

AA1H, ASSIGNMENT 1

You may talk to others about the problems. But you are expected to write out the solutions yourself, with no one else indicating what to write, and without help from anyone else's notes. If someone else had a major input into your solution, you should indicate this.

The assignment is due in by March 19, Friday 4pm (**NOT** 5pm as indicated in the general information sheet), in the AA1H collection box in the foyer of the Mathematics Department.

Mathematics is the ultimate form of careful and precise argument. *Keep your answers brief and to the point.* In each of your solutions you should usually follow the following template:

Proof.

1. Write down any information given to you in the Exercise. That is, write:
Assume ... *given information* ...
2. Argue carefully and precisely, using only standard logic and
 - (a) the given information,
 - (b) the axioms or standard properties of the real numbers (depending on whether the question asks for an argument from the axioms or not)
 - (c) previously proved facts and theorems
3. End with the desired conclusion.

□

Write your proofs in good English (although you will frequently be using symbols rather than words). Sentences should normally have a verb (it may be a symbol, such as “=”), a subject and an object. Do not omit words such as “hence”, “we assume”, “because”, “therefore”, “and”, “or”, etc.

For Exercises 2–4 you should study the style of argument used for proofs from the axioms in the relevant sections of the Notes.

In Exercise 6.1 you might begin the proof with

Proof. Let $n > 0$ be an integer.

After dividing by 3, the remainder is either 0, 1 or 2. ...

Sometimes you will prove a statement by assuming it is false and obtaining a contradiction. Thus in Exercise 6.3 you might begin the proof with

Proof. Suppose $x > 0$ and $x^2 = 3$.

Assume x is rational.

Let $x = m/n$ where m and n are positive integers with no common factors. ... ”

Then argue to obtain a contradiction.

The last line of the proof would be;

Hence x is not rational.

□

Exercise 1. Read Sections 2.1–2.3, Section 2.5 to the beginning of Theorem 2.12, and Section 2.6.

Exercise 2. Prove from the axioms that *if a , b and c are real numbers, $c \neq 0$, and $ac = bc$, then $a = b$.* HINT: Use an argument similar to that in Theorem 2.2.

Exercise 3. Prove from the axioms that for any number a , $a0 = 0$. HINT: This is just Theorem 2.3.2, and is done on page 18. So all you have to do is to justify the 4 steps in terms of the axioms, rules of logic, or something else already proved from the axioms.

Exercise 4.

1. Prove from the axioms that $a + (b - a) = b$. *HINT*: What was the definition of $b - a$?
2. Prove from the axioms that *if* $a + x = b$ *then* $x = b - a$.
The first part of the question shows that $b - a$ is a solution of $a + x = b$. The second part shows that it is the *only* solution. Notice this subtle but important distinction.
3. Deduce that 0 is the *only* real number that has the property $a + 0 = 0 + a = a$. for all a .

In the following exercises, do *not* give proofs from the axioms. Use all the usual properties of addition, multiplication, subtraction, division and inequalities, without further justification.

Exercise 5. *An exercise in clear thinking.* For each of the following statements, say if it is true or false, and give a short justification.

1. For every real number $x > 0$ there exists a real number y such that $0 < y < x$.
2. There exists a real number y such that for every real number $x > 0$, $0 < y < x$.
3. For every real number $x > 0$ and for every real number y , $0 < y < x$.
4. For every real number y and for every real number $x > 0$, $0 < y < x$.
5. There exists a real number y and there exists a real number $x > 0$ such that $0 < y < x$.
6. There exists a real number $x > 0$ and there exists a real number y such that $0 < y < x$.

Exercise 6.

1. Prove that for every integer $n > 0$ one has

$$n = 3m, 3m + 1 \text{ or } 3m + 2$$

for some integer m .

2. Prove that the square of any integer of the first kind is again of the first kind, but that the square of any integer of the second or third kind is of the second kind.
3. Prove there is no rational number $x > 0$ such that $x^2 = 3$ by a similar argument to that in Theorem 2.11 (except that now, instead of working with odd and even integers, one works with integers of the first, second, or third kind.)