AA1H, SOLUTIONS TO ASSIGNMENT 2

Exercise 1.

- Read my TEX'd notes on Linear Algebra to the end of Section 1.3.
- Study Lay to the end of Section 1.3, *particularly* the Examples..
- Study Section 4.1 in Lay to the end of Example 9.
- Read Section 2.6 of the AA1H 1998 calculus Notes.
- Read the AA1H 1999 calculus Notes to the end of Section 4.2.

Exercise 2. Lay, Q30 p11, Q32 page 12.

Solution.

• Q30. To go from the first to the second matrix, interchange R1 and R2. To go from the second to the first matrix, interchange R1 and R2.

• Q32. To go from the first to the second matrix, replace R3 by R3 - 3R1. To go from the second to the first matrix, replace R3 by R3 + 3R1. ☐

Exercise 3. Lay, Q22 p25. *First* find the reduced row echelon form. *Then* answer the question.

Solution. The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -h & k \end{bmatrix}.$$

Performing row operations, this is equivalent to

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 6-h & k-2 \end{bmatrix}. \tag{1}$$

In order to put the matrix into reduced echelon form we need to distinguish various cases. *This is the main point to the Question.*

If $h \neq 6$ then the augmented matrix is equivalent to

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & \frac{k-2}{6-h} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{3k-h}{6-h} \\ 0 & 1 & \frac{k-2}{6-h} \end{bmatrix}.$$

In this case the system has a unique solution, namely $x_1 = \frac{3k-h}{6-h}$, $x_2 = \frac{k-2}{6-h}$.

If $h = 6$ then (1) is

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & k-2 \end{bmatrix}.$$

If $k \neq 2$ then this implies the oiginal system is inconsistent, and so it has no solutions.

Finally, *if* $h = 6$ and $k = 2$ then the system has infinitely many solutions. The augmented matrix is equivalent to

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means x_2 is a free variable and the set of all solutions is given by $x_1 = 1 + 3s$, $x_2 = s$, for any real number s.

In summary, there are no solutions if $h = 6$ and $k \neq 2$, a unique solution if $h \neq 6$, and many solutions if $h = 6$ and $k = 2$. ☐

Exercise 4. Read and understand the solution to Lay, Q31, 33 p26. Then do Lay, Q32 p26.

Solution.

$$
\begin{aligned}
x + y + z &= 1 \\
x + y + z &= 2
\end{aligned}
$$

☐

Exercise 5. Lay, Q6, 8, p217.

Solution.

- Q6 This is not a subspace as it is not closed under vector addition. For example, $3 + t^2$ and $4 + t^2$ are of the required form, but their sum $7 + 2t^2$ is not. (The set is not closed under scalar multiplication either, why?).
- Q8 This is a subspace. The zero polynomial has this property. If $\mathbf{p}_1(t) = 0$ for all t and $\mathbf{p}_2(t) = 0$ for all t, then $(\mathbf{p}_1 + \mathbf{p}_2)(t) = 0$ for all t. If $\mathbf{p}(t) = 0$ for all t and $c \in \mathbb{R}$ then $c\,\mathbf{p}(t) = 0$ for all t. ☐

Exercise 6. Lay, Q 256, 28, p219.

Note that Lay takes a slightly different version of the Axioms. If you look at his axioms on p211, I did not include "Axioms 1, 6" explicitly, but I did include them implicitly; see lines 14 and 15 from the bottom of p9 of my notes on linear algebra.

He also takes a slightly different form of his Axiom 5.

Use the version in Lay for these exercises.

Solution.

- Q26 Axioms 3, 5, 4 respectivley.
- Q28 Axioms 4, 7, 3, 5, 4 respectivley. ☐

Exercise 7. Use Maple or a calculator to compute enough terms of the following sequences to guess what their limits are:

1. $a_n = n \sin \frac{1}{n}$,
2. $a_n = (1 + \frac{1}{n})^n$,
3. $a_{n+1} = \frac{1}{2}a_n + 2,\ a_1 = .5$,
4. $a_{n+1} = 2.5a_n(1 - a_n),\ a_1 = 3$.

Solutions. The first sequence converges to 1 and the second to 2.71828... (in fact to the number denoted by e.)

