AA1H, Assignment 5

You may talk to others about the problems. But you should write out the solutions yourself, with no one else indicating what to write, and without help from anyone elses notes. If someone else had a major input in your solution, you should indicate this.

The assignment is due in by June 5, Friday 10 am, in the AA1H collection box in the foyer of the Mathematics Department (so this means before the lecture!).

Read Chapter 5, and Adams Section 1.5 and Appendix 3.

Exercise 1.

1. For which of the followiing functions f is there a function F with domian $\mathbb{R}$ such that $F(x) = f(x)$ for all x in the domain of f?
   (a) $f(x) = \frac{x^2-4}{x-2}$
   (b) $f(x) = \frac{x}{|x|}$
2. Suppose f is a function satsifying $f(x) \leq |x|$ for all x. Prove f is contionuous at 0. *HINT*: Note that $f(0) = 0$; the proof is easy if you aply the right theorem.
3. Give an example of a function f (with domain $\mathbb{R}$) such that f is continuous nowhere and $|f|$ is continuous everywhere.
4. Give a function f which is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \ldots$, but continuous everywhere else.
5. Give a function f which is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \ldots$ and at 0, but continuous everywhere else.
6. Find an integer n such that $x^5 + 5x^4 + 2x + 1 = 0$ for some x between n and $n = +1$.
7. Prove there is some number x such that
$$x^{179} + \frac{163}{1 + x^2 + \sin^2 x} = 0.$$

Exercise 2.

1. Prove that if f is continuous on $[a, b]$ then there is a function F which is defined and continuous on all of $\mathbb{R}$, and such that $F(x) = f(x)$ for $x \in [a, b]$. *HINT*: Since there is a great deal of choice, try making F constant on $(-\infty, a]$ and on $[b, \infty)$.
2. Give an example to show the previous assertion is false if $[a, b]$ is replaced by (a, b). Explain.

Exercise 3.

1. Prove that if a continuous function f defined on $[a, b]$ takes only rational values, then it is constant. *HINT*: If you proof is not very short, it is not the right one.
2. Suppose f and g are continuous on $[a, b]$ and $f(a) < g(a)$ but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some $x \in [a, b]$. *HINT*: If you proof is not very short, it is not the right one.