AA1H, SOLUTIONS TO ASSIGNMENT 5

Exercise 1. Read Chapter 5 of the 1999 Calculus Notes.

Exercise 2. Let $f(x)$ be a continuous function and prove $|f(x)|$ is continuous.

Proof. Let $g(x) = |f(x)|$. Then g is the composition of the absolute value function with the function f. Since f is continuous, and the absolute value function is continuous from Example 5.3 page 20, it follows g is continuous from Theorem 5.7 page 22. ☐

Exercise 3. Where is the function $\ln(\sin x)$ defined and continuous?

Solution. The function is defined for those x such that both

1. x is in the domain of the sin function (i.e. any x), and
2. $\sin x$ is in the domain of the ln function (i.e. $\sin x > 0$).

Thus $x \in (2n\pi, (2n+1)\pi)$ for some natural number n.

That is, the domain of the function is

$$\{\, x \mid x \in (2n\pi, (2n+1)\pi) \text{ for some } n \in \mathbb{N} \,\}$$

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Exercise 4. Suppose f is continuous on $\mathbb{R}$ and $f(q) = 0$ for all rationals q. Prove $f(x) = 0$ for all $x \in \mathbb{R}$.

Proof. For any real number x there is a sequence of *rational* numbers (x_n) such that $x_n \to x$ (take the decimal approximations to x).

Since $f(x_n) = 0$ for all n (by assumption) and $f(x_n) \to f(x)$ (as f is continuous), it follows $f(x) = 0$. ☐

Exercise 5. Suppose f is defined only on the integers. Explain why it is continuous.

Proof. We need to show that if $x_n \to x$ where x_n, x are *in the domain of* f (i.e. are integers), then $f(x_n) \to f(x)$. In fact we will show there is an integer N such that whenever $n \geq N$ then $x_n = x$ and so $f(x_n) = f(x)$.

(To see this choose N such that whenever $n \geq N$ then $|x_n - x| \leq \frac{1}{2}$. Since x_n and x are integers, this means that $x_n = x$. It follows that $f(x_n) = f(x)$ and this gives the result. ☐

Exercise 6. Find sequences (a_n) and (b_n) such that $a_n \to 0$ and $b_n \to 0$, but $\sin(1/a_n) \to 1$ and $\sin(1/b_n) \to 0$.

Proof. Let

$$a_n = \frac{1}{\left(2n + \frac{1}{2}\right)\pi}, \quad b_n = \frac{1}{2n\pi}.$$

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Exercise 7. Prove that $f(x) = x^3 - 4x + 2$ has a zero in the interval $[0, 1]$.

Proof. Since $f(0) = 2$ and $f(1) = -1$, it follows from Theorem 5.19 (the "Intermediate Value Theorem") that $f(c) = 0$ for some $c \in (0, 1)$. ☐

Exercise 8. Prove that all cubic polynomials have at least one real root.

Proof. Let

$$f(x) = ax^3 + bx^2 + cx + d$$

for all $x \in \mathbb{R}$.

If we can find p such that $f(p) < 0$ and q such that $f(q) > 0$, then by Theorem 5.19 there is some r between p and q such that $f(r) = 0$.

First suppose $a > 0$. Write

$$f(x) = x^3 \left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} \right).$$

Choose q sufficiently large (and positive) that $\left|\frac{b}{q}\right|$, $\left|\frac{c}{q^2}\right|$ and $\left|\frac{d}{q^3}\right|$ are each less than $\frac{a}{3}$. It follows that the term in parentheses is > 0, and since $q^3 > 0$ it follows that $f(q) > 0$.

Similarly we can choose p sufficiently large (and *negative*) that $\left|\frac{b}{p}\right|$, $\left|\frac{c}{p^2}\right|$ and $\left|\frac{d}{p^3}\right|$ are each less than $\frac{a}{3}$. It again follows that the term in parentheses is > 0, but since $p^3 < 0$ it now follows that $f(p) < 0$.

This completes the proof in case $a > 0$.

A similar proof works if $a > 0$.

Alternatively, note that $-f(x)$ is a cubic polynomial for which the coefficient of x^3 is now > 0 and so by the previous case there is some x such that $-f(x) = 0$, and hence $f(x) = 0$. ☐

Exercise 9. Let f be a continuous function defined on a finite interval $[a, b]$. Suppose that $f(x) > 0$ for all $x \in [a, b]$. Prove that there is an $\alpha > 0$ such that $f(x) > \alpha$ for all $x \in [a, b]$.

Give a simple example to show this is not true if f is not continuous.

Proof. By Theorem 5.17 there is some $c \in [a, b]$ such that $f(x) \geq f(c)$ for all $x \in [a, b]$. Let $\alpha = \frac{f(c)}{2}$. Since $f(c) > 0$, it follows that $\alpha > 0$ and so we are done.

