Solutions to AA1H, Assignment 3

Exercise 1. Read the AA1H 1999 Calculus Notes from Remark 4.5 on page 5 to the end of 4.5 on page 13.

Exercise 2. Read Examples 4.13 and 4.14. Prove that the following sequences converge, and find their limits, by using Theorem 4.12.

1. $a_n = 5\left(1 + \frac{1}{\sqrt[3]{n}}\right)^2$.
2. $a_n = \frac{3n+1}{n+2}$.
3. $a_n = \frac{n^2+6}{3n^2-2}$.
4. $a_n = \frac{5+\left(\frac{2}{3^n}\right)^2}{2+\frac{2n+5}{3n-2}}$.

Solutions.

1. First note that $\lim \frac{1}{\sqrt[3]{n}} = 0$ (This may be established directly from the definition of convergence for a sequence. More precisely, if $\epsilon > 0$ is any given real number, then $\frac{1}{\sqrt[3]{n}} \leq \epsilon$ whenever $n \geq \left[\frac{1}{\epsilon^3}\right] + 1$. But, as indicated in class, it is okay to assume $\lim \frac{1}{\sqrt[3]{n}} = 0$ at this stage.)

It follows that

$$\lim\left(1 + \frac{1}{\sqrt[3]{n}}\right) = 1 + \lim \frac{1}{\sqrt[3]{n}} = 1$$

by the theorem about the limit of the sum of two sequences (the first sequence being the constant sequence 1).

It then follows that

$$\lim \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 = \lim\left(1 + \frac{1}{\sqrt[3]{n}}\right) \times \lim\left(1 + \frac{1}{\sqrt[3]{n}}\right) = 1$$

by the theorem about the limit of the product of two sequences.

It then follows that

$$\lim 5 \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 = 5 \times \lim \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 = 5$$

by the theorem about the limit of the product of a real number and a sequence.

Alternatively, one could argue as follows:

We have

$$\lim 5 \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 = 5 \lim \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2$$

provided $\lim \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2$ exists, by the theorem about the limit of the product of a real number and a sequence,

$$= 5 \left(\lim\left(1 + \frac{1}{\sqrt[3]{n}}\right)\right)^2$$

provided $\lim\left(1 + \frac{1}{\sqrt[3]{n}}\right)$ exists, by the theorem about the limit of the product of two sequences,

$$= 5 \left(1 + \lim \frac{1}{\sqrt[3]{n}}\right)^2$$

provided $\lim \frac{1}{\sqrt[3]{n}}$ exists, by the theorem about the limit of the sum of two sequences (the first sequence being the constant sequence 1),

$$= 5(1+0)^2 = 5$$

since $\lim \frac{1}{\sqrt[3]{n}} = 0$. (The fact $\lim \frac{1}{\sqrt[3]{n}} = 0$ may be established directly from the definition of convergence for a sequence. More precisely, if $\epsilon > 0$ is any given real number, then $\frac{1}{\sqrt[3]{n}} \le \epsilon$ whenever $n \ge \left[\frac{1}{\epsilon^3}\right] + 1$. But, as indicated in class, it is okay to assume $\lim \frac{1}{\sqrt[3]{n}} = 0$ at this stage.)

Each equality in the previous argument is justified *provided* the limits which occur on the right of the "=" sign exist. Finally we come to a simple limit which we know *does* exist, and for this reason by working backwards we see that all the previous limits exist (and equal 5).

In each of the following solutions I have used the second type of argument, but you could just as well use the first type instead.

2. We have

$$\lim \frac{3n+1}{n+2} = \lim \frac{3+\frac{1}{n}}{1+\frac{2}{n}} = \frac{\lim\left(3+\frac{1}{n}\right)}{\lim\left(1+\frac{2}{n}\right)} = \frac{3+\lim\frac{1}{n}}{1+\lim\frac{2}{n}} = \frac{3+0}{1+0} = 3.$$

In each case the expression to the left of "=" exists *provided* the expression to the right exists. The first "=" is justified by algebra. The next two by the theorem about limits for quotients and for sums of sequences respectively. The fourth by the results that $\lim \frac{2}{n} = 0$ and $\lim \frac{1}{n} = 0$ (which can be easily shown from the definition, but you were not required to do so in this question).

