mathbf{b}_h \in B_h^d$. With this decomposition of $\mathbf{v}_h$, we have

$$b(\mathbf{v}_h, q_h - I_h^* q_h) = \sum_{T \in \mathscr{T}_h} \int_T \operatorname{div} \mathbf{s}_h \, (q_h - I_h^* q_h) \, dx + \int_T \operatorname{div} \mathbf{b}_h \, (q_h - I_h^* q_h) \, dx.$$

Using the fact that $\operatorname{div} \mathbf{s}_h$ is constant in each element T, the first integral in the right side of the above equation is zero. Therefore, we obtain

$$b(\mathbf{v}_h, q_h - I_h^* q_h) = b(\mathbf{b}_h, q_h - I_h^* q_h).$$

Assuming $S_i := \operatorname{supp} \phi_i$, for $\mathbf{b}_h \in [B_h]^d$, we can write

$$b(\mathbf{b}_h, q_h) = \sum_{i=1}^{N} q_i \int_{V_i} \operatorname{div} \mathbf{b}_h \, dx, \quad \text{and} \quad b(\mathbf{b}_h, I_h^* q_h) = \sum_{i=1}^{N} q_i \int_{S_i} \operatorname{div} \mathbf{b}_h \, \phi_i \, dx.$$

Since $\mathbf{b}_h$ belongs to the space of bubble functions, restricted to an element T, it can be written as $\mathbf{b}_h = \mathbf{a}_T b_T$ for some constant vector $\mathbf{a}_T$. Now, we decompose the integrals inside both sums into each element

$$b(\mathbf{b}_h, q_h) = \sum_{i=1}^{N} q_i \sum_{T \subset S_i} \mathbf{a}_T \cdot \int_{T \cap V_{T^i}} \nabla b_T \, dx, \quad \text{and} \quad b(\mathbf{b}_h, I_h^* q_h) = \sum_{i=1}^{N} q_i \sum_{T \subset S_i} \mathbf{a}_T \cdot \int_T \nabla b_T \, \phi_i \, dx.$$

Defining $\mathbf{v}_h^* := \mathbf{s}_h + \alpha \hat{\theta} \mathbf{b}_h$ and using the result of Lemma 3.2, we get

$$b(\mathbf{v}_h^*, q_h) = b(\mathbf{v}_h, I_h^* q_h).$$

Since $\mathbf{s}_h$ and $\mathbf{b}_h$ are linearly independent, and $\alpha \hat{\theta}$ is a positive and fixed constant, it easily follows that $\|\mathbf{v}_h^*\|_1 \leqslant C \|\mathbf{v}_h\|_1$ proving the first condition (3.3). The second condition (3.4) follows easily by using the fact that $\|I_h^* q_h\|_0^2$, $\|q_h\|_0^2$ and $\sum_{i=1}^N q_i^2 h_i^2$ are equivalent, where h_i is the local mesh-size at the i-th node of $\mathscr{T}_h$. □

Hence, the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ satisfy all **three conditions of well-posedness** on the discrete subspaces $\mathbf{V}_h^B$ and S_h^* given in Section 2, and therefore, the discrete problem (3.1) is well-posed and the following theorem holds, see Brezzi & Fortin (1991); Braess (1996).

THEOREM 3.2 The discrete problem (3.1) has exactly one solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^B \times S_h^*$, which is uniformly stable with respect to the data $\mathbf{f}$, and there exists a constant C independent of Lamé parameter λ such that

$$\|\mathbf{u}_h\|_1 + \|p_h\|_0 \leqslant C \|\mathbf{f}\|_0.$$

The convergence theory is provided by an abstract result about the approximation of saddle point problems (Brezzi & Fortin, 1991, Section II.1.2).

THEOREM 3.3 Assume that $(\mathbf{u}, p)$ and $(\mathbf{u}_h, p_h)$ be the solutions of problems (2.7) and (3.1), respectively. Then, we have the following error estimate uniform with respect to λ:

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h^B} \|\mathbf{u} - \mathbf{v}_h\|_1 + \inf_{q_h \in S_h^*} \|p - q_h\|_0 \right). \quad (3.6)$$

Since the space $\mathbf{V}_h^B$ contains the space of piece-wise linear polynomials with respect to the primal mesh $\mathscr{T}_h$ and S_h^* contains the space of piece-wise constant functions with respect to the dual mesh $\mathscr{T}_h^*$, Theorem 3.3 yields the linear convergence of the energy norm of the error with respect to the mesh-size if $\mathbf{u} \in [H^2(\Omega)]^d$.

