SELFSIMILAR FRACTALS
AND
SELFSIMILAR RANDOM FRACTALS

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Abstract. We survey the application of contraction mapping argments to selfsimilar (nonrandom) fractal sets, measures and functions. We review the results for selfsimilar random fractal sets and measures and show how the method and extensions also work for selfsimilar random fractal functions.

1. Introduction

Contraction mapping methods for showing the existence and uniqueness of (nonrandom) selfsimilar fractal sets, measures and functions were first used in [Hut81]. In [Fal86] and [Gra87], contraction methods were used to obtain random selfsimilar fractal sets, and in [Ols94] to obtain random selfsimilar fractal measures, by essentially applying the nonrandom metrics to a.e. realisation in the random setting. In [HR98a, HR98b] we introduced new probability metrics for random measures which give natural and much stronger results, including the results of [Arb91] previously obtained by martingale techniques.

In this paper we review and extend these results. We also discuss the case of random selfsimilar fractal functions and indicate how Brownian motion and other stochastic processes with certain scaling properties can be included in the present framework. (Selfsimilar integral flat chains were developed in an analogous manner in [Hut81]; one could obtain similar results in the random setting by using the methods reviewed here.)

Of major importance is the structure of selfsimilar fractals. For this we refer the reader to the two books [Fal90, Fal97] and the references therein. For motivation and diagrams we refer to these two books, the elementary survey paper [Hut99], and the papers [Hut81] and [HR98a].

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2. Selfsimilar fractal sets

Definition 2.1. A *scaling law* $\mathbb{S}$ is an N-tuple $(S_1, \dots, S_N)$ $(N \geq 2)$ of Lipschitz maps $S_i : \mathbb{R}^n \to \mathbb{R}^n$. We denote the Lipschitz constants by $r_i = \operatorname{Lip} S_i$.

If $K \subset \mathbb{R}^n$ then $\mathbb{S}K \subset \mathbb{R}^n$ is defined by

$$\mathbb{S}K = \bigcup_i S_i(K).$$

We say $K \subset \mathbb{R}^n$ *satisfies the scaling law* $\mathbb{S}$, or is a *selfsimilar fractal set*, if

$$K = \mathbb{S}K.$$

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In future we make the *convention* that all compact sets are nonempty.

The following was essentially proved in [Hut81].

Theorem 1. *If $\mathbb{S}$ is a scaling law with $r = \max r_i < 1$ then there is a unique compact $K^* \subset \mathbb{R}^n$ which satisfies $\mathbb{S}$. Moreover, for any compact $K_0 \subset \mathbb{R}^n$,*

$$d_{\mathcal{H}}(\mathbb{S}^k K_0, K^*) \leq \frac{r^k}{1-r} d_{\mathcal{H}}(K_0, \mathbb{S}K_0) \to 0$$

as $k \to \infty$, where $d_{\mathcal{H}}$ is the Hausdorff metric.

Proof. One checks that

$$d_{\mathcal{H}}(\mathbb{S}A, \mathbb{S}B) \leq r d_{\mathcal{H}}(A, B),$$

and so $\mathbb{S}$ is a contraction map on the complete metric space $\mathcal{C}$ of (nonempty) compact subsets of $\mathbb{R}^n$ endowed with the Hausdorff metric. The result follows. □

One can obtain a random version of the above.

For this let $(\Omega, \mathcal{A}, \Sigma)$ be the underlying probability space. If X is a random variable (set, measure, etc.), we denote by $\operatorname{dist} X$ the corresponding probability distribution on $\mathbb{R}$ (sets, measures, etc.). By $\overset{d}{=}$ we denote equality at the probability distribution level.

Definition 2.2. A *random scaling law* $\mathbb{S} = (S_1, \ldots, S_N)$ is a random variable whose values are scaling laws. We write $\mathcal{S} = \operatorname{dist} \mathbb{S}$ for the probability distribution determined by $\mathbb{S}$.

If K is a random set, then the random set $\mathbb{S}K$ is defined (up to probability distribution) by

$$\mathbb{S}K = \bigcup_i S_i K^{(i)},$$

where $\mathbb{S} = (S_1, \ldots, S_N)$, $K^{(1)}, \ldots, K^{(N)}$ are independent of one another and $K^{(i)} \overset{d}{=} K$. If $\mathcal{K} = \operatorname{dist} K$ is the probabiltiy distribution on sets determined by K, we define

$$\mathcal{S}\mathcal{K} = \operatorname{dist} \mathbb{S}K.$$

We say K *(or more precisely $\mathcal{K}$) satisfies the scaling law* $\mathbb{S}$, or is a *selfsimilar random fractal set*, if

$$\mathbb{S}K \overset{d}{=} K, \quad \text{or equivalently } \mathcal{S}\mathcal{K} = \mathcal{K}.$$

We remark that in the previous definition the random maps $S_1, \ldots, S_N$ are normally *not* independent of one another.

One generates random sets in the following manner, c.f. [HR98a].

Definition 2.3. Beginning with a (nonrandom) set K_0 one defines a sequence of random sets

$$\begin{aligned}
\mathbb{S}K_0 &= \bigcup_i S_i\, K_0, \\
\mathbb{S}^2 K_0 &= \bigcup_{i,j} S_i \circ S^i_j\, K_0, \\
\mathbb{S}^3 K_0 &= \bigcup_{i,j,k} S_i \circ S^i_j \circ S^{ij}_k\, K_0,
\end{aligned}$$

etc.; where $\mathbb{S}^i = (S^i_1, \ldots, S^i_N)$ for $i = 1, \ldots, N$ are independent of each other and of $\mathbb{S}$, the $\mathbb{S}^{ij} = (S^{ij}_1, \ldots, S^{ij}_N)$ for $i, j = 1, \ldots, N$ are independent of each other and of $\mathbb{S}$ and the $\mathbb{S}^i$, etc.