The third sequence converges to 4 and the fourth diverges to $+\infty$. ☐

Remark: If we already knew somehow that the third sequence converges, then we could find the limit as follows. Let the limit be a. Then looking at the given equality $a_{n+1} = \frac{1}{2}a_n + 2$, we see $a_{n+1} \to a$ and $\frac{1}{2}a_n + 2 \to \frac{1}{2}a + 2$. Hence $a = \frac{1}{2}a + 2$ and so $a = 4$.

Similarly, from $a_{n+1} = 2.5a_n(1 - a_n)$ and assuming $a_n \to a$, taking limits of both sides we see $a = 2.5a(1 - a)$, which leads to $a = 0$ or $a = .6$. However, in this case the sequence does not converge. On the other hand, if we begin with any $a_1 \in (0, 1)$ then $a_n \to .6$. (In fact, I meant to set $a_1 = .3$, rather than $a_1 = 3$.)

Exercise 8. Prove directly that each of the following sequences converges by letting $\epsilon > 0$ be given and finding N such that (4.1), on p4 of the AA1H 1999 Notes, holds.

1. $a_n = 1 + \frac{10}{\sqrt{n}}$,

2. $a_n = 1 + \frac{1}{\sqrt[3]{n}}$,
3. $a_n = 3 + 2^{-n}$,
4. $a_n = \sqrt{\frac{n}{n+1}}$.

Solution.

1. We want to show $a_n \to 1$.

Let $\epsilon > 0$ be given. We want to show $|a_n - 1| \leq \epsilon$ for all sufficiently large n.

But $|a_n - 1| \leq \epsilon$ is equivalent to $\frac{10}{\sqrt{n}} \leq \epsilon$, which in turn is equivalent to $n \geq \frac{100}{\epsilon^2}$.

So we can choose $N = 1 + \left[\frac{100}{\epsilon^2}\right]$.

2. We want to show $a_n \to 1$.

Let $\epsilon > 0$ be given. We want to show $|a_n - 1| \leq \epsilon$ for all sufficiently large n.

But $|a_n - 1| \leq \epsilon$ is equivalent to $\frac{1}{\sqrt[3]{n}} \leq \epsilon$, which in turn is equivalent to $n \geq \frac{1}{\epsilon^3}$.

So we can choose $N = 1 + \left[\frac{1}{\epsilon^3}\right]$.

3. We want to show $a_n \to 3$.

Let $\epsilon > 0$ be given. We want to show $|a_n - 3| \leq \epsilon$ for all sufficiently large n.

But $|a_n - 3| \leq \epsilon$ is equivalent to $2^{-n} \leq \epsilon$, which is equivalent to $-n \ln 2 \leq \ln \epsilon$, which is equivalent to $n \ln 2 \geq \ln \frac{1}{\epsilon}$, which is equivalent to $n \geq \frac{\ln \epsilon^{-1}}{\ln 2}$.

So we can choose $N = 1 + \left[\frac{\ln \epsilon^{-1}}{\ln 2}\right]$.

4. We will show $a_n \to 1$. (It is clear what the limit should be, namely 1. We will use the definition to show that this is indeed the case.)

Let $\epsilon > 0$ be given. In order to find N such that $|a_n - 1| \leq \epsilon$ for all $n \geq N$, we calculate (multiplying and dividing by $\sqrt{\frac{n+1}{n}} + 1$)

$$\begin{aligned} |a_n - 1| = \sqrt{\frac{n+1}{n}} - 1 &= \frac{\left(\sqrt{\frac{n+1}{n}} - 1\right)\left(\sqrt{\frac{n+1}{n}} + 1\right)}{\sqrt{\frac{n+1}{n}} + 1} \\ &= \frac{\frac{n+1}{n} - 1}{\sqrt{\frac{n+1}{n}} + 1} = \frac{\frac{1}{n}}{\sqrt{\frac{n+1}{n}} + 1} \leq \frac{1}{n}. \end{aligned}$$

But $\frac{1}{n} \leq \epsilon$ iff[1] $n \geq \frac{1}{\epsilon}$. Thus we can take $N = 1 + \left[\frac{1}{\epsilon}\right]$. ☐

---

[1] By "iff" we mean "if and only if", i.e. "is equivalent to".