

# A Mixed Finite Element Method Based on a Biorthogonal System for Nearly Incompressible Elastic Problems

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## Abstract

We consider a non-symmetric saddle point problem having a particular structure. A mixed finite element method based on a Petrov-Galerkin formulation is analyzed within the framework of the non-symmetric saddle point problem. As a result, we obtain a robust and efficient numerical scheme for nearly incompressible linear elasticity using linear finite elements. A numerical example is provided to demonstrate the robustness of the approach.

*Key words: Mixed finite elements, elliptic problems, biorthogonal system*

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# 1 Introduction and abstract setting

It is well-known that low order finite elements based on standard displacement formulation exhibit a locking effect when applied to nearly incompressible problems [1, 2]. A standard remedy for this problem is to work with mixed methods introducing some extra variables leading to a saddle point problem [3, 4]. Here we consider a simple mixed formulation in displacement-pressure form [4], and apply a Petrov-Galerkin formulation for the pressure equation. The pressure equation is discretized using different trial and test spaces, where the bases of the trial and test spaces form a biorthogonal system. This results in a diagonal mass matrix for the pressure equation, and it is easy to statically condense out the pressure inverting a diagonal matrix. We have proposed this formulation in the recent work [5] as an alternative approach to the method based on primal and dual meshes. However, the mathematical analysis presented in that study covers only the case of primal and dual meshes. We note that the method based on primal and dual meshes results in a symmetric saddle point system, whereas the Petrov-Galerkin formulation for the pressure equation yields a non-symmetric saddle point system. The novel idea of this paper is to show the well-posedness of this formulation by combining the results on the non-symmetric saddle point problem [6, 7, 8] and the stability result of mini-element [9].

We start with an abstract setting. Let  $V$ ,  $W$ ,  $P$  and  $Q$  be Hilbert spaces with inner products  $(\cdot, \cdot)_V$ ,  $(\cdot, \cdot)_W$ ,  $(\cdot, \cdot)_P$  and  $(\cdot, \cdot)_Q$ , respectively. Let  $a(\cdot, \cdot) : V \times W \rightarrow \mathbb{R}$ ,  $b_1(\cdot, \cdot) : W \times P \rightarrow \mathbb{R}$ ,  $b_2(\cdot, \cdot) : V \times Q \rightarrow \mathbb{R}$ , and  $c(\cdot, \cdot) : P \times Q \rightarrow \mathbb{R}$  be bilinear forms. We consider a non-symmetric saddle point problem with penalty: given  $f \in W'$  and  $g \in Q'$ , find  $(u, p) \in V \times P$  so that

$$\begin{aligned} a(u, w) + b_1(w, p) &= f(w), & w &\in W, \\ b_2(u, q) - t c(p, q) &= g(q), & q &\in Q, \end{aligned} \tag{1}$$

where  $t$  is a positive small parameter, and  $W'$  and  $Q'$  denote the space of continuous linear functionals on  $W$  and  $Q$ , respectively. We are interested in analyzing the well-posedness of the problem (1) when  $t \rightarrow 0$ . There are a number of publications devoted to the analysis of this problem in different particular forms. For example, when  $b_1(\cdot, \cdot)$  and  $b_2(\cdot, \cdot)$  are the same, and  $V = W$  and  $P = Q$ , see references [3, 4]. The case with  $t = 0$  is analyzed by Nicolaides [6] and Bernardi et al. [7], whereas the case with  $V = W$ ,  $P = Q$  and  $t = 1$  is considered by Ciarlet et al. [8]. We show the stability of the problem (1) by combining the ideas presented by Ciarlet et al. [8] and Bernardi et al. [7]. To this end, we assume that the bilinear forms  $a(\cdot, \cdot)$ ,