Definition 2.4. Let $\mathbb{C}$ be the set of random compact sets K such that

$$\operatorname*{ess\,sup}_{\omega} d_{\mathcal{H}}(K^{\omega}, \delta_B^{\omega}) < \infty,$$

for some, and hence any, fixed compact set $B \subset \mathbb{R}^n$. By δ_B we mean the random set equal a.s. to B.

Let $\mathcal{C}$ be the set of probability distributions of members of $\mathbb{C}$, i.e.

$$\mathcal{C} = \{ \operatorname{dist} K \mid K \in \mathbb{C} \}.$$

Our goal is to show that under natural conditions there is a unique random compact set $K^* \in \mathbb{C}$ (up to probability distribution level) which satisfies $\mathbb{S}$, and that for any initial (nonrandom) compact K_0 one has $\mathbb{S}^k K_0 \to K^*$ a.s. in the Hausdorff metric. In order to establish the a.s. convergence, we need the following natural probability space for the iteration construction.

Definition 2.5. The N-fold tree of all finite sequences from $\{1, \ldots, N\}$, including the empty sequence $\emptyset$, is denoted by $C = C_N$.

A *construction tree* (or *construction process*) is a map $\omega : C \to \Upsilon$, where Υ is the set of (nonrandom) scaling laws. The *sample space* of all construction trees is denoted by $\widetilde{\Omega} = \{ \omega \mid \omega : C \to \Upsilon \}$.

The underlying *probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\Sigma})$ for the iteration procedure is generated by selecting iid (independent and identically distributed) scaling laws $\omega(\sigma) \overset{d}{=} \mathbb{S}$ for each $\sigma \in C$.

The following result is due to Falconer [Fal86] and Graf [Gra87].

Theorem 2. *If $\mathbb{S} = (S_1, \ldots, S_N)$ is a random scaling law with $\lambda := \operatorname{ess\,sup}_\omega r^\omega < 1$ (where $r^\omega = \max_i r_i^\omega = \max_i \operatorname{Lip} S_i^\omega$), then for any (nonrandom) compact set K_0,*

$$\operatorname{ess\,sup} d_{\mathcal{H}}(\mathbb{S}^k K_0, K^*) \le \frac{\lambda^k}{1-\lambda} \operatorname{ess\,sup} d_{\mathcal{H}}(K_0, \mathbb{S}K_0) \to 0$$

as $k \to \infty$, where K^* does not depend on K_0. In particular, $\mathbb{S}^k K_0 \to K^*$ a.s.

Moreover, up to probability distribution, K^* is the unique random compact set which satisfies $\mathbb{S}$.

Remarks. The probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\Sigma})$ is required in order to establish the a.s. convergence result. On the other hand, the existence and uniqueness of $\operatorname{dist} K^*$ does not depend on the choice of the underlying probability space $(\Omega, \mathcal{A}, \Sigma)$.

One could begin with a *random* compact set K_0, but the limit random set K^* will no longer be independent of K_0 (although it will still be independent up to distribution). The argument is similar, except that one works in a probability space which has $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\Sigma})$ as a factor.

Proof. Take $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\Sigma})$ as the underlying probability space.

Define

$$d^*_{\mathcal{H}}(E, F) = \operatorname*{ess\,sup}_{\omega} d_{\mathcal{H}}(E^\omega, F^\omega)$$

for $E, F \in \mathbb{C}$. Then $(\mathbb{C}, d^*_{\mathcal{H}})$ is a complete metric space.

Define $\mathbb{S} : \mathbb{C} \to \mathbb{C}$ by

$$\mathbb{S}K\,(\omega) = \bigcup_i S_i(\omega)\, K^{(i)}(\omega), \tag{2.1}$$

using notation we now explain. (This will be a definition at the random set level, not just at the probability distribution level.)

First recall that we are working with the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\Sigma})$. The "top node" $\omega(\emptyset)$ of each $\omega \in \widetilde{\Omega}$ is a scaling law which we denote by $(S_1(\omega), \ldots, S_N(\omega))$.

By $K^{(i)}$ we mean that member of $\mathbb{C}$ defined by $K^{(i)}(\omega) = K(\omega^{(i)})$ where $\omega^{(i)}$ is ω shifted by i, i.e. $\omega^{(i)}(\sigma) = \omega(i * \sigma)$ for $\sigma \in C_N$ with $*$ denoting concatenation. Loosely speaking, one can think of $K^{(i)}$ as being the random set determined by K and by the ith main branch $\omega^{(i)}$ of ω.

Note that by construction the scaling law $(S_1, \dots, S_N)$ and the random sets $K^{(1)}, \dots, K^{(N)}$ are independent of each other, $(S_1, \dots, S_N) \stackrel{d}{=} \mathbb{S}$ and $K^{(i)} \stackrel{d}{=} K$ for each i. Thus the above definition of $\mathbb{S}K$ is consistent with Definition 2.2.

Moreover, one can see that $\mathbb{S}^k K_0 := \overbrace{\mathbb{S} \circ \cdots \circ \mathbb{S}}^{k} K_0$ agrees with Definition 2.3.

One easily checks that $\mathbb{S}$ is a contraction map on $\mathbb{C}$ with contraction ratio λ. In fact

$$\begin{aligned} d^*_{\mathcal{H}}(\mathbb{S}E, \mathbb{S}F) &= \operatorname*{ess\,sup}_{\omega} d_{\mathcal{H}}\Big(\bigcup_i S_i(\omega)E^{(i)}(\omega), \bigcup_i S_i(\omega)F^{(i)}(\omega)\Big) \\ &\le \operatorname*{ess\,sup}_{\omega}\Big(r^{\omega} \max_i d_{\mathcal{H}}\big(E^{(i)}(\omega), F^{(i)}(\omega)\big)\Big) \\ &\le \lambda \max_i \operatorname*{ess\,sup}_{\omega} d_{\mathcal{H}}\big(E^{(i)}(\omega), F^{(i)}(\omega)\big) \\ &= \lambda d^*_{\mathcal{H}}(E, F), \end{aligned}$$

where the last step comes from the fact $E^{(i)} \stackrel{d}{=} E$. This gives the claims in the first paragraph of the theorem.