REMARK 3.1 We point out that the node-based uniform strain elements for simplices introduced in Dohrmann *et al.* (2000) can be recovered by defining the displacement field on the primal mesh and the stress field in the dual mesh in the Hellinger-Reissner problem of finding $(\mathbf{u}_h, \sigma_h) \in \mathbf{V}_h \times \mathbf{S}_h^*$ such that

$$\begin{array}{llll} \int_\Omega \mathscr{C}^{-1}\sigma_h : \tau_h \, dx & -\int_\Omega \varepsilon(\mathbf{u}_h) : \tau_h \, dx & = & 0, & \tau_h \in \mathbf{S}_h^*, \\ \int_\Omega \varepsilon(\mathbf{v}_h) : \sigma_h \, dx & & = & \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx, & \mathbf{v}_h \in \mathbf{V}_h, \end{array}$$

where $\mathbf{S}_h^* := [S_h^*]^{d\times d}$. However, the coercivity of the bilinear form does not hold uniformly in this case.

In the rest of this section, we briefly recall how the scheme can be implemented for a nearly incompressible elasticity problem in the displacement-based formulation. From the second equation of (3.1), we can write $p_h = \sum_{i=1}^N p_i \chi_i$ with

$$p_i = \frac{\lambda}{|V_i|} \int_{V_i} \nabla \cdot \mathbf{u}_h \, dx.$$

Hence, after condensing out the pressure from the formulation, we arrive at a problem of finding $\mathbf{u}_h \in \mathbf{V}_h^B$ so that

$$a(\mathbf{u}_h, \mathbf{v}_h) + \sum_{i=1}^N \frac{\lambda}{|V_i|} \left(\int_{V_i} \nabla \cdot \mathbf{u}_h \, dx \right) \left(\int_{V_i} \nabla \cdot \mathbf{v}_h \, dx \right) = \ell(\mathbf{v}_h), \; \mathbf{v}_h \in \mathbf{V}_h^B. \tag{3.7}$$

Since the solution of equation (3.7) is the same as the solution $\mathbf{u}_h$ of the saddle point problem (3.1), the discrete solution of (3.7) converges linearly to the exact solution in the energy norm independently of λ.

We point out that since the displacement is enriched with bubble functions $\operatorname{div} \mathbf{v}_h$ is not constant in each element. However, using the result of Lemma 3.2, we find that

$$\int_{V_i} \nabla \cdot \mathbf{b}_h \, dx = \sum_{T \subset S_i} \int_{T \cap V_i} \nabla \cdot \mathbf{b}_h \, dx = \sum_{T \subset S_i} \int_{T \cap V_i} \mathbf{a}_T \cdot \nabla b_T \, dx = -\frac{1}{\alpha} \sum_{T \subset S_i} \mathbf{a}_T \cdot \nabla \phi_i J_T.$$

Hence, there is no difficulty in computing the pressure-contribution of the bubble function. After eliminating the degree of freedom corresponding to the pressure variable, the static condensation of the bubble function becomes difficult during the assembling process. However, if the control volumes $|V_i|$, $1 \leqslant i \leqslant N$, are computed a priori, the coupling between the pressure and the bubble can be separated, and the degree of freedom corresponding to the bubble functions can be statically condensed out from the system during the assembling process.

REMARK 3.2 The scheme can also be implemented in the framework of saddle point problem as in the case of mini-element. In this case, the algebraic formulation of the problem can be written as

$$\begin{pmatrix} A & B^T \\ B & -\frac{1}{\lambda} C \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h \\ 0 \end{pmatrix}, \tag{3.8}$$

where A, B and C are matrices associated with the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$, respectively, and $\mathbf{f}_h$ is associated with the linear forms $\ell(\cdot)$. However, in contrast to mini-element, the matrix C is diagonal and the degree of freedom corresponding to the pressure can easily be condensed out from the system leading to a positive definite formulation.