$b_1(\cdot, \cdot)$ ,  $b_2(\cdot, \cdot)$  and  $c(\cdot, \cdot)$  satisfy

$$\begin{aligned} |a(v, w)| &\leq \mathbf{a} \|v\|_V \|w\|_W, & v \in V, w \in W, \\ |b_1(w, p)| &\leq \mathbf{b}_1 \|w\|_W \|p\|_P, & w \in W, p \in P, \\ |b_2(v, q)| &\leq \mathbf{b}_2 \|v\|_V \|q\|_Q, & v \in V, q \in Q, \\ |c(p, q)| &\leq \mathbf{c} \|p\|_P \|q\|_Q, & p \in P, q \in Q, \end{aligned} \quad (2)$$

where  $\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2$  and  $\mathbf{c}$  are continuity constants of the bilinear forms  $a(\cdot, \cdot)$ ,  $b_1(\cdot, \cdot)$ ,  $b_2(\cdot, \cdot)$  and  $c(\cdot, \cdot)$ , respectively, and  $\|\cdot\|_V$ ,  $\|\cdot\|_W$ ,  $\|\cdot\|_P$ , and  $\|\cdot\|_Q$  are induced inner products from the associated norms.

We define  $U_W \subset W$  and  $U_V \subset V$  as

$$\begin{aligned} U_W &:= \{w \in W : b_1(w, p) = 0, p \in P\}, \\ U_V &:= \{v \in V : b_2(v, q) = 0, q \in Q\}, \end{aligned}$$

and assume that

$$\begin{aligned} \sup_{w \in U_W} \frac{a(v, w)}{\|w\|_W} &\geq \alpha \|v\|_V, & v \in U_V, \\ \sup_{v \in U_V} a(v, w) &> 0, & w \in U_W \setminus \{0\}, \\ \sup_{w \in W} \frac{b_1(w, p)}{\|w\|_W} &\geq \beta_1 \|p\|_P, & p \in P, \\ \sup_{v \in V} \frac{b_2(v, q)}{\|v\|_V} &\geq \beta_2 \|q\|_Q, & q \in Q \end{aligned} \quad (3)$$

hold for some constants  $\alpha, \beta_1, \beta_2 > 0$ , where the supremum is taken only over the non-trivial elements of the underlying sets. In order to show the existence and uniqueness of the problem (1), we need the following theorem, which is proved by Nicolaides [6] and Bernardi et al. [7].

**Theorem 1.** *Let assumptions (2) and (3) be satisfied. Then for any  $f \in W'$  and  $g \in Q'$ , there exists a unique solution  $(u, p) \in V \times P$  to the saddle point problem of finding  $(u, p) \in V \times P$  so that*

$$\begin{aligned} a(u, w) + b_1(w, p) &= f(w), & w \in W, \\ b_2(u, q) &= g(q), & q \in Q, \end{aligned} \quad (4)$$

which satisfies the following stability estimates:

$$\|u\|_V \leq \beta_2^{-1} (1 + \alpha^{-1} \mathbf{a}) \|g\|_{Q'} + \alpha^{-1} \|f\|_{W'}, \quad \|p\|_P \leq \beta_1^{-1} (\|f\|_{W'} + \mathbf{a} \|u\|_V). \quad (5)$$

The existence and uniqueness of the problem (1) is then established by the following theorem. Although we use non-symmetric formulation in contrast to the work of Ciarlet et al. [8], the proof of this theorem is similar to that of [8, Theorem 3.2].

**Theorem 2.** *Let assumptions (2) and (3) be satisfied, and*

$$\delta := \beta_1^{-1}\beta_2^{-1}\mathbf{a}(1 + \alpha^{-1}\mathbf{a})t\mathbf{c} < 1. \quad (6)$$

*Then for any  $f \in V'$  and  $g \in Q'$ , there exists a unique solution  $(u, p) \in V \times P$  to the saddle point problem (1) satisfying the following stability estimates:*

$$\|p\|_P \leq \frac{1}{1 - \delta}\|\tilde{p}\|_P, \quad \|u\|_V \leq \|\tilde{u}\|_V + \frac{\beta_2(1 + \alpha^{-1}\mathbf{a})t\mathbf{c}}{1 - \delta}\|\tilde{p}\|_P, \quad (7)$$

*where  $(\tilde{u}, \tilde{p})$  is the solution to (4) and satisfies the bounds*

$$\|\tilde{u}\|_V \leq \beta_2^{-1}(1 + \alpha^{-1}\mathbf{a})\|g\|_{Q'} + \alpha^{-1}\|f\|_{W'}, \quad \|\tilde{p}\|_P \leq \beta_1^{-1}(\|f\|_{W'} + \mathbf{a}\|\tilde{u}\|_V).$$