Since $\mathbb{S}K^* = K^*$, where $\mathbb{S}K^*$ is defined as in (2.1), we see by taking the distribution of each side that $\operatorname{dist} K^*$ satisfies $\mathbb{S}$ in the sense of Definition 2.2. We next show the uniqueness of a probability distribution satisfying $\mathbb{S}$, regardless of the underlying probability space.

For this define

$$d^{**}_{\mathcal{H}}(\mathcal{E}, \mathcal{F}) = \inf\{\, d^*_{\mathcal{H}}(E, F) \mid E \stackrel{d}{=} \mathcal{E}, F \stackrel{d}{=} \mathcal{F} \,\}.$$

It is straightforward to check that $(\mathcal{C}, d^{**}_{\mathcal{H}})$ is a complete metric space, and that $\mathcal{S} : \mathcal{C} \to \mathcal{C}$, where the operator $\mathcal{S}$ is defined in Definition 2.2.

We claim

$$d^{**}_{\mathcal{H}}(\mathcal{S}\mathcal{E}, \mathcal{S}\mathcal{F}) \le \lambda d^{**}_{\mathcal{H}}(\mathcal{E}, \mathcal{F})$$

and so $\mathcal{S}$ is a contraction map. To see this, choose $E_i \stackrel{d}{=} \mathcal{E}$, $F_i \stackrel{d}{=} \mathcal{F}$ for $i = 1, \dots, N$, such that the pairs (E_i, F_i) are independent of one another and such that $d^{**}_{\mathcal{H}}(\mathcal{E}, \mathcal{F}) = d^*_{\mathcal{H}}(E_i, F_i)$. Choose $(S_1, \dots, S_N) \stackrel{d}{=} \mathcal{S}$ independent of the (E_i, F_i). The proof of the above inequality is now similar to that for $d^*_{\mathcal{H}}$. This establishes the rest of the theorem. □

3. Selfsimilar fractal measures

It is usually more convenient to work with measures rather than sets. This leads to more useful metrics, and for applications such as to image compression it is convenient to consider "grey-scales".

Extensions of the results in this section, and further details of proofs, can be found in [HR98a, HR98b]

We first extend Definition 2.1.

Definition 3.1. A *scaling law* $\mathbb{S}$ (with weights) is a $2N$-tuple $(p_1, S_1, \dots, p_n, S_N)$ of positive real numbers p_i such that $\sum p_i = 1$, and of Lipschitz maps S_i as before.

If μ is a Radon measure on $\mathbb{R}^n$, then the measure $\mathbb{S}\mu$ is defined by

$$\mathbb{S}\mu = \sum_{i=1}^{N} p_i S_i \mu,$$

where $S_i\mu$ is the usual pushforward measure. We say μ satisfies the scaling law $\mathbb{S}$, or is a *selfsimilar fractal measure*, if

$$\mathbb{S}\mu = \mu.$$

Definition 3.2. For $p > 0$ let

$$M_p = \left\{ \mu \;\middle|\; \mu \text{ is a measure on } \mathbb{R}^n, \mu(\mathbb{R}^n) = 1, \int |x|^p \, d\mu < \infty \right\}.$$

The *minimal metric* ℓ_p on M_p is the complete metric defined by

$$\begin{aligned} \ell_p(\mu, \nu) &= \inf\left\{ (E|X - Y|^p)^{\frac{1}{p}\wedge 1} \;\middle|\; X \overset{d}{=} \mu,\, Y \overset{d}{=} \nu \right\} \\ &= \inf\left\{ \left(\int |x - y|^p \, d\gamma(x, y) \right)^{\frac{1}{p}\wedge 1} \;\middle|\; \pi_1\gamma = \mu,\ \pi_2\gamma = \nu \right\}, \end{aligned}$$

where $\wedge$ denotes the minimum of the relevant numbers and $\pi_i\gamma$ denotes the i-th marginal of γ, i.e. projection of the measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ onto the i-th component.

It will be convenient to extend the second form of the definition to $\ell_p(\mu, \nu)$ in case μ and ν have equal masses other than one.

Suppose α is a positive real, $S : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz, and $\vee$ denotes the maximum of the relevant numbers. Then for $p > 0$

$$\begin{aligned} \ell_p^{p\vee 1}(\alpha\mu, \alpha\nu) &= \alpha \ell_p^{p\vee 1}(\mu, \nu), \\ \ell_p^{p\vee 1}(\mu_1 + \mu_2, \nu_1 + \nu_2) &\le \ell_p^{p\vee 1}(\mu_1, \nu_1) + \ell_p^{p\vee 1}(\mu_2, \nu_2), \\ \ell_p(S\mu, S\nu) &\le (\mathrm{Lip}\, S)^{p\wedge 1} \ell_p(\mu, \nu). \end{aligned}$$

The first follows from the definition by setting $\gamma = c\widetilde{\gamma}$ where $\widetilde{\gamma}$ is optimal for (μ, ν), and the third by setting $\gamma = S\widetilde{\gamma}$. The second follows by setting $\gamma = \gamma_1 + \gamma_2$ where γ_i is optimal for (μ_i, ν_i), and by also noting $(a + b)^p \le a^p + b^p$ if $a, b \ge 0$ and $0 < p < 1$.

The following theorem was proved in [Hut81] in case $p = 1$ by using the Monge-Kantorovich metric, and essentially in [BDEG88, BDEG89] in general by using Markov process arguments. The simple argument here gives additional information concerning the rate of convergence.

Theorem 3. *If $\mathbb{S} = (p_1, S_1, \ldots, p_n, S_N)$ is a scaling law with $\lambda_p := \sum_i p_i r_i^p < 1$ for some $p > 0$ then there is a unique measure $\mu^* \in M_p$ which satisfies $\mathbb{S}$. Moreover, for any $\mu_0 \in M_p$,*

$$\ell_p(\mathbb{S}^k \mu_0, \mu^*) \le \frac{\lambda_p^{k(\frac{1}{p}\wedge 1)}}{1 - \lambda_p^{\frac{1}{p}\wedge 1}} \, \ell_p(\mu_0, \mathbb{S}\mu_0) \to 0$$

as $k \to \infty$.