3. Here we have

$$\lim \frac{n^2+6}{3n^2-2} = \lim \frac{1+\frac{6}{n^2}}{3-\frac{2}{n^2}} = \frac{\lim\left(1+\frac{6}{n^2}\right)}{\lim\left(3-\frac{2}{n^2}\right)} = \frac{1+\lim\frac{6}{n^2}}{3-\lim\frac{2}{n^2}} = \frac{1}{3}.$$

In each case the expression to the left of "=" exists provided the expression to the right exists, and we finally ended with something which we do know exists.

4.

$$\lim \frac{5+\left(\frac{2}{3^n}\right)^2}{2+\frac{2n+5}{3n-2}} = \frac{\lim\left(5+\left(\frac{2}{3^n}\right)^2\right)}{\lim\left(2+\frac{2n+5}{3n-2}\right)} = \frac{5+\lim\left(\frac{2}{3^n}\right)^2}{2+\lim\frac{2n+5}{3n-2}}$$

$$= \frac{5+\left(\lim\frac{2}{3^n}\right)^2}{2+\lim\frac{2+\frac{5}{n}}{3-\frac{2}{n}}} = \frac{5}{2+\frac{\lim\left(2+\frac{5}{n}\right)}{\lim\left(3-\frac{2}{n}\right)}} = \frac{5}{2+\frac{2}{3}} = \frac{15}{8}.$$

The argument is justified as in the previous examples. □

Exercise 3. Read Example 4.17. Prove that the following sequences converge, and find their limits, by using Theorem 4.16. You may use any of the standard properties of the sine, logarithm and exponential functions.

1. $a_n = e^{-n}\sin n$.
2. $a_n = (\sin n)\sin\frac{1}{n}$.
3. $a_n = (\cos n)(\ln n)^{-1}$ for $n \ge 2$.

Solution.

1. Since $-1 \le \sin n \le 1$, it follows that

$$-e^{-n} \le e^{-n} \sin n \le e^{-n}.$$

But $\lim -e^{-n} = \lim e^{-n} = 0$, and so $\lim e^{-n} \sin n$ exists and equals 0 by the Squeeze Theorem.

2. Since $|\sin x| \le |x|$, we have

$$\left|(\sin n) \sin \frac{1}{n}\right| = |\sin n| \left|\sin \frac{1}{n}\right| \le 1 \cdot \frac{1}{n} = \frac{1}{n}.$$

Hence

$$-\frac{1}{n} \le (\sin n) \sin \frac{1}{n} \le \frac{1}{n}.$$

But $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$, so by the Squeeze Theorem, $(\sin n) \sin \frac{1}{n} \to 0$.

3.

$$-\frac{1}{\ln n} \le \frac{\cos n}{\ln n} \le \frac{1}{\ln n}.$$

But $-\frac{1}{\ln n} \to 0$ and $\frac{1}{\ln n} \to 0$ (since $\ln n$ diverges to infinity — in fact for any K, $\ln n \ge K$ for $n \ge \left[e^K\right] + 1$.) □

Exercise 4. Prove that the sequence in Exercise 7 of Assignment 2 converges to 4. That is, prove that if $a_{n+1} = \frac{1}{2} a_n + 2$, $a_1 = .5$, then $a_n \to 4$. *HINT*: subtract 4 from both sides.

Consider the sequence defined by the relation $a_{n+1} = \alpha a_n + 2$. Prove that if $|\alpha| < 1$ then the sequence has a limit independent of a_1. What is the limit?

Solution.

1. Motivated by the hint,

$$a_{n+1} - 4 = \frac{1}{2} a_n + 2 - 4 = \frac{1}{2} a_n - 2 = \frac{1}{2}(a_n - 4),$$

if $n \ge 1$.

Hence

$$a_2 - 4 = \frac{1}{2}(a_1 - 4) = \frac{1}{2} \times (-3.5)$$
$$a_3 - 4 = \frac{1}{2}(a_2 - 4) = \frac{1}{4} \times (-3.5)$$
$$a_4 - 4 = \frac{1}{2}(a_3 - 4) = \frac{1}{8} \times (-3.5)$$

etc.