*Proof.* Letting  $p_0 = 0 \in P$ , we define a sequence  $\{(u_n, p_n)\}$  for  $n \in \mathbb{N}$  by

$$\begin{aligned} a(u_{n+1}, w) + b_1(w, p_{n+1}) &= f(w), & w \in W \\ b_2(u_{n+1}, q) &= g(q) + tc(p_n, q), & q \in Q. \end{aligned} \quad (8)$$

The sequence is well-defined from Theorem 1, and for  $n \in \mathbb{N}$  we have

$$\begin{aligned} a(u_{n+1} - u_n, w) + b_1(w, p_{n+1} - p_n) &= 0, & w \in W \\ b_2(u_{n+1} - u_n, q) &= tc(p_n - p_{n-1}, q), & q \in Q. \end{aligned} \quad (9)$$

Theorem 1 yields the existence and uniqueness of the solution of (9) with the estimates

$$\begin{aligned} \|u_{n+1} - u_n\|_V &\leq \beta_2^{-1}(1 + \alpha^{-1}\mathbf{a})t\mathbf{c}\|p_n - p_{n-1}\|_P, \\ \|p_{n+1} - p_n\|_P &\leq \beta_1^{-1}\mathbf{a}\|u_{n+1} - u_n\|_V, \end{aligned} \quad (10)$$

and hence

$$\|p_{n+1} - p_n\|_P \leq \beta_1^{-1}\beta_2^{-1}\mathbf{a}(1 + \alpha^{-1}\mathbf{a})t\mathbf{c}\|p_n - p_{n-1}\|_P \leq \delta^n\|p_1\|_P. \quad (11)$$

Now taking  $n \in \mathbb{N}$  and an integer  $m > n$ , we have

$$\|p_m - p_n\|_P \leq \sum_{i=n}^{m-1} \|p_{i+1} - p_i\|_P \leq \sum_{i=n}^{m-1} \delta^i \|p_1\|_P \leq \frac{\delta^n}{1 - \delta} \|p_1\|_P, \quad (12)$$

which shows that  $\{p_n\}$  is a Cauchy sequence, and so converges to a  $p \in P$ . The stability estimate for  $p$  can be obtained by taking  $n = 0$  in (12). Using the first inequality of (10) and the estimate (11), the sequence  $\{u_n\}$  can be shown to be a Cauchy sequence, and stability estimate for  $u$  can be obtained similarly as for  $p$ . The uniqueness and other details can be worked out as in the work of Ciarlet et al. [8, Theorem 3.2].  $\square$

Our primary concern is to find a robust approximation scheme based on linear finite elements and simplicial triangulation for the nearly incompressible elastic problem. In particular, we are interested in a mixed scheme where the pressure variable can easily be eliminated and obtain a formulation based only on the displacement variable.

Let  $\Omega \subset \mathbb{R}^d$  with  $d \in \{2, 3\}$ ,  $L^2(\Omega)$  be the space of square-integrable functions defined on  $\Omega$  with the inner product and norm being denoted by  $(\cdot, \cdot)_0$  and  $\|\cdot\|_0$ , respectively, and  $L_0^2(\Omega) := \{p \in L^2(\Omega) : \int_{\Omega} p \, dx = 0\}$ . The space  $H_0^1(\Omega)$  consists of functions in  $H^1(\Omega)$  which vanish on the boundary in the sense of traces. To write the weak or variational formulation of the boundary value problem, we introduce the space  $\mathbf{V} := [H_0^1(\Omega)]^d$  of displacements with inner product  $(\cdot, \cdot)_1$  and norm  $\|\cdot\|_1$  defined in the standard way; that is,  $(\mathbf{u}, \mathbf{v})_1 := \sum_{i=1}^d (u_i, v_i)_1$ , with the norm being induced by this inner product. A mixed formulation of the linear elastic problem is given by introducing pressure as an extra variable leading to penalized Stokes equations [4]. Defining  $p := \lambda \operatorname{div} \mathbf{u}$ , a mixed variational formulation of linear elastic problem is: given  $\ell \in [L^2(\Omega)]^d$ , find  $(\mathbf{u}, p) \in \mathbf{V} \times L_0^2(\Omega)$  such that