Proof. It is straightforward to check that $\mathbb{S} : M_p \to M_p$. Moreover,

$$\begin{aligned} \ell_p^{p\vee 1}(\mathbb{S}\mu, \mathbb{S}\nu) &= \ell_p^{p\vee 1}\Big(\sum_i p_i S_i \mu, \sum_i p_i S_i \nu \Big) \\ &\le \sum_i p_i \ell_p^{p\vee 1}(S_i\mu, S_i\nu) \\ &\le \sum_i p_i r_i^p \ell_p^{p\vee 1}(\mu, \nu) \end{aligned}$$

from the properties of ℓ_p. Hence $\mathbb{S}$ is a contraction map with contraction constant $\lambda_p^{\frac{1}{p}\wedge 1}$. This implies the theorem. $\square$

Remarks. Since $\left(\int |x|^p\, d\mu\right)^{1/p} \to \exp \int \log |x|\, d\mu$ as $p \downarrow 0$, it follows that

$$M_0 := \bigcup_{p>0} M_p = \left\{ \mu \;\middle|\; \mu(\mathbb{R}^n) = 1, \int \log |x|\, d\mu < \infty \right\}.$$

Since $\lambda_p^{1/p} \downarrow \prod_i r_i^{p_i}$ as $p \downarrow 0$, it follows that if $\prod_i r_i^{p_i} < 1$ (i.e. $\sum_i p_i \log r_i < 0$), then there is a unique measure $\mu^* \in M_0$ which satisfies $\mathbb{S}$. Moreover, for any $\mu_0 \in M_0$, $\mathbb{S}^k \mu_0 \to \mu^*$ in the weak sense of measures as $k \to \infty$.

It also follows that the μ^* in the theorem is unique in the larger class M_0.

Since $\lambda_p^{1/p} \uparrow \max_i r_i$ as $p \uparrow \infty$, we can regard Theorem 1 as a limit case of Theorem 3. More precisely, if $\max_i r_i < 1$ in Theorem 3 then $\operatorname{spt} \mu^*$ is compact and is the unique compact set satisfying $(S_1, \dots, S_N)$. Moreover, if $\operatorname{spt} \mu_0$ is compact then $\operatorname{spt} \mathbb{S}^k \mu_0 \to \operatorname{spt} \mu^*$ in the Hausdorff metric sense.

Analogous to Definition 2.2, one has the following.

Definition 3.3. A *random scaling law* $\mathbf{S} = (p_1, S_1, \dots, p_n, S_N)$ is a random variable whose values are scaling laws. We write $\mathcal{S} = \operatorname{dist} \mathbb{S}$ for the probability distribution determined by $\mathbb{S}$.

If μ is a random measure, then the random measure $\mathbb{S}\mu$ is defined (up to probability distribution) by

$$\mathbb{S}\mu = \sum_i p_i S_i \mu^{(i)},$$

where $\mathbb{S}, \mu^{(1)}, \dots, \mu^{(N)}$ are independent of one another and $\mu^{(i)} \overset{d}{=} \mu$. If $\mathcal{P} = \operatorname{dist} \mu$ we define

$$\mathcal{S}\mathcal{P} = \operatorname{dist} \mathbb{S}\mu.$$

We say μ *(or more precisely $\mathcal{P}$) satisfies the scaling law* $\mathbb{S}$, or is a *selfsimilar random fractal measure*, if

$$\mathbb{S}\mu \overset{d}{=} \mu, \quad \text{or equivalently } \mathcal{S}\mathcal{P} = \mathcal{P}.$$

One can now proceed in a manner analogous to that for random sets.

Fix $p > 0$. Beginning from any $\mu_0 \in M_p$ one defines a sequence of random measures

$$\begin{aligned}
\mathbb{S}\mu_0 &= \sum_i p_i S_i\, \mu_0, \\
\mathbb{S}^2\mu_0 &= \sum_{i,j} p_i p_j^i S_i \circ S_j^i\, \mu_0, \\
\mathbb{S}^3\mu_0 &= \sum_{i,j,k} p_i p_j^i p_k^{ij} S_i \circ S_j^i \circ S_k^{ij}\, \mu_0,
\end{aligned}$$

etc.; with independencies analogous to those in Definition 2.3.

The following analogue of Theorem 2 extends results of Arbeiter [Arb91] established with the use of martingale theory, c.f. [HR98b].

Theorem 4. *If $\mathbb{S} = (p_1, S_1, \dots, p_N, S_N)$ is a random scaling law which satisfies $\lambda_p := E \sum p_i r_i^p < 1$ and $E \sum p_i |S_i(0)|^p < \infty$ for some $p > 0$, then for any $\mu_0 \in M_p$,*

$$\mathbb{E}^{\frac{1}{p}} \ell_p^p(\mathbb{S}^k \mu_0, \mu^*) \le \frac{\lambda_p^{\frac{k}{p}}}{1 - \lambda_p^{\frac{1}{p}}}\, \mathbb{E}^{\frac{1}{p}} \ell_p^p(\mu_0, \mathbb{S}\mu_0) \to 0 \qquad p \ge 1$$

$$\mathbb{E}\ell_p(\mathbb{S}^k\mu_0,\mu^*) \le \frac{\lambda_p^k}{1-\lambda_p}\,\mathbb{E}\ell_p(\mu_0,\mathbb{S}\mu_0)\to 0 \qquad\qquad 0<p<1$$

as $k\to\infty$, where μ^* does not depend on μ_0. In particular, $\mathbb{S}^k\mu_0\to\mu^*$ a.s. in the sense of weak convergence of measures.

Moreover, up to probability distribution, μ^* is the unique unit mass random measure with $\mathbb{E}_\omega\int\log|x|\,d\mu^\omega<\infty$ which satisfies $\mathbb{S}$.