Thus (one could use induction, but it is clear)

$$a_n - 4 = \frac{1}{2^{n-1}} \times (-3.5). \tag{1}$$

for all n. In particular,

$$\lim a_n = 4.$$

(This is clear, but to justify it from the definition we argue as follows. Let $\epsilon > 0$ be given. Choose N so $\frac{3.5}{2^{n-1}} \le \epsilon$ if $n \ge N$. Then from (1) $|a_n - 4| \le \epsilon$ if $n \ge N$. So $\lim a_n = 4$ by the definition of limit.)

2. For the second part, motivated by the previous example, with β to be selected, we have

$$a_{n+1} - \beta = \alpha a_n + 2 - \beta,$$

i.e.

$$a_{n+1} - \beta = \alpha \left(a_n - \frac{\beta - 2}{\alpha} \right).$$

Now choose β so $\beta = \frac{\beta-2}{\alpha}$, which gives $\beta = \frac{2}{1-\alpha}$. Hence

$$a_{n+1} - \frac{2}{1-\alpha} = \alpha \left(a_n - \frac{2}{1-\alpha} \right).$$

Hence

$$\begin{aligned}
a_2 - \frac{2}{1-\alpha} &= \alpha \left(a_1 - \frac{2}{1-\alpha} \right) \\
a_3 - \frac{2}{1-\alpha} &= \alpha \left(a_2 - \frac{2}{1-\alpha} \right) = \alpha^2 \left(a_1 - \frac{2}{1-\alpha} \right) \\
a_4 - \frac{2}{1-\alpha} &= \alpha \left(a_3 - \frac{2}{1-\alpha} \right) = \alpha^3 \left(a_1 - \frac{2}{1-\alpha} \right)
\end{aligned}$$

etc., and in general

$$a_n - \frac{2}{1-\alpha} = \alpha^{n-1} \left(a_1 - \frac{2}{1-\alpha} \right). \tag{2}$$

Since $|\alpha| < 1$ it follows that $\lim |\alpha|^{n-1} = 0$, and so $\lim \alpha^{n-1} \left(a_1 - \frac{2}{1-\alpha} \right) = 0$. Hence from (2), $\lim a_n - \frac{2}{1-\alpha} = 0$ and so $\lim a_n = \frac{2}{1-\alpha}$.

Thus we have shown that (a_n) converges *and* the limit is $\frac{2}{1-\alpha}$ (which is independent of a_1). □

Exercise 5. ★ Let

$$f(x) = \lim_{n\to\infty} \left(\lim_{k\to\infty} \left(\cos(n!\pi x) \right)^{2k} \right).$$

Compute $f(x)$ for each real number x.

Solution. One needs to distinguish the cases x is rational and x is irrational.

If x is rational, then $n!\pi x$ is a multiple of π for all sufficiently large n, i.e. for all $n \geq N$ where $N = N(x)$ depends on x. Hence

$$n \geq N \Rightarrow \left(\cos(n!\pi x) \right)^{2k} = 1 \quad \text{for all } k.$$

It follows that

$$n \geq N \Rightarrow \lim_{k\to\infty} \left(\cos(n!\pi x) \right)^{2k} = 1,$$

and so

$$\lim_{n\to\infty} \left(\lim_{k\to\infty} \left(\cos(n!\pi x) \right)^{2k} \right) = 1.$$

If x is irrational, then for each n, $n!\pi x$ is *not* a multiple of π, and so

$$|\cos(n!\pi x)| < 1.$$

Hence (since $|\alpha| < 1$ implies $\alpha^{2k} \to 0$ as $k \to \infty$)

$$\lim_{k\to\infty} \left(\cos(n!\pi x) \right)^{2k} = 0,$$

and so

$$\lim_{n\to\infty}\left(\lim_{k\to\infty}\big(\cos(n!\pi x)\big)^{2k}\right)=0.$$

Hence

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational,}\\ 0 & \text{if } x \text{ is irrational.}\end{cases}$$

□