AA1H, SOLUTIONS TO ASSIGNMENT 4

Exercise 1. Read Sections 4.5 and 4.6 of the Calculus 1999 Notes.

Exercise 2.

1. As in Example 4.19, show from the definition that the sequence $a_n = \frac{n+1}{n-3}$ for $n \geq 4$ is Cauchy.
2. Use Theorem 4.20 (and any properties of limits) to show that $(a_n)_{n\geq 4}$ is Cauchy.

Proof. 1. In order to prove the sequence is Cauchy, compute

$$|a_n - a_m| = \left|\frac{n+1}{n-3} - \frac{m+1}{m-3}\right| = \left|\frac{4(m-n)}{(n-3)(m-3)}\right|$$
$$\leq \left|\frac{4m}{(n-3)(m-3)}\right| + \left|\frac{4n}{(n-3)(m-3)}\right|$$

There are many ways to proceed from here. The main point is that $\frac{4m}{m-3}$ and $\frac{4n}{n-3}$ are bounded. For example, we can argue that

$$\frac{4m}{m-3} \leq \frac{4m}{\frac{1}{2}m} = 8$$

if $m - 3 \geq \frac{1}{2}m$, i.e. if $m \geq 6$. Similarly, $\frac{4n}{n-3} \leq 8$ if $n \geq 6$. Hence

$$|a_n - a_m| \leq \frac{8}{n-3} + \frac{8}{m-3}$$

if $m, n \geq 6$.

We want to ensure $\frac{8}{n-3} \leq \frac{\epsilon}{2}$ and $\frac{8}{m-3} \leq \frac{\epsilon}{2}$. This is the case if $\frac{n-3}{8} \geq \frac{2}{\epsilon}$, i.e. $n \geq 3 + \frac{16}{\epsilon}$, and if similarly $m \geq 3 + \frac{16}{\epsilon}$.

Thus $|a_m - a_n| \leq \epsilon$ if

$$m, n \geq \max\left\{6,\ 3 + \left[\frac{16}{\epsilon}\right]\right\}.$$

Hence we see (a_n) is Cauchy by taking $N = \max\left\{6,\ 3 + \left[\frac{16}{\epsilon}\right]\right\}$ in the definition of a Cauchy sequence.

2. We have

$$a_n = \frac{n+1}{n-3} = \frac{1 + \frac{1}{n}}{1 - 3\frac{1}{n}}.$$

But $\lim \frac{1}{n} = 0$ from previous examples. By the properties of limits for scalar multiples of sequences, sums of sequences and quotients of limits, it follows that $\lim a_n$ exists (and equals 1). In particular, by Theorem 4.20 the sequence is Cauchy. □

Exercise 3. Recall Example 4.23.

Let $a_n = \sqrt[n]{n} - 1$. Use the binomial theorem to prove that

$$n = (1 + a_n)^n \geq \frac{n(n-1)}{2}\, a_n^2.$$

Rearrange the inequality and deduce that $\sqrt[n]{n} \to 1$.

Proof. Since $1 + a_n = n^{\frac{1}{n}}$, by the binomial theorem,

$$n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2}\, a_n^2 + \cdots \geq \frac{n(n-1)}{2}\, a_n^2.$$

Hence $\frac{2}{n-1} \geq a_n^2$ and so $a_n \leq \sqrt{\frac{2}{n-1}}$ (if $n \geq 2$).

But clearly $a_n > 0$ (because $\sqrt[n]{n} \geq 1$, which in turn comes from the fact $n \geq 1$ by taking nth roots of each side).

Thus we have

$$0 \leq a_n \leq \sqrt{\frac{2}{n-1}}$$

and so $\lim a_n = 0$ by the Squeeze Theorem. Hence $\lim(1 + a_n) = 1$, i.e. $\lim \sqrt[n]{n} = 1$. ☐

Exercise 4.

1. What can be said about the sequence (a_n) if it converges and if every a_n is an integer?
2. Find all convergent subsequences of the sequence

   $$1, -1, 1, -1, 1, -1, \ldots.$$

   (There are an infinite number of such subsequences, but only two limits which such subsequences can have). Express your answer in as succinct a form as possible.
3. Find all convergent subsequences of the sequence

   $$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots.$$

   (There are infinitely many limits such subsequences can have.) Express your answer in as succinct a form as possible.
4. Consider the sequence

   $$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots.$$

   For which numbers x is there a subsequence converging to x.

Solution.

1. The sequence must be eventually constant. That is, there exists an integer N such that a_n is constant for $n \geq N$.

(*Reason*: The sequence is Cauchy, and hence there exists N such that $|a_m - a_n| \leq \frac{1}{2}$ if $m, n \geq N$. Since a_m and a_n are integers, this means they are equal.)

2. The convergent subsequence must eventually equal 1, or it must eventually equal -1, by the previous part of the question. Moreover, any such sequence of 1's and -1's is possible.

3. *Any* sequence of positive integers which is eventually constant is possible.