Proof. The proof is analogous to that in Theorem 2.

One defines

$$\mathbb{M}_p=\left\{\mu\;\middle|\;\mu \text{ a random measure with } \mu(\mathbb{R}^n)=1 \text{ a.s., } E_\omega\!\int|x|^p\,d\mu^\omega(x)<\infty\right\}.$$

For $\mu,\nu\in\mathbb{M}_p$, define

$$\ell_p^*(\mu,\nu)=\begin{cases}E_\omega^{\frac1p}\ell_p^p(\mu^\omega,\nu^\omega) & p\ge 1\\ E_\omega\ell_p(\mu^\omega,\nu^\omega) & 0<p<1.\end{cases}$$

Then one can check that $(\mathbb{M}_p,\ell_p^*)$ is a complete metric space, and using the various independencies, that $\mathbb{S}:\mathbb{M}_p\to\mathbb{M}_p$ and is a contraction map with contraction constant $\lambda_p^{\frac1p\wedge 1}$. Now argue as in Theorem 2 to establish the claims in the first paragraph.

For the uniqueness of $\operatorname{dist}\mu^*$ satisfying $\mathbb{S}$, define a metric on the set $\mathcal{M}_p$ of probability distributions of members of $\mathbb{M}_p$ by

$$\ell_p^{**}(\mathcal{P},\mathcal{Q})=\inf\left\{\,\ell_p^*(\mu,\nu)\;\middle|\;\mu\overset{d}{=}\mathcal{P},\ \nu\overset{d}{=}\mathcal{Q}\,\right\}.$$

Then $(\mathcal{M}_p,\ell_p^{**})$ is a complete metric space and $\mathcal{S}$ is a contraction map with contraction constant $\lambda_p^{\frac1p\wedge 1}$. The result follows. ∎

Remarks. In a manner analogous to the remarks following Theorem 3, by letting $p\downarrow 0$ one can deduce existence, uniqueness and convergence results provided $\mathbb{E}\sum_i p_i\log r_i<0$ and $\mathbb{E}\sum_i p_i\log|S_i(0)|<\infty$. By letting $p\uparrow\infty$ we can consider Theorem 2 as a limit case of Theorem 4.

One can also replace the condition $\sum p_i=1$ a.s., implicit in Definition 3.3, by $\mathbb{E}\sum p_i=1$. Similar results still hold, but the proofs involve a number of new ideas, see [HR98b].

4. Selfsimilar fractal functions

For simplicity we consider functions $f:I\to\mathbb{R}^n$ where $I\subset\mathbb{R}$ is a closed bounded interval. (One can easily consider more general domains in $\mathbb{R}^m$ for $m>1$, as in [Hut81] in the nonrandom case.)

Let $I=I_1\cup\cdots\cup I_N$ be a partition of I into disjoint subintervals. Let $\phi_i:I\to I_i$ be increasing Lipschitz maps with $p_i=\operatorname{Lip}\phi_i$. (Note that $\sum_i p_i\ge 1$, and if the ϕ_i are affine then $\sum_i p_i=1$.) If $g_i:I_i\to\mathbb{R}^n$ for $i=1,\dots,N$ define $\bigsqcup_i g_i:I\to\mathbb{R}^n$ by

$$\Big(\bigsqcup_i g_i\Big)(x)=g_j(x)\quad\text{for } x\in I_j.$$

Definition 4.1. Let $\mathbb{S}=(S_1,\dots,S_N)$ be a scaling law (as in Definition 2.1). For $f:I\to\mathbb{R}^n$ define $\mathbb{S}f:I\to\mathbb{R}^n$ by

$$\mathbb{S}f=\bigsqcup_i S_i\circ f\circ\phi_i^{-1}.$$

We say f *satisfies the scaling law* $\mathbb{S}$, or is a *selfsimilar fractal function* if

$$f=\mathbb{S}f.$$

Definition 4.2. For $0 < p \leq \infty$ let

$$L_\infty = \{ f : I \to \mathbb{R}^n \mid \operatorname{ess\,sup} |f| < \infty \},$$
$$L_p = \{ f : I \to \mathbb{R}^n \mid \int |f|^p < \infty \} \qquad \text{if } 0 < p < \infty.$$

The metric d_p on L_p is the complete metric defined by

$$d_\infty(f,g) = \operatorname*{ess\,sup}_x |f(x) - g(x)|,$$
$$d_p(f,g) = \left(\int |f-g|^p \right)^{\frac{1}{p} \wedge 1} \qquad \text{if } 0 < p < \infty.$$

Let $\lambda_\infty = \max_i r_i$ and $\lambda_p = \sum_i p_i r_i^p$ for $0 < p < \infty$. One easily obtains the following.

Theorem 5. *If $\mathbb{S}$ is a scaling law with $\lambda_p < 1$ for some $0 < p \leq \infty$ then there is a unique $f^* \in L^p$ such that f^* satisfies $\mathbb{S}$. Moreover, for any $f_0 \in L^p$,*

$$d_\infty(\mathbb{S}^k f_0, f^*) \leq \frac{\lambda_\infty^k}{1-\lambda_\infty} d_\infty(f_0, \mathbb{S}f_0) \to 0$$
$$d_p(\mathbb{S}^k f_0, f^*) \leq \frac{\lambda_p^{k(\frac{1}{p}\wedge 1)}}{1-\lambda_p^{\frac{1}{p}\wedge 1}} d_p(f_0, \mathbb{S}f_0) \to 0 \qquad 0 < p < \infty$$

as $k \to \infty$.

Proof. Simple calculations show

$$d_\infty(\mathbb{S}f, \mathbb{S}g) \leq \lambda_\infty d(f,g), \quad d_p(\mathbb{S}f, \mathbb{S}g) \leq \lambda_p^{\frac{1}{p}\wedge 1} d_p(f,g) \text{ for } 0 < p < \infty.$$

□

Corollary. *Suppose $\lambda_\infty < 1$. Let α and β be the fixed points of S_1 and S_N respectively and assume $S_i(\beta) = S_{i+1}(\alpha)$ for $i = 1, \ldots, N-1$.*

Then f^* is continuous and $f^*(a) = \alpha$, $f^*(b) = \beta$ where $I = [a,b]$. If $a = a_0 < a_1 < \cdots < a_N = b$ and $I_i = [a_{i-1}, a_i]$ for each i, then $f^*(a_i) = S_i(\beta)$.

