AA1H, Solutions to Assignment 1

Read Sections 2.1–2.5.

In the following exercises, you should *not* give rigorous proofs from the axioms, *unless* indicated otherwise. You may use all the usual properties of addition, multiplication, subtraction, division, and the usual properties of inequalities. However, you *should* indicate whenever the Completeness Axiom is required.

Exercise 1.

1. Explain why for every integer $n > 0$ one has
$$n = 3m,\ 3m + 1 \text{ or } 3m + 2$$
for some integer m.
2. Show that the square of any integer of the first kind is again of the first kind, but that the square of any integer of the second or third kind is of the second kind.
3. Prove that $\sqrt{3}$ is not rational, by a similar argument to that in Theorem 2.11 (except that now, instead of working with odd and even integers, one works with integers of the first, second, or third kind.)

Solution.

1. If n is divided by 3, the remainder will be either 0, 1 or 2. That is,
$$n = 3m,\ 3m + 1 \text{ or } 3m + 2.$$
$\Big($ The above is all that was required. But in more detail, let m be the largest integer such that $3m \le n$. Then $n - 3m = 0, 1,$ or 2, as otherwise we could take a larger m. This gives the result. $\Big)$
2.
$$(3m)^2 = 9m^2 = 3(3m^2) \quad \text{which is of the first kind}$$
$$(3m + 1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1 \quad \text{which is of the second kind}$$
$$(3m + 2)^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1 \quad \text{which is also of the second kind}$$
3. Suppose that $\sqrt{3}$ is rational.[1] This means that, after cancellation if necessary, we can write
$$\sqrt{3} = m/n$$
where m and n have no common factors. Squaring both sides of the equation, we have
$$3 = m^2/n^2$$
and hence
$$m^2 = 3n^2.$$
It follows that m^2 is of the first kind (i.e. is divisible by 3), and so from Part 2, m is also of the first kind (as otherwise its square would be of the second kind). But since m is of the first kind, we can write
$$m = 3p$$

[1] By $\sqrt{3}$ we mean the positive number whose square is 3. To show that there *is* such a number requires an argument similar to that used in Theorem 2.12. (Later, see Theorem 5.17, we will prove a much more general fact.)

Here we are showing that *if* there is a number x whose square is 3, then x *must* be irrational.

for some integer p, and hence

$$m^2 = 9p^2.$$

Substituting this into $m^2 = 3n^2$ gives

$$9p^2 = 3n^2,$$

and hence

$$3p^2 = n^2.$$

But now we can argue as we did before for m, and deduce that n is also of the fist kind (divisible by 3). Thus m and n both have the common factor 3, which contradicts the fact they have no common factors.

This contradiction implies that our original assumption was wrong, and so $\sqrt{3}$ is not rational.

□

Exercise 2. Prove from the axioms that *if* a, b *and* c *are real numbers,* $c \neq 0$, *and* $ac = bc$, *then* $a = b$. HINT: Use an argument similar to that in Theorem 2.2.

Solution. Assume

$$ac = bc$$

and $c \neq 0$. Since ac and bc are equal, we obtain the same number if we multiply ac by c^{-1} or bc by c^{-1}; i.e.

$$(ac)c^{-1} = (bc)c^{-1}.$$

(This uses the existence of the number c^{-1} from A8.) Hence

$$a(cc^{-1}) = b(cc^{-1})$$

from A6 applied twice, once to each side of the equation. Hence

$$a1 = b1$$

from A8 applied twice. Finally,

$$a = b$$

from A7. □

Exercise 3. Prove from the axioms that for any number a, $a0 = 0$. HINT: This is just Theorem 2.3.2, and is done on page 18. So all you have to do is to justify the 4 steps in terms of the axioms, rules of logic, or something else already proved from the axioms.