4. Alternative approach based on dual bases

Here, we show that we can also work with the space constructed by dual basis functions instead of a dual mesh to obtain the diagonal structure of the matrix C in (3.8). We start with the space of the standard linear finite element functions S_h spanned by the basis $\{\phi_1, \cdots, \phi_n\}$ and construct a dual basis M_h spanned by the basis $\{\mu_1, \cdots, \mu_n\}$ so that the basis functions of S_h and M_h satisfy a condition of biorthogonality relation

$$\int_\Omega \mu_i \; \phi_j \, dx = c_j \delta_{ij}, \; c_j \neq 0, \; 1 \leqslant i, j \leqslant n, \tag{4.1}$$

where $n := \dim M_h = \dim S_h$, δ_{ij} is the Kronecker symbol, and c_j a scaling factor. This scaling factor c_j can be chosen as proportional to the area $|\mathrm{supp}\phi_j|$. It is easy to show that a local basis on the reference element $\hat{T}$ can be easily constructed so that the equation (4.1) holds. Explicitly, for the reference triangle $\hat{T} :== \{(x,y) : 0 \leqslant x, 0 \leqslant y, x+y \leqslant 1\}$, we have

$$\hat{\mu}_1 := 3 - 4x - 4y, \hat{\mu}_2 := 4x - 1, \text{ and } \hat{\mu}_3 := 4y - 1,$$

and for the reference tetrahedron $\hat{T} := \{(x,y,z) : 0 \leqslant x, 0 \leqslant y, 0 \leqslant z, x+y+z \leqslant 1\}$, we have

$$\hat{\mu}_1 := 4 - 5x - 5y - 5z, \hat{\mu}_2 := 5x - 1, \text{ and } \hat{\mu}_3 := 5y - 1, \hat{\mu}_4 := 5z - 1.$$

Then, the finite element approximation of (2.7) is defined as a solution to the following problem: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^B \times S_h \cap L_0^2(\Omega)$ such that

$$\begin{array}{rclcrcl} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) & = & \ell(\mathbf{v}_h), & & \mathbf{v}_h & \in & \mathbf{V}_h^B, \\ b(\mathbf{u}_h, q_h) - \frac{1}{\lambda} c(p_h, q_h) & = & 0, & & q_h & \in & M_h. \end{array} \tag{4.2}$$

We point out that the trial space is spanned by the continuous linear finite element basis functions, whereas the test functions are not continuous. However, the approach is conforming as $M_h \subset L^2(\Omega)$. Because of the biorthogonality relation (4.1) between the basis functions of S_h and M_h, the matrix C in (3.8) is diagonal, and therefore, static condensation of the pressure can be done as before. We note that the Petrov-Galerkin discretization is used for the pressure as the trial and test spaces are different resulting in a non-symmetric saddle point system. By recourse to the mini-element, it is possible to show that both the pairs $(\mathbf{V}_h^B, S_h)$ and $(\mathbf{V}_h^B, M_h)$ satisfy the inf-sup condition. Since $\sum_{i=1}^{d+1} \hat{\mu}_i = 1$, it is easy to prove that the space M_h contains all piece-wise constant functions with respect to the mesh $\mathscr{T}_h$, see also Kim *et al.* (2001). Hence, all spaces S_h, $\mathbf{V}_h^B$ and M_h have optimal approximation properties. Therefore, combining the ideas of Braess (1996) and Nicolaides (1982); Bernardi *et al.* (1988), stability of the discrete system (4.2) and the optimal approximation property of the discrete solution can be shown. Details of the mathematical analysis will be presented elsewhere.

5. Numerical Results

In this section, we illustrate the performance of the formulation discussed in the preceding sections in some numerical examples. In particular, we show the locking-free response in the incompressible limit of the proposed formulation by comparing the results with the analytical solution and the results obtained from the standard displacement approach and the mini-element formulation. We point out that the continuous pressure space is to be used in case of the mini-element, whereas we use discontinuous pressure space so that the degree of freedom corresponding to the pressure can be statically condensed out to obtain the system only based on the displacement. Assuming isotropy in all examples, the two-dimensional test examples are computed with plane strain assumption.