Proof. $\mathbb{S}$ preserves continuity and the conditions $f(a) = z_0$ and $f(b) = z_N$, while d_∞ is complete on the set of continuous functions.

Hence

$$f^*(a_i) = (\mathbb{S}f)^*(a_i) = S_i \circ f^* \circ \phi_i^{-1}(a_i) = S_i \circ f^*(b) = S_i(z_N) = z_i.$$

□

Remark. It follows from the definitions that if f^* is selfsimilar as in Theorem 5 then $\operatorname{spt} f[I]$ ($= f[I]$ if f is continuous as in the Corollary) is the selfsimilar compact set in Theorem 1, and the pushforward measure $f_\# \mathcal{L}^1 \lfloor I$ of f applied to Lebesgue measure restricted to I is the selfsimilar measure in Theorem 3.

As before one has a random version.

Definition 4.3. Let $\mathbb{S} = (S_1, \ldots, S_N)$ be a random scaling law (as in Definition 2.2). If $f : I \to \mathbb{R}^n$ is a random function, then the random function $\mathbb{S}f$ is defined (up to probability distribution) by

$$\mathbb{S}f = \bigsqcup_i S_i \circ f^{(i)} \circ \phi_i^{-1},$$

where $\mathbb{S}$, $f^{(1)}, \ldots, f^{(N)}$ are independent of one another and $f^{(i)} \overset{d}{=} f$. If $\mathcal{F} = \operatorname{dist} f$ we define

$$\mathcal{S}\mathcal{F} = \operatorname{dist} \mathbb{S}f.$$

We say f *(or more precisely $\mathcal{F}$) satisfies the scaling law* $\mathbb{S}$, or is a *selfsimilar random function*, if

$$\mathbb{S}f \overset{d}{=} f, \quad \text{or equivalently } \mathcal{S}\mathcal{F} = \mathcal{F}.$$

One now proceeds in a manner analogous to before.

Fix $0 < p \le \infty$. Beginning from any $f_0 \in L_p$ one defines a sequence of random functions

$$\begin{aligned} \mathbb{S}f_0 &= \bigsqcup_i S_i \circ f_0 \circ \phi_i^{-1}, \\ \mathbb{S}^2 f_0 &= \bigsqcup_{i,j} S_i \circ S_j^i \circ f_0 \circ \phi_j^{-1} \circ \phi_i^{-1}, \\ \mathbb{S}^3 f_0 &= \bigsqcup_{i,j,k} S_i \circ S_j^i \circ S_k^{ij} \circ f_0 \circ \phi_k^{-1} \circ \phi_j^{-1} \circ \phi_i^{-1}, \end{aligned} \tag{4.1}$$

etc.; with independencies analogous to those in Definition 2.3.

We then have the following analogue of previous results.

Theorem 6. *If $\mathbb{S} = (S_1, \dots, S_N)$ is a random scaling law which satisfies either $\lambda_p := \mathbb{E}\sum p_i r_i^p < 1$ and $E\sum p_i |S_i(0)|^p < \infty$ for some $0 < p < \infty$, or satisfies $\lambda_\infty := \operatorname{ess\,sup}_\omega \max_i r_i < 1$ and $\operatorname{ess\,sup}_\omega \max_i |S_i(0)| < \infty$, then for any $f_0 \in L_p$,*

$$\begin{aligned} \operatorname{ess\,sup} d_\infty(\mathbb{S}^k f_0, f^*) &\le \frac{\lambda_\infty^k}{1-\lambda_\infty} \operatorname{ess\,sup} d_\infty(f_0, \mathbb{S}f_0) \to 0 \\ \mathbb{E}^{\frac{1}{p}} d_p^p(\mathbb{S}^k f_0, f^*) &\le \frac{\lambda_p^{\frac{k}{p}}}{1-\lambda_p^{\frac{1}{p}}} \mathbb{E}^{\frac{1}{p}} d_p^p(f_0, \mathbb{S}f_0) \to 0 \qquad 0 \le p < \infty \\ \mathbb{E} d_p(\mathbb{S}^k f_0, f^*) &\le \frac{\lambda_p^k}{1-\lambda_p} \mathbb{E} d_p(f_0, \mathbb{S}f_0) \to 0 \qquad 0 < p < 1 \end{aligned}$$

as $k \to \infty$, where f^* does not depend on f_0. In particular, $\mathbb{S}^k f_0 \to f^*$ a.s.

Moreover, up to probability distribution, f^* is the unique function such that $\mathbb{E}\int \log|f^*| < \infty$ and which satisfies $\mathbb{S}$.

Proof. Let

$$\begin{aligned} \mathbb{L}_\infty &= \left\{ \text{random } f : I \to \mathbb{R}^n \;\middle|\; \operatorname*{ess\,sup}_\omega \operatorname*{ess\,sup}_x |f^\omega(x)| < \infty \right\}, \\ \mathbb{L}_p &= \left\{ \text{random } f : I \to \mathbb{R}^n \;\middle|\; \mathbb{E}_\omega \int |f^\omega|^p < \infty \right\} \qquad 0 < p < \infty. \end{aligned}$$

For $f, g \in \mathbb{L}_p$, define

$$d_p^*(f,g) = \begin{cases} \operatorname{ess\,sup}_\omega d_\infty(f^\omega, g^\omega) & p = \infty \\ E_\omega^{\frac{1}{p}} d_p^p(f^\omega, g^\omega) & 1 \le p < \infty \\ E_\omega d_p(f^\omega, g^\omega) & 0 < p < 1. \end{cases}$$

Then one can check that $(\mathbb{L}_p, d_p^*)$ is a complete metric space, and using the various independencies that $\mathbb{S} : \mathbb{L}_p \to \mathbb{L}_p$ is a contraction map with contraction constant λ_∞, or $\lambda_p^{\frac{1}{p}\wedge 1}$ if $p < \infty$. Now argue as in Theorem 2 to establish the claims in the first paragraph.