$$\begin{aligned} A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) &= \ell(\mathbf{v}), & \mathbf{v} &\in \mathbf{V}, \\ B(\mathbf{u}, q) - \frac{1}{\lambda} C(p, q) &= 0, & q &\in L_0^2(\Omega), \end{aligned} \quad (13)$$

where  $A(\mathbf{u}, \mathbf{v}) := 2\mu \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx$ ,  $B(\mathbf{v}, q) := \int_{\Omega} \operatorname{div} \mathbf{v} \, q \, dx$ ,  $C(p, q) := \int_{\Omega} p \, q \, dx$ , and  $\ell(\mathbf{v}) := \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx$ . Here,  $\lambda$  and  $\mu$  are Lamé parameters,  $\boldsymbol{\varepsilon}(\mathbf{u})$  is the strain of the displacement defined as  $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + [\nabla \mathbf{u}]^T)$  and  $\mathbf{f}$  is the prescribed body force. The existence and uniqueness of the solution of the problem (13) can be shown by using a standard theory of saddle point problem [3, 4].

## 2 Finite element discretization

We consider a quasi-uniform triangulation  $\mathcal{T}_h$  of the polygonal or polyhedral domain  $\Omega$ , where  $\mathcal{T}_h$  consists of simplices, either triangles or tetrahedra. Making use of the standard linear finite element space  $S_h$  defined on the triangulation  $\mathcal{T}_h$

$$S_h := \{v \in H^1(\Omega) : v|_T \in \mathcal{P}_1(T), T \in \mathcal{T}_h\}$$

and the space of bubble functions

$$B_h := \{b_T \in H^1(T) : b_{T|_{\partial T}} = 0, \text{ and } \int_T b_T \, dx > 0, T \in \mathcal{T}_h\},$$

we introduce our finite element space for the displacement as  $\mathbf{V}_h = (S_h \oplus B_h)^d \cap \mathbf{V}$ . The bubble function on an element  $T$  can be defined as  $b_T(x) = c_b \prod_{i=1}^{d+1} \lambda_{T^i}(x)$ , where  $\lambda_{T^i}(x)$  are the barycentric coordinates of the element  $T$  associated with vertices  $x_{T^i}$  of  $T$ ,  $i = 1, \dots, d+1$ . The constant  $c_b$  is computed in such a way that the value of the bubble function at the barycenter of  $T$  is one. Let  $N$  be the number of nodes in the finite element mesh, and  $\{\phi_1, \dots, \phi_N\}$  be the finite element basis of  $S_h$ . Starting with the basis of  $S_h$ , we construct a dual space  $Q_h$  spanned by the basis  $\{\mu_1, \dots, \mu_N\}$  so that the basis functions of  $S_h$  and  $Q_h$  satisfy a condition of biorthogonality relation

$$\int_{\Omega} \mu_i \phi_j dx = c_j \delta_{ij}, \quad c_j \neq 0, \quad 1 \leq i, j \leq N, \quad (14)$$

where  $\delta_{ij}$  is the Kronecker symbol, and  $c_j$  a scaling factor, which can be chosen so that  $\int_T \mu_i dx = \int_T \phi_i dx$ . Hence, the sets of basis functions of  $S_h$  and  $Q_h$  form a biorthogonal system. The basis functions of  $Q_h$  can be constructed locally on the reference element  $\hat{T}$  so that basis functions of  $S_h$  and  $Q_h$  have the same support [5]. We also need a subspace of  $S_h$  and a subspace of  $Q_h$  having zero average on  $\Omega$  defined as

$$S_h^0 := \{v_h \in S_h : \int_{\Omega} v_h dx = 0\}, \quad Q_h^0 := \{q_h \in Q_h : \int_{\Omega} q_h dx = 0\}.$$