EXAMPLE 1: COOK'S MEMBRANE PROBLEM. This is a popular benchmark problem for the nearly incompressible elasticity, see Simo & Rifai (1990); Küssner & Reddy (2001); Kasper & Taylor (2000a). Let $\Omega := \text{conv}\{(0,0),(48,44),(48,60),(0,44)\}$, where conv$\xi$ is the convex hull of the set ξ. The left boundary of Ω is clamped, and the right one is subjected to an in-plane shearing load of 100N along the y-direction, as shown in the left picture of Figure 2. The material properties are taken to be $E = 250$ and $\nu = 0.4999$, so that a nearly incompressible response is obtained. The vertical tip displacements

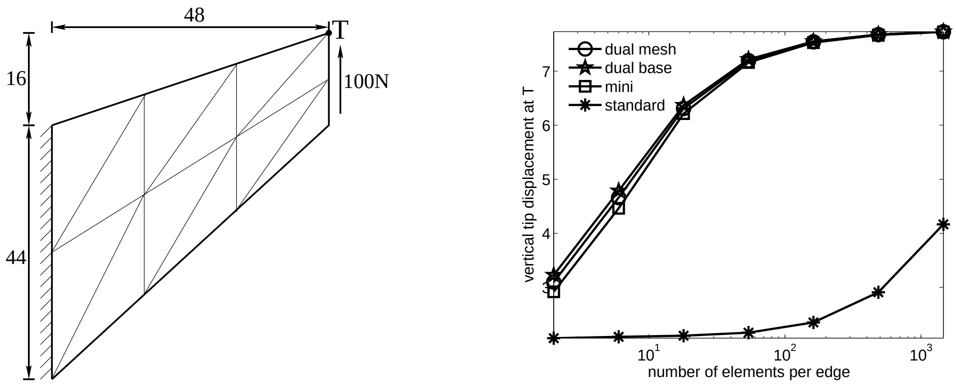

FIG. 2. Cook's membrane problem with initial triangulation (left) and the vertical tip displacement versus number of elements per edge

at the point T computed using different finite element formulations are presented in the right picture of Figure 2, for different levels of uniform refinement, starting with the initial triangulation shown in the left picture of Figure 2. As can be seen from the right picture of Figure 2, the standard displacement approach exhibits extreme locking whereas all other approaches show rapid convergence. We can see that both of our approaches perform equally well as the mini-element approach.

EXAMPLE 2: RECTANGULAR BEAM. In this second example, we consider a linear elastic beam of rectangular size subjected to a couple at one end, as shown in Figure 3. Along the edge $x = 0$, the horizontal displacement and vertical surface traction are zero. At the point $(0,0)$, the vertical displacement is also zero. The exact solution is given by

$$u(x,y) = \frac{2f(1-\nu^2)}{El}x\left(\frac{l}{2}-y\right), \text{ and } \quad v(x,y) = \frac{f(1-\nu^2)}{El}\left[x^2 + \frac{\nu}{1-\nu}y(y-l)\right].$$

We set $L = 10$, $l = 2$, $E = 1500$, $\nu = 0.4999$, and $f = 3000$. We have shown the setting of the problem in Figure 3, and the discretization errors with respect to the number of elements are presented in Figure 4. As can be seen from Figure 4, the standard approach locks completely, whereas we get very good numerical approximations with primal-dual approaches and mini-element.

EXAMPLE 3: THICK WALLED SPHERE UNDER INTERNAL PRESSURE. In this numerical example, the proposed formulation is tested for volumetric locking in the three-dimensional case. A thick walled sphere having internal radius $r_i = 7.5mm$ and external radius $r_e = 10mm$ subjected to a uniform internal pressure $P = 1N/mm^2$ is considered (Timoshenko & Goodier (1970); Kasper & Taylor (2000a)). The