For the uniqueness of dist f^* satisfying $\mathbb{S}$, define a metric on the set $\mathcal{L}_p$ of probability distributions of members of $\mathbb{L}_p$ by

$$d_p^{**}(\mathcal{F}, \mathcal{G}) = \inf\left\{ d_p^*(f,g) \;\middle|\; f \overset{d}{=} \mathcal{F},\; g \overset{d}{=} \mathcal{G} \right\}.$$

Then $(\mathcal{L}_p, d_p^{**})$ is a complete metric space and $\mathcal{S}$ is a contraction map with contraction constant λ_∞, or $\lambda_p^{\frac{1}{p}\wedge 1}$ if $p < \infty$.

Note that

$$\left\{ \text{random } f : I \to \mathbb{R}^n \;\middle|\; \mathbb{E} \int \log |f| < \infty \right\} = \bigcup_{0<p<\infty} \mathbb{L}_p.$$

The result follows. □

The random version of the previous corollary is established similarly.

Corollary. *Suppose* $\lambda_\infty := \operatorname{ess\,sup}_\omega \max_i r_i < 1$ *and* $\operatorname{ess\,sup}_\omega \max_i |S_i(0)| < \infty$. *Let* S_1 *and* S_N *in* (4.1) *have random fixed points* α *and* β *respectively, and assume* $S_i(\beta) = S_{i+1}(\alpha)$ *a.s. for* $i = 1, \ldots, N-1$.

Then f^* *is continuous a.s. and* $f^*(a) = \alpha$, $f^*(b) = \beta$ *where* $I = [a, b]$. *If* $a = a_0 < a_1 < \cdots < a_N = b$ *and* $I_i = [a_{i-1}, a_i]$ *for each* i, *then* $f^*(a_i) = S_i(\beta)$ *a.s.*

We next briefly indicate how the previous framework can be modified to include Brownian motion. It will be clear that the ideas can be considerably extended, but we leave details for elsewhere. In [Gra91], using different notation, it was shown that Brownian motion can be characterised as the fixed point of a scaling operator analogous to the following setting. Here we show that moreover one has contraction maps at the random process and distribution level.

For each $\alpha > 0$, let $B^\alpha(t)$ denote Brownian motion in $\mathbb{R}$ characterised by

$$\begin{aligned} B^\alpha(0) &= 0 && \text{a.s.,} \\ B^\alpha(t+h) - B^\alpha(t) &\overset{d}{=} N(0, \alpha h) && \text{for } t > 0 \text{ and } h > 0, \end{aligned}$$

where $N(a, \sigma^2)$ denotes the normal distribution with mean a and variance σ^2.

Let $X^\alpha : [0,1] \to \mathbb{R}$ be the corresponding *constrained* Brownian motion given by

$$X^\alpha(0) = 0 \text{ a.s.}, \quad X^\alpha(1) = 1 \text{ a.s.}.$$

In particular,

$$X^\alpha\left(\frac{1}{2}\right) \overset{d}{=} N\left(0, \frac{\alpha}{2}\right),$$

by a standard property of Brownian motion.

Next *fix* $p \in \mathbb{R}$.

Consider Brownian motion $X^\alpha\big|_{X^\alpha(\frac{1}{2})=p}$, which is just X^α *further constrained* by $X^\alpha(1/2) = p$.

Let $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ be the unique affine transformations characterised by

$$S_1(0) = 0, \quad S_1(1) = S_2(0) = p, \quad S_2(1) = 1.$$

Let

$$r_1 := \operatorname{Lip} S_1 = |p|, \quad r_2 := \operatorname{Lip} S_2 = |1-p|.$$

Then

$$X^\alpha\big|_{X^\alpha(\frac{1}{2})=p}(t) \overset{d}{=} S_1 \circ X^{\frac{\alpha}{2r_1^2}}(2t), \quad t \in [0, 1/2],$$

essentially since scaling time by 2 and distance by r_1 (because of composition with S_1) means the variance of Brownian increments is scaled by $2r_1^2$. Similarly

$$X^\alpha\big|_{X^\alpha(\frac{1}{2})=p}(t) \overset{d}{=} S_2 \circ X^{\frac{\alpha}{2r_2^2}}(2t-1), \quad t \in [1/2, 1].$$

Next let $I = [0,1]$ and define

$$\phi_1 : I \to [0, 1/2], \quad \phi_1(s) = s/2,$$

$$\phi_2 : I \to [1/2, 1], \quad \phi_2(s) = (s+1)/2,$$

It follows that

$$X^\alpha\big|_{X^\alpha(\frac{1}{2})=p} \stackrel{d}{=} \bigsqcup_{i=1}^{2} S_i \circ X^{\frac{\alpha}{2r_i^2}} \circ \phi_i^{-1}, \quad t \in [0,1].$$

Motivated by the above, now let p^α be a *random* point in $\mathbb{R}$ with distribution $N(0, \alpha/2)$ (this is just the distribution of $X^\alpha(1/2)$).

For each $\alpha > 0$, let $\mathbb{S}^\alpha = (S_1^\alpha, S_2^\alpha)$ be the random scaling law obtained by defining (S_1^α, S_2^α) from the random point p^α in the same manner (S_1, S_2) was previously defined from the fixed point p. Since p^α is a random point whose distribution depends on α, the distribution of the random scaling law $\mathbb{S}^\alpha$ likewise depends on α. Let $r_i^\alpha = \operatorname{Lip} S_i^\alpha$ for $i = 1, 2$ and $r^\alpha = \max\{r_1^\alpha, r_2^\alpha\}$. Write $\mathbb{S}$ for $\{\, \mathbb{S}^\alpha \mid \alpha > 0 \,\}$.