Next let

$$f(x) = \begin{cases} |x| & -1 \leq x \leq 1 \text{ and } x \neq 0, \\ 1 & x = 0. \end{cases}$$

There is no $\alpha > 0$ such that $f(x) > \alpha$ for all $x \in [-1, 1]$. ☐

Exercise 10. Let f and g be continuous functions of $[a, b]$ and suppose $f(x) > g(x)$ for all $x \in [a, b]$. Prove that there is an $\alpha > 0$ such that $f(x) > \alpha + g(x)$ for all $x \in [a, b]$.

(You may use the previous result, even if you cannot prove it.)

Proof. Since f and g are both continuous, so is $f - g$. Moreover, $(f - g)(x) = f(x) - g(x) > 0$ for all $x \in [a, b]$.

By the previous exercise, there is an $\alpha > 0$ such that $f(x) - g(x) > \alpha$ for all $x \in [a, b]$. That is, $f(x) > \alpha + g(x)$ for all $x \in [a, b]$. ☐

Exercise 11. Suppose f is continuous on $[0, 1]$ and that $f(0) > 0$, $f(1) < 1$. Prove that $f(x) = x$ for some $x \in [0, 1]$.

Proof. We want to show there is some $c \in [0,1]$ such that $f(c) - c = 0$. So we first define the function g by

$$g(x) = f(x) - x.$$

Then g is continuous on $[0,1]$, $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$.

It follows from Theorem 5.19 that there is some $c \in [0,1]$ such that $g(c) = 0$. In other words $f(c) = c$. This proves the result. ☐

Exercise 12. ★ Prove that at any time there are two antipodal points on the equator with the same temperature.

You may reformulate the problem as follows: Assume f is a continuous function on $[0, L]$ (think of L as the distance around the equator and f as the temperature). Assume $f(0) = f(L)$. Prove there exists $x \in [0, L/2]$ such that $f(x) = f(x + L/2)$.

Proof. We want to prove there exists $c \in [0, L/2]$ such that $f(c) - f(c + L/2) = 0$. So we first define the function g on $[0, L/2]$ by

$$g(x) = f(x) - f(x + L/2)\,.$$

Then g is continuous. (To see this, first note that $f(x+L/2)$ defines a continuous function on $[0, L/2]$ since it is the composition of the continuous function obtained by adding $L/2$ with the continuous function f. Hence g is the difference of two continuous functions and so is continuous.)

Moreover,

$$g(0) = f(0) - f(L/2)$$

and

$$g(L/2) = f(L/2) - f(L) = f(L/2) - f(0) = -g(0)\,.$$

If $g(0) = 0$ then $f(0) - f(L/2) = 0$ and we are done. If $g(0) \neq 0$ then *either* $g(0) < 0$ and so $g(L/2) > 0$ *or* $g(0) > 0$ and so $g(L/2) < 0$. In either case, by Theorem 5.19 there is some $c \in [0, L/2]$ such that $g(c) = 0$, i.e. such that $f(c) - f(c + L/2) = 0$. This completes the proof. ☐

Exercise 13. ★ Assume that f is continuous at some point c. Prove that for every $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - f(c)| \le \epsilon \quad \text{whenever} \quad |x - c| \le \delta \text{ and } x \in \mathcal{D}(f). \tag{1}$$

HINT Assume for some $\epsilon > 0$ there is no such δ and obtain a contradiction.

Proof. Following the hint, in order to obtain a contradiction we make the *assumption* that the statement to be proved is false, i.e. for some “bad” $\epsilon > 0$ there is *no* $\delta > 0$ such that (1) is true.

This means that for this particular “bad” $\epsilon > 0$, whatever $\delta > 0$ we consider, there will be *at least* one real number x such that

$$|x - c| \le \delta \text{ and } x \in \mathcal{D}(f) \quad \text{but} \quad |f(x) - f(c)| > \epsilon\,.$$

Let $\delta = 1$ and choose x_1 so $|x_1 - c| \le 1$, $x_1 \in \mathcal{D}(f)$, and $|f(x_1) - f(c)| > \epsilon$.
Let $\delta = \frac{1}{2}$ and choose x_2 so $|x_2 - c| \le \frac{1}{2}$, $x_2 \in \mathcal{D}(f)$, and $|f(x_2) - f(c)| > \epsilon$.
Let $\delta = \frac{1}{3}$ and choose x_3 so $|x_3 - c| \le \frac{1}{3}$, $x_3 \in \mathcal{D}(f)$, and $|f(x_3) - f(c)| > \epsilon$.
. . .
Let $\delta = \frac{1}{n}$ and choose x_n so $|x_n - c| \le \frac{1}{n}$, $x_n \in \mathcal{D}(f)$, and $|f(x_n) - f(c)| > \epsilon$.
. . .

It follows that $x_n \to c$, and so $f(x_n) \to f(c)$ by the continuity of f. But this contradicts the fact that $|f(x_n) - f(c)| > \epsilon$ for all n.

Hence the *assumption* is wrong and so the statement to be proved is true. ☐