The first equation in (13) is discretized by using a Galerkin formulation, and the second equation is discretized by using a Petrov-Galerkin formulation. The Petrov-Galerkin formulation is chosen so that the pressure solution is taken from  $S_h$ , whereas the test functions are taken from  $Q_h$ . Hence the discrete formulation of variational equation (13) can be written as: find  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times S_h^0$  such that

$$\begin{aligned} A(\mathbf{u}_h, \mathbf{v}_h) + B_1(\mathbf{v}_h, p_h) &= \ell(\mathbf{v}_h), & \mathbf{v}_h &\in \mathbf{V}_h, \\ B_2(\mathbf{u}_h, q_h) - \frac{1}{\lambda} C(p_h, q_h) &= 0, & q_h &\in Q_h^0, \end{aligned} \quad (15)$$

where the bilinear forms  $B_1(\cdot, \cdot) : \mathbf{V}_h \times S_h^0 \rightarrow \mathbb{R}$  and  $B_2(\cdot, \cdot) : \mathbf{V}_h \times Q_h^0 \rightarrow \mathbb{R}$  have different domains of definition but are defined exactly as the bilinear form  $B(\cdot, \cdot)$  in (13). The goal of choosing the Petrov-Galerkin formulation for the pressure is to obtain a diagonal matrix corresponding the bilinear form  $C(\cdot, \cdot)$  so that the degree of freedom corresponding to the pressure variable can be eliminated easily.

We show the existence and uniqueness of the solution of the mixed formulation (15) using Theorem 2.

The continuity of the bilinear form  $A(\cdot, \cdot)$  on  $\mathbf{V}_h \times \mathbf{V}_h$ , of  $B_1(\cdot, \cdot)$  on  $\mathbf{V}_h \times S_h^0$ , and  $B_2(\cdot, \cdot)$  on  $\mathbf{V}_h \times Q_h^0$  and of  $C(\cdot, \cdot)$  on  $S_h^0 \times Q_h^0$  is straightforward. By

using the Korn's inequality, it is standard that the ellipticity of the bilinear form  $A(\cdot, \cdot)$  holds on  $\mathbf{V}_h \times \mathbf{V}_h$ . It remains to show that the uniform inf-sup condition holds for the bilinear form  $B_1(\cdot, \cdot)$  on  $\mathbf{V}_h \times S_h^0$ , and for the bilinear form  $B_2(\cdot, \cdot)$  on  $\mathbf{V}_h \times Q_h^0$ . As it is well-known that the bilinear form  $B_1(\cdot, \cdot)$  on  $\mathbf{V}_h \times S_h^0$  satisfies the inf-sup condition uniformly with respect to the mesh-size [9], we turn our attention to prove a uniform inf-sup condition for  $B_2(\cdot, \cdot)$  on  $\mathbf{V}_h \times Q_h^0$ . That means we have to show the existence of a constant  $\beta > 0$  independent of the mesh-size such that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{B_2(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq \beta \|q_h\|_0, \quad q_h \in Q_h^0. \quad (16)$$

To this end, we define an operator  $I_h : Q_h^0 \rightarrow S_h^0$  mapping every element  $\mu_h = \sum_{i=1}^n c_i \mu_i$  of  $Q_h^0$  to the element  $\phi_h = \sum_{i=1}^n c_i \phi_i$  of  $S_h^0$ . Using the fact that  $\int_T \phi_i dx = \int_T \mu_i dx$ ,  $1 \leq i \leq d+1$ , we have  $\int_T \phi_h dx = \int_T I_h \mu_h dx$ , and hence the operator is well-defined.

For the reference triangle or reference tetrahedron, using Gauss divergence theorem, we have the following identity for any bubble function  $b_T \in B_h$

$$\int_T \nabla b_T \mu_i dx = (d+2) \int_T \nabla b_T \phi_i dx. \quad (17)$$

Using (17), we can show the following properties of the operator  $I_h$ .

**Lemma 3.** *Given  $\mathbf{v}_h \in \mathbf{V}_h$ , we can construct an element  $\tilde{\mathbf{v}}_h \in \mathbf{V}_h$  so that there exists a constant  $c > 0$  with*

$$\|\tilde{\mathbf{v}}_h\|_1 \leq c \|\mathbf{v}_h\|_1, \quad B_2(\tilde{\mathbf{v}}_h, q_h) = B_2(\mathbf{v}_h, I_h q_h), \quad q_h \in Q_h^0, \quad (18)$$