It follows for each $\alpha > 0$ that

$$X^\alpha \stackrel{d}{=} \bigsqcup_{i=1}^{2} S_i^\alpha \circ X^{\frac{\alpha}{2(r_i^\alpha)^2}(i)} \circ \phi_i^{-1}, \tag{4.2}$$

where $\mathbb{S}^\alpha = (S_1^\alpha, S_2^\alpha)$ is first chosen as above, and then after conditioning on $\mathbb{S}^\alpha$, $X^{\frac{\alpha}{2(r_1^\alpha)^2}(1)} \stackrel{d}{=} X^{\frac{\alpha}{2(r_1^\alpha)^2}}$ and $X^{\frac{\alpha}{2(r_2^\alpha)^2}(2)} \stackrel{d}{=} X^{\frac{\alpha}{2(r_2^\alpha)^2}}$ are chosen independently of one another. We emphasise that even for fixed α, the $X^{\frac{\alpha}{2(r_i^\alpha)^2}(i)}$ depend on $\mathbb{S}^\alpha$ via $r_i^\alpha = \operatorname{Lip} S_i^\alpha$.

Thus the family of constrained Brownian motions (or Brownian bridges) $\{\, X^\alpha \mid \alpha > 0 \,\}$ satisfies the family of scaling laws $\mathbb{S} = \{\, \mathbb{S}^\alpha \mid \alpha > 0 \,\}$ in a manner generalising Definition 4.3.

For random functions

$$f^{\omega,\alpha}(t) = f^\omega(\alpha, t) : (0, \infty) \times I \to \mathbb{R}$$

such that

$$\sup_\alpha \alpha^{-1/2} E_\omega \int_I |f^\omega(\alpha, t)|\, dt < \infty,$$

and similarly for g, define

$$d^*(f, g) = \sup_\alpha \alpha^{-1/2} E_\omega \int_I |f^\omega(\alpha, t) - g^\omega(\alpha, t)|\, dt.$$

The factor $\alpha^{-1/2}$ is the appropriate one to capture the scaling behaviour of Brownian motion.

In the following, with some abuse of notation, we suppress ω and write $f(\alpha, t) = f^\alpha(t)$ for $f^\omega(\alpha, t) = f^{\omega,\alpha}(t)$. Motivated by (4.2), we define *up to distribution*

$$(\mathbb{S}f)^\alpha \stackrel{d}{=} \bigsqcup_{i=1}^{2} S_i^\alpha \circ f^{\frac{\alpha}{2(r_i^\alpha)^2}(i)} \circ \phi_i^{-1},$$

where $\mathbb{S}^\alpha = (S_1^\alpha, S_2^\alpha)$ are first chosen as before, and then after conditioning on $\mathbb{S}^\alpha$, choose $f^{\frac{\alpha}{2(r_1^\alpha)^2}(1)} \stackrel{d}{=} f^{\frac{\alpha}{2(r_1^\alpha)^2}}$ and $f^{\frac{\alpha}{2(r_2^\alpha)^2}(2)} \stackrel{d}{=} f^{\frac{\alpha}{2(r_2^\alpha)^2}}$ independently of one another. To turn this into a definition at the random variable level (rather than just the distribution level), one needs an analogue of the previous tree construction so that the choices of the $f^{\frac{\alpha}{2(r_i^\alpha)^2}(i)}$ are determined by f, and so that the same independencies are maintained.

Assuming this has been done, in order to obtain a contraction map, compute for $\alpha > 0$

$$\begin{aligned}
&\alpha^{-\frac{1}{2}}\mathbb{E}\int_I \big|(\mathbb{S}f)(\alpha,t) - (\mathbb{S}g)(\alpha,t)\big|\,dt \\
&\le \alpha^{-\frac{1}{2}}\mathbb{E}\int_I \Big|\bigsqcup_{i=1}^{2} S_i^\alpha \circ f^{\frac{\alpha}{2(r_i^\alpha)^2}(i)} \circ \phi_i^{-1} - \bigsqcup_{i=1}^{2} S_i^\alpha \circ g^{\frac{\alpha}{2(r_i^\alpha)^2}(i)} \circ \phi_i^{-1}\Big| \\
&\le \alpha^{-\frac{1}{2}}\mathbb{E}\bigg(\frac{1}{2}\sum_i r_i^\alpha \int_I \Big|f^{\frac{\alpha}{2(r_i^\alpha)^2}(i)} - g^{\frac{\alpha}{2(r_i^\alpha)^2}(i)}\Big|\bigg) \quad \text{as } |\phi_i'| = \frac{1}{2} \\
&\le \alpha^{-\frac{1}{2}}\frac{1}{2}\sum_i \mathbb{E}r_i^\alpha\, \mathbb{E}|_{r_i} \int_I \Big|f^{\frac{\alpha}{2(r_i^\alpha)^2}(i)} - g^{\frac{\alpha}{2(r_i^\alpha)^2}(i)}\Big| \\
&= \alpha^{-\frac{1}{2}}\frac{1}{2}\sum_i \mathbb{E}r_i^\alpha\, \mathbb{E}|_{r_i} \int_I \Big|f^{\frac{\alpha}{2(r_i^\alpha)^2}} - g^{\frac{\alpha}{2(r_i^\alpha)^2}}\Big| \\
&\qquad\qquad\qquad\qquad \text{by definition of } f^{\frac{\alpha}{2(r_i^\alpha)^2}(i)}, g^{\frac{\alpha}{2(r_i^\alpha)^2}(i)} \\
&\le \alpha^{-\frac{1}{2}}\frac{1}{2}\sum_i \mathbb{E}\, r_i^\alpha \left(\frac{\alpha}{2(r_i^\alpha)^2}\right)^{\frac{1}{2}} d^*(f,g) \quad \text{by definition of } d^* \\
&= \frac{1}{\sqrt{2}} d^*(f,g).
\end{aligned}$$

Taking the sup over $\alpha > 0$ gives the contraction. Existence and uniqueness results follow.

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