

AA1H, ASSIGNMENT 5

You may talk to others about the problems. But you are expected to write out the solutions yourself, with no one else indicating what to write, and without help from anyone else's notes. If someone else had a major input into your solution, you should indicate this.

The assignment is due in by June 4, Friday 4pm (**NOT** 5pm as indicated in the general information sheet), in the AA1H collection box corresponding to your tutorial group, in the foyer of the Mathematics Department.

As usual, the ★ questions are for extra credit.

Remark 1. There are some very interesting websites available about mathematics, and careers in areas which use significant amounts of mathematics.

1) <http://www.maths.anu.edu.au/MathematicsHandbook/> describes Math Department courses

2) At the top of the first page of the above there is a *link* to: <http://www.maths.anu.edu.au/MathematicsHandbook/> which gives a few general ideas about mathematics in combination with other disciplines

3) The red arrow at the top of the previous address is a *link* to <http://cs.jsu.edu/ms/ms-home.html> which is an excellent comprehensive US site which has links to just about everything on the Web that is mathematical (and not too flaky). Of particular interest is the careers heading, which gives some idea of the variety of fields which you can combine with mathematics.

4) An excellent site for career ideas is <http://www.ams.org/careers/>

Exercise 2. Read Sections 4.5 and 4.6 of the Calculus 1999 Notes.

Exercise 3.

1. As in Example 4.19, show from the definition that the sequence $a_n = \frac{n+1}{n-3}$ for $n \geq 4$ is Cauchy.
2. Use Theorem 4.20 (and any properties of limits) to show that $(a_n)_{n \geq 4}$ is Cauchy.

Exercise 4. Recall Example 4.23.

Let $a_n = \sqrt[n]{n} - 1$. Use the binomial theorem to prove that

$$n = (1 + a_n)^n \geq \frac{n(n-1)}{2} a_n^2.$$

Rearrange the inequality and deduce that $\sqrt[n]{n} \rightarrow 1$.

Exercise 5.

1. What can be said about the sequence (a_n) if it converges and if every a_n is an integer?
2. Find all convergent subsequences of the sequence

$$1, -1, 1, -1, 1, -1, \dots$$

(There are an infinite number of such subsequences, but only two limits which such subsequences can have). Express your answer in as succinct a form as possible.

3. Find all convergent subsequences of the sequence

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots$$

(There are infinitely many limits such subsequences can have.) Express your answer in as succinct a form as possible.

4. Consider the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots$$

For which numbers x is there a subsequence converging to x .

Exercise 6. ★ Recall that if $a_{n+1} - a_n \rightarrow 0$, it need not be the case that the sequence (a_n) is Cauchy. See the example at the bottom of page 13 and the top of page 14 of Calculus 1999 Notes, where $a_n = \sqrt{n}$.

However, if $a_{n+1} - a_n \rightarrow 0$ “sufficiently fast”, then it *is* true that (a_n) is Cauchy. More precisely, prove that if

$$|a_{n+1} - a_n| \leq 2^{-n}$$

for all n , then (a_n) is Cauchy.

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Exercise 7. ★ Give a simple example of a nested sequence of open bounded intervals

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \dots$$

such that there is no number x with the property $x \in (a_n, b_n)$ for every n . By “nested” we mean that each interval contains the next, i.e.

$$(1) \quad a_1 \leq a_2 \leq a_3 \leq a_4 \leq \dots \leq b_4 \leq b_3 \leq b_2 \leq b_1.$$

Also, assume $a_n \neq b_n$ for every n (so every interval does have some numbers in it; otherwise the result is immediate).

Prove that for any nested sequence of closed bounded intervals

$$[a_1, b_1], [a_2, b_2], [a_3, b_3], [a_4, b_4], \dots$$

there is always at least one number x such that $x \in [a_n, b_n]$ for all n . By “nested” we again mean (1). This is called the *Nested Intervals Theorem*. *HINT*: Use the Bolzano-Weierstrass Theorem and Remark 4.25.

Give a simple example where there is more than one such number x .