Solution. The trick here is to use the fact $0 + 0 = 0$ (from A3), together with the distributive axiom A9. The proof is as follows:

One has $a(0 + 0) = a0$ (since $0 + 0 = 0$ from A4)
But the left side equals $a0 + a0$ (from A9)
and the right side equals $0 + a0$ (from A4)
Hence $a0 + a0 = 0 + a0$
Hence $a0 = 0$ (from Theorem 2.2). □

Exercise 4. *An exercise in clear thinking.* For each of the following statments, say if it is true or false, and give a short justification.

1. For every real number m there exists a real number n such that $m < n$.
2. There exists a real number n such that for every real number m, $m < n$.

3. For every human x there exists (or existed) a human y such that x is the child of y.
4. There exists (or existed) a human y such that for every human x, x is the child of y.

Solution. The point to this problem is that the order of the statements "for every real number m" and "there exists a real number n" is critical.

The first statement is *true*. For example, for any particular m we could take $n = m + 1$. Then certainly $m < n$. (Note that n depends on m).

The second statement is *false*. There is no (single) real number which is greater than *every* real number!

The third statement is true since every human does indeed have a parent.

The fourth is false since there is certainly no human who is the parent of everyone. □

Remark: It is convenient to use the standard logical notation

$$\forall \quad \text{to mean "for all"},$$
$$\exists \quad \text{to mean "there exists"}.$$

Then the first statement can be written

$$\forall m \; \exists n \text{ such that } (m < n),$$

and the second can be written

$$\exists n \text{ such that } \forall m \; (m < n).$$

The symbols $\forall$ and *exists* are called *quantifiers*. We frequently omit the words "such that" after $\exists$, but they are understood from the context to be there. Thus we often write

$$\forall m \; \exists n \; (m < n), \quad \exists n \; \forall m \; (m < n).$$

The main point to this exercise is that the *order* of the quantifiers $\forall$ and $\exists$ is critical.

Remark: It is *important* to realise that Statement 1 has *exactly* the same meaning as

for every real number x there exists a real number y such that $x < y$,

i.e. the same as

$$\forall x \; \exists y \; (x < y),$$

and even *exactly* the same meaning as

for every real number n there exists a real number m such that $n < m$,

i.e. the same as

$$\forall n \; \exists m \; (n < m).$$

We say that x, y, m, n in the above are *dummy variables*. It is a similar situation with statements such as $\int_0^1 x^2\,dx = 1/3$ and $\int_0^1 y^2\,dy = 1/3$. The variables x and y are again dummy variables, and the two equalities have *exactly* the same meaning.

Note, however, that the statement

for every real number n there exists a real number n such that $n < n$,

i.e.

$$\forall n \; \exists n \; n < n,$$

has quite a different meaning (and in fact is false). In fact we would not use the same variable in this way after more than one quantifier.

Exercise 5. For each of the following sets, decide if they have a least upper bound, and if so say what it is. Do the same for the greatest lower bound.

Also, decide if the least upper bound and greatest lower bound are members of the set.

No explanation is needed, just give the answers.

1. $\{ 1/n : n \in \mathbb{N} \}$.
2. $\{ 1/n : n \in \mathbb{Z}$ and $n \neq 0 \}$.
3. $\{ x : x = 0$ or $x = 1/n$ for some $n \in \mathbb{N} \}$.
4. $\{ x : 0 \leq x \leq \sqrt{2}$ and x is rational $\}$.
5. $\{ x : x^2 + x + 1 \geq 0 \}$.
6. $\{ x : x^2 + x - 1 < 0 \}$.
7. $\{ x : x < 0$ and $x^2 + x - 1 < 0 \}$.
8. $\{ 1/n + (-1)^n : n \in \mathbb{N} \}$.

HINT: In most cases it may help if you write out an alternative description of the set in question.

Solution.