and there exist two constants  $c_1 > 0$  and  $c_2 > 0$  with

$$c_1 \|q_h\|_0 \leq \|I_h q_h\|_0 \leq c_2 \|q_h\|_0. \quad (19)$$

*Proof.* We start with

$$B_2(\mathbf{v}_h, q_h - I_h q_h) = \int_{\Omega} \operatorname{div} \mathbf{v}_h (q_h - I_h q_h) dx = \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div} \mathbf{v}_h (q_h - I_h q_h) dx$$

for  $\mathbf{v}_h \in \mathbf{V}_h$  and  $q_h \in Q_h^0$ . Let  $\mathbf{v}_h = \mathbf{s}_h + \mathbf{b}_h$  with  $\mathbf{s}_h \in S_h^d$ , and  $\mathbf{b}_h \in B_h^d$ . Then

$$B_2(\mathbf{v}_h, q_h - I_h q_h) = \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div} \mathbf{s}_h (q_h - I_h q_h) dx + \int_T \operatorname{div} \mathbf{b}_h (q_h - I_h q_h) dx.$$

Using the fact that  $\operatorname{div} \mathbf{s}_h$  is constant in each element  $T$ , the first integral in the right side of the above equation is zero, and hence

$$B_2(\mathbf{v}_h, q_h - I_h q_h) = B_2(\mathbf{b}_h, q_h - I_h q_h).$$

As  $q_h \in Q_h^0$  can be written as  $q_h = \sum_{i=1}^N q_i \mu_i$ , we obtain

$$B_2(\mathbf{b}_h, q_h) = \sum_{i=1}^N q_i \int_{S_i} \operatorname{div} \mathbf{b}_h \mu_i dx, \quad \text{and} \quad B_2(\mathbf{b}_h, I_h q_h) = \sum_{i=1}^N q_i \int_{S_i} \operatorname{div} \mathbf{b}_h \phi_i dx,$$

where  $S_i$  denotes the support of  $\phi_i$  or  $\mu_i$ . Since  $\mathbf{b}_h$  belongs to the space of bubble functions, restricted to an element  $T$ , it can be written as  $\mathbf{b}_h = \mathbf{a}_T b_T$  for some constant vector  $\mathbf{a}_T$ . Now we decompose the integrals inside both sums into each element

$$\begin{aligned} B_2(\mathbf{b}_h, q_h) &= \sum_{i=1}^N q_i \sum_{T \subset S_i} \mathbf{a}_T \cdot \int_T \nabla b_T \mu_i dx, \\ B_2(\mathbf{b}_h, I_h q_h) &= \sum_{i=1}^N q_i \sum_{T \subset S_i} \mathbf{a}_T \cdot \int_T \nabla b_T \phi_i dx, \end{aligned}$$

and use the property of a bubble function (17) to get

$$B_2(\mathbf{b}_h, q_h) = (d+2)B_2(\mathbf{b}_h, I_h q_h).$$

Hence defining  $\tilde{\mathbf{v}}_h := \mathbf{s}_h + \frac{1}{d+2} \mathbf{b}_h$ , and noting that  $\mathbf{s}_h$  and  $\mathbf{b}_h$  are linearly independent the first condition (18) is proved. The second condition (19) follows easily by using the fact that  $\|I_h q_h\|_0^2$ ,  $\|q_h\|_0^2$  and  $\sum_{i=1}^N q_i^2 h_i^2$  are equivalent, where  $h_i$  is the local mesh-size at the  $i$ -th node of  $\mathcal{T}_h$ .  $\square$

An easy consequence of the above lemma is the following theorem.

**Theorem 4.** *The finite element pair  $(\mathbf{V}_h, Q_h^0)$  satisfies the inf-sup condition (16).*

*Proof.* Let  $q_h \in Q_h^0$ , and  $I_h q_h \in S_h^0$ . Since the pair  $(\mathbf{V}_h, S_h^0)$  satisfies the inf-sup condition, we can find an element  $\mathbf{v}_h \in \mathbf{V}_h$  satisfying

$$B_2(\mathbf{v}_h, I_h q_h) \geq c \|I_h q_h\|_0^2, \quad \text{and} \quad \|\mathbf{v}_h\|_1 \leq c \|I_h q_h\|_0.$$

Hence, using the properties (18) and (19) of the interpolation operator  $I_h$ , we can find an element  $\tilde{\mathbf{v}}_h \in \mathbf{V}_h$  with

$$B_2(\tilde{\mathbf{v}}_h, q_h) = B_2(\mathbf{v}_h, I_h q_h) \geq c \|I_h q_h\|_0^2 \geq c \|q_h\|_0^2,$$

and

$$\|\tilde{\mathbf{v}}_h\|_1 \leq c \|\mathbf{v}_h\|_1 \leq c \|I_h q_h\|_0 \leq c \|q_h\|_0.$$

$\square$

Thanks to Theorems 2 and 4, the following corollary holds, see references [6, 7, 3, 10].