(*Reason*: We already know from the first part that the sequence is eventually constant. This means it must be of the form

$$b_1, b_2, \ldots, b_{N-1}, k, k, k, \ldots.$$

To show there is indeed a subsequence of this form, first choose a member of the original sequence equal to b_1. Then choose a *later* member equal to b_2 (this is possible by the form of the original sequence). Etc.

4. Any real number $x \in [0, 1]$ is possible.

To see this, first suppose $0 < x \leq 1$ and let $x = \cdot x_1 x_2 x_3 \dots$ be the decimal expansion (for $x = 1$ take the decimal expansion $\cdot 999 \dots$). Now take the subsequence

$$\frac{x_1}{10}, \frac{10x_1 + x_2}{100}, \frac{100x_1 + 10x_2 + x_3}{1000}, \dots .$$

This converges to x.

If $x = 0$ take the subsequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots .$$

We have thus shown any $x \in [0, 1]$ is the limit of some subsequence. We cannot obtain any other value of x since we know that if a convergent sequence has all its terms between 0 and 1 (inclusive), then the same is true for its limit by Proposition 5.13. ☐

Exercise 5. ★ Recall that if $a_{n+1} - a_n \to 0$, it need not be the case that the sequence (a_n) is Cauchy. See the example at the bottom of page 13 and the top of page 14 of Calculus 1999 Notes, where $a_n = \sqrt{n}$.

However, if $a_{n+1} - a_n \to 0$ "sufficiently fast", then it *is* true that (a_n) is Cauchy. More precisely, prove that if

$$|a_{n+1} - a_n| \leq 2^{-n}$$

for all n, then (a_n) is Cauchy.

Proof. In order to prove (a_n) is Cauchy suppose that $m, n \geq N$, where we will choose N later. Assume $m > n$ (a similar argument will apply if $m < n$.

Then

$$\begin{aligned}
|a_n - a_m| &= |(a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + \cdots + (a_{m-1} + a_m)| \\
&\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} + a_m| \\
&\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} \\
&\leq \frac{1}{2^n}\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{m-n-1}\right) \\
&= \frac{1}{2^n}\frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}} \\
&\leq \frac{1}{2^n}\frac{1}{1 - \frac{1}{2}} = \frac{1}{2^{n+1}} \leq \frac{1}{2^{N+1}}.
\end{aligned}$$

It follows that $|a_m - a_n| \leq \epsilon$ whenever $m, n \geq N$, provided $\frac{1}{2^{N+1}} \leq \epsilon$. For this we need $2^{N+1} \geq \frac{1}{\epsilon}$, i.e. $N + 1 \geq \frac{\ln\frac{1}{\epsilon}}{\ln 2}$, and so we can take $N = \left\lceil \frac{\ln\frac{1}{\epsilon}}{\ln 2} \right\rceil$.

Hence (a_n) is Cauchy. ☐

(*Remark*: The same result and proof applies if $|a_{n+1} - a_n| \leq r^{-n}$ for any number r such that $|r| < 1$.)

Exercise 6. ★ Give a simple example of a nested sequence of open bounded intervals

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \dots$$

such that there is no number x with the property $x \in (a_n, b_n)$ for every n. By "nested" we mean that each interval contains the next, i.e.

$$a_1 \le a_2 \le a_3 \le a_4 \le \cdots \le b_4 \le b_3 \le b_2 \le b_1. \tag{1}$$

Also, assume $a_n \ne b_n$ for every n (so every interval does have some numbers in it; otherwise the result is immediate).

Prove that *for any nested sequence of closed bounded intervals*

$$[a_1, b_1], [a_2, b_2], [a_3, b_3], [a_4, b_4], \ldots$$

there is always at least one number x such that $x \in [a_n, b_n]$ for all n. By "nested" we again mean (1). This is called the *Nested Intervals Theorem*. *HINT*: Use the Bolzano-Weierstrass Theorem and Remark 4.25.

Give a simple example where there is more than one such number x.

Solution.

1. Take the sequence

$$(0,1)\,, \left(0, \frac{1}{2}\right), \left(0, \frac{1}{3}\right), \left(0, \frac{1}{4}\right), \ldots .$$

There is no x such that $x \in (0, \frac{1}{n})$ for all n.

2. For each n, choose some $x_n \in [a_n, b_n]$. By the Bolzano-Weierstass theorem and Remark 4.25, since each term of the sequence (x_n) belongs to $[a_1, b_1]$, the sequence (x_n) has a convergent subsequence with limit x (say), and $x \in [a_1, b_1]$.

However, more is true. For any k, the original sequence (x_n), and in particular the subsequence, eventually belongs to $[a_k, b_k]$. This implies, again by Remark 4.25, that $x \in [a_k, b_k]$.

In other words, $x \in [a_k, b_k]$ for all k, i.e. $x \in [a_n, b_n]$ for all n.

3. The sequence

$$\left[-\frac{1}{2}, 1 + \frac{1}{2}\right], \left[-\frac{1}{3}, 1 + \frac{1}{3}\right], \left[-\frac{1}{4}, 1 + \frac{1}{4}\right], \ldots$$

is an example where there is more than one x such that $x \in [a_n, b_n]$ for all n; in fact any $x \in [0, 1]$ will do . ☐