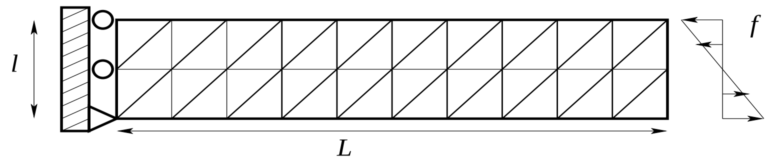

FIG. 3. The rectangular beam with initial mesh and problem setting

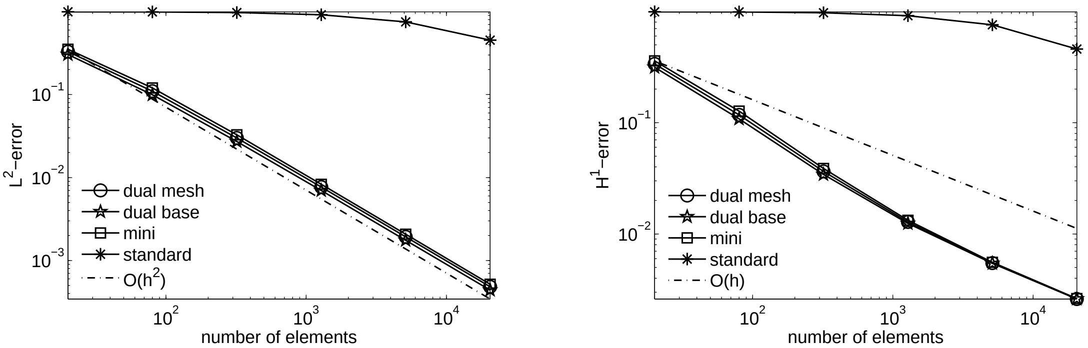

FIG. 4. Error plot versus number of elements, L^2-norm (left) and H^1-norm (right), rectangular beam

linear elastic material is assumed with modulus of elasticity $E = 250N/mm^2$, and different Poisson's ratios ν as shown in Table 1. Only one octant of the sphere is discretized as shown in the right picture of Figure 5 with symmetrical boundary conditions. The radial displacement (with respect to the analytical solution given in Timoshenko & Goodier (1970)) at point A is compared using different discretization schemes in Table 1. As expected, standard finite elements show locking in the incompressible range of ν, while all other formulations show a good behavior.

TABLE 1 *The radial displacement at A for thick walled sphere*

Poisson's ratio ν	dual mesh	dual base	mini	P_1	exact
0.490000	0.022404	0.022394	0.022425	0.024205	0.022276
0.499000	0.022014	0.022001	0.022038	0.028152	0.022407
0.499900	0.021975	0.021962	0.021999	0.024972	0.022421
0.499990	0.021971	0.021958	0.021995	0.013638	0.022422
0.499999	0.021971	0.021957	0.021995	0.004587	0.022422

6. Conclusion

We have presented two finite element approaches based on primal and dual meshes and primal and dual bases to solve the problem of nearly incompressible elasticity. Working with the space of linear finite elements enriched with bubble functions for displacements, we have shown that the uniform inf-sup condition holds for both cases. Since the degree of freedom for the pressure can be statically

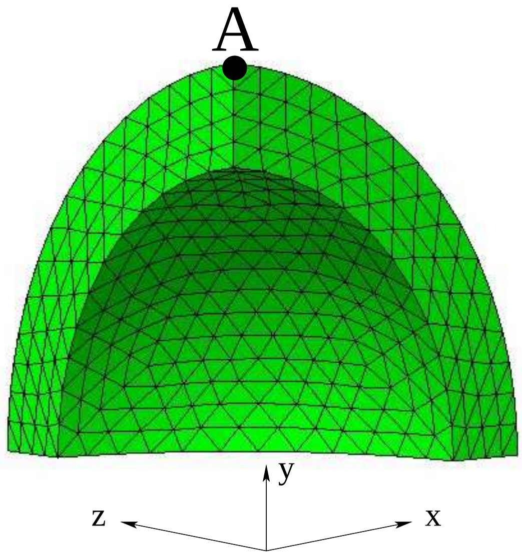

FIG. 5. Domain with triangulation for Example 3

condensed out from the system in both cases, the resulting scheme is based only on displacements. The numerical results show that both approaches perform equally well as the mini-element approach.

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