1. This is the set $\{1, 1/2, 1/3, \dots\}$. The l. u. b. $= 1$ and belongs to the set. The g. l. b. $= 0$ and does not belong to the set.
2. This is the set $\{\pm 1, \pm 1/2, \pm 1/3, \dots\}$. The l. u. b. $= 1$ and belongs to the set. The g. l. b. $= -1$ and belongs to the set.
3. This is the set $\{0, 1, 1/2, 1/3, \dots\}$. The l. u. b. $= 1$ and belongs to the set. The g. l. b. $= 0$ and belongs to the set.
4. The l. u. b. $= \sqrt{2}$ and does not belong to the set. The g. l. b. $= 0$ and belongs to the set.
5. $x^2 + x + 1 = 0$ has no real roots and so $x^2 + x + 1 > 0$ for *all* real numbers x. Thus the given set is the same as $\mathbb{R}$ and so has no l. u. b. and no g. l. b..
6. $x^2 + x - 1 = 0$ iff[2] $x = (-1 \pm \sqrt{5})/2$ and so the given set is the same set as $\{ x : (-1 - \sqrt{5})/2 < x < (-1 + \sqrt{5})/2 \}$, which is just the interval $((-1 - \sqrt{5})/2, (-1 + \sqrt{5})/2)$.

   This set has l. u. b. $= (-1 + \sqrt{5})/2$, which is not a member of the set, and g. l. b. $= (-1 - \sqrt{5})/2$, which is not a member of the set.
7. The given set consists of all numbers which are < 0 and belong to the set in the part 6. Thus the given set is the interval $((-1 - \sqrt{5})/2, 0)$.

   This set has l. u. b. $= 0$, which is not a member of the set, and g. l. b. $= (-1 - \sqrt{5})/2)$, which is not a member of the set.
8. The given set is

$$\{ 0,\ 3/2,\ -2/3,\ 5/4,\ -4/5,\ 7/6,\ -6/7,\ 9/8,\ -8/9,\ 11/10,\ -10/11,\ 13/12,\ -12/13,\ \dots \}.$$

   The l. u. b. $= 3/2$ and belongs to the set, the g. l. b. $= -1$ and does not belong to the set.

□

Exercise 6. Consider a sequence of closed intervals

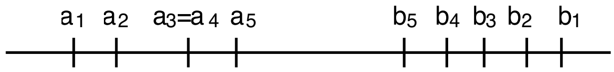
$$I_1 = [a_1, b_1],\ I_2 = [a_2, b_2],\ I_3 = [a_3, b_3],\ \dots\ , I_n = [a_n, b_n],\ \dots$$

which are *nested*, that is

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots \supset [a_n, b_n] \supset \cdots .$$

[2] "Iff" is an abbreviation for "if and only if".

(By $A \supset B$ we mean that B is a subset of A; that is every member of B is also a member of A).



This is the same as

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots \quad \cdots \le b_n \le \cdots \le b_3 \le b_2 \le b_1.$$

In particular, every a_n is $\le$ every b_m.

Prove there exists a real number x which is in every I_n. HINT: You will need the Completeness Axiom. You should define x to be the least upper bound of a suitable set of real numbers.

Give an example to show that the result is not true if the intervals are (non-empty) *open* intervals (a, b). HINT: Remember that we do not require $a_1 < a_2 < \ldots$, only that $a_1 \le a_2 \le \ldots$.

Solution. The set of numbers $S = \{a_1, a_2, a_3, \ldots\}$ is certainly not empty, and is bounded above (by any of the b_m's) and so has a l. u. b. x (say) by the Completeness Axiom. Thus

(1) $$a_n \le x \quad \text{for every } n.$$

Let m be any natural number. Then b_m is an upper bound for S (since $a_n \le b_m$ for all n). Since x is the *least* upper bound for S, it follows that $x \le b_m$. Since we could take m to be *any* natural number, it follows that

$$x \le b_m, \quad \text{for every } m.$$

This has *exactly* the same meaning as the statement

(2) $$x \le b_n, \quad \text{for every } n.$$

It follows from (1) and (2) that

$$x \in I_n, \quad \text{for every } n.$$

This completes the proof of the first part of the Question.

If $I_n = (0, 1/n)$ for $n \in \mathbb{N}$, then there is no real number x such that $x \in I_n$ for every n. (Because no matter which $x > 0$ we take, there is always some natural number n such that $0 < 1/n < x$ and hence such that $x \notin I_n$.) □