AA1H, Assignment 4

You may talk to others about the problems. But you are expected to write out the solutions yourself, with no one else indicating what to write, and without help from anyone elses notes. If someone else had a major input into your solution, you should indicate this.

The assignment is due in by May 28, Friday 4pm (**NOT** 5pm as indicated in the general information sheet), in the AA1H collection box corresponding to your tutorial group, in the foyer of the Mathematics Department.

As usual, the ★ questions are for extra credit.

Exercise 1. Read Sections 4.5 and 4.6 of the Calculus 1999 Notes.

Exercise 2.

1. As in Example 4.19, show from the definition that the sequence $a_n = \frac{n+1}{n-3}$ for $n \geq 4$ is Cauchy.
2. Use Theorem 4.20 (and any properties of limits) to show that $(a_n)_{n\geq 4}$ is Cauchy.

Exercise 3. Recall Example 4.23.

Let $a_n = \sqrt[n]{n} - 1$. Use the binomial theorem to prove that

$$n = (1 + a_n)^n \geq \frac{n(n-1)}{2}\, a_n^2.$$

Rearrange the inequality and deduce that $\sqrt[n]{n} \to 1$.

Exercise 4.

1. What can be said about the sequence (a_n) if it converges and if every a_n is an integer?
2. Find all convergent subsequences of the sequence

$$1, -1, 1, -1, 1, -1, \ldots .$$

   (There are an infinite number of such subsequences, but only two limits which such subsequences can have). Express your answer in as succinct a form as possible.
3. Find all convergent subsequences of the sequence

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots .$$

   (There are infinitely many limits such subsequences can have.) Express your answer in as succinct a form as possible.
4. Consider the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots .$$

   For which numbers x is there a subsequence converging to x.

Exercise 5. ★ Recall that if $a_{n+1} - a_n \to 0$, it need not be the case that the sequence (a_n) is Cauchy. See the example at the bottom of page 13 and the top of page 14 of Calculus 1999 Notes, where $a_n = \sqrt{n}$.

However, if $a_{n+1} - a_n \to 0$ "sufficiently fast", then it *is* true that (a_n) is Cauchy. More precisely, prove that if

$$|a_{n+1} - a_n| \leq 2^{-n}$$

for all n, then (a_n) is Cauchy. /OVER

Exercise 6. ★ Give a simple example of a nested sequence of open bounded intervals

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), \ldots$$

such that there is no number x with the property $x \in (a_n, b_n)$ for every n. By "nested" we mean that each interval contains the next, i.e.

$$a_1 \le a_2 \le a_3 \le a_4 \le \cdots \le b_4 \le b_3 \le b_2 \le b_1. \tag{1}$$

Also, assume $a_n \neq b_n$ for every n (so every interval does have some numbers in it; otherwise the result is immediate).

Prove that *for any nested sequence of closed bounded intervals*

$$[a_1, b_1], [a_2, b_2], [a_3, b_3], [a_4, b_4], \ldots$$

there is always at least one number x such that $x \in [a_n, b_n]$ for all n. By "nested" we again mean (1). This is called the *Nested Intervals Theorem*. *HINT*: Use the Bolzano-Weierstrass Theorem and Remark 4.25.

Give a simple example where there is more than one such number x.