**Corollary 5.** *The discrete problem (15) has exactly one solution  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times S_h^0$ , and there exists a constant  $c$  independent of Lamé parameter  $\lambda$  such that*

$$\|\mathbf{u}_h\|_1 + \|p_h\|_0 \leq c \|\mathbf{f}\|_0.$$

Furthermore, if  $(\mathbf{u}, p)$  is the solution to the problem (13), we have the following error estimate uniform with respect to  $\lambda$ :

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq c_1 \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_1 + c_2 \inf_{q_h \in S_h^0} \|p - q_h\|_0,$$

where the constants  $c_1$  and  $c_2$  are independent of the mesh-size.

Using the standard approximation properties of the spaces  $\mathbf{V}_h$  and  $S_h^0$ , it is easy to see that the approximation to the displacement converges to the exact solution with  $O(h)$  in  $H^1$ -norm.

### 3 Numerical results

In this section, we illustrate the performance of the formulation discussed in the preceding sections in a numerical example [11] showing a comparison of different formulations using  $L^2$ - and  $H^1$ -norms. Here, the exact solution  $\mathbf{u} = (u_1, u_2)$  is

$$\begin{aligned} u_1(x, y) &:= \sin(2\pi y) (-1 + \cos(2\pi x)) + \frac{\sin(\pi x) \sin(\pi y)}{1 + \lambda}, \\ u_2(x, y) &:= \sin(2\pi x) (1 - \cos(2\pi y)) + \frac{\sin(\pi x) \sin(\pi y)}{1 + \lambda} \end{aligned}$$

with  $\lambda = 2499666.644443238$  and  $\mu = 500.0333355557037$  (corresponding Poisson ratio and Young's modulus are  $\nu = 0.4999$  and  $E = 1500$ ) so that a nearly incompressible response is obtained. We compute the solution by taking  $\Omega$  as a unit square, where the right hand side and the Dirichlet boundary conditions are computed by using the exact solution. We have shown the discretization errors with respect to the number of elements in Figure 1. As can be seen from Figure 1, the standard approach locks completely, whereas we get very good numerical approximations with our approach and mini-element. However, we note that our formulation is more efficient as we can reduce the problem to the displacement-based formulation by inverting a diagonal matrix.

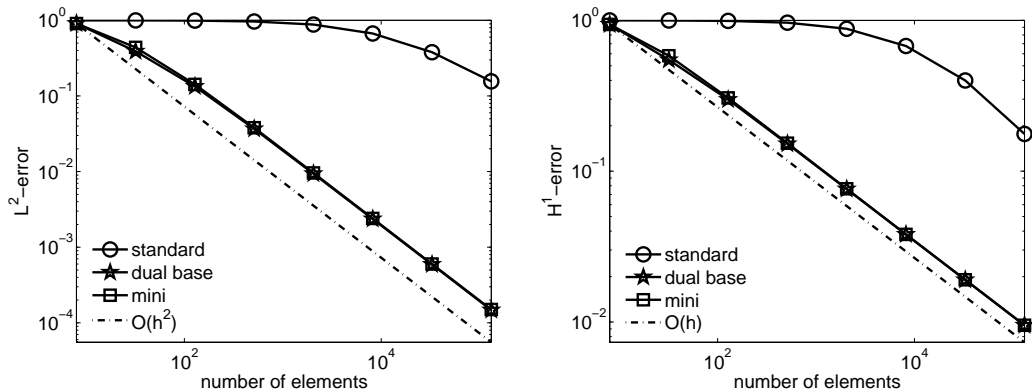


Figure 1: Error plot versus number of elements,  $L^2$ -norm (left) and  $H^1$ -norm (right)

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