AA1H, Assignment 4

You may talk to others about the problems. But you should write out the solutions yourself, with no one else indicating what to write, and without help from anyone elses notes. If someone else had a major input in your solution, you should indicate this.

The assignment is due in by May 22, Friday 10 am, in the AA1H collection box in the foyer of the Mathematics Department (so this means before the lecture!).

Read Chapter 4 and Chapter 5 to end of Section 2.

Exercise 1. Suppose $f(x)$ is *continuous* at a. By essentially repeating the proof of Theorem 4.12 show that if $a_n \to a$ then $f(a_n) \to f(a)$. (So this is just a matter of writing out in detail the proof of one direction in Theorem 5.3. Note that we allow $a_n = a$.)

Hence show that $\sin(\sin(1 + (-1)^n \frac{1}{n})) \to \sin(\sin(1))$ as $n \to \infty$.

Exercise 2.

1. Use the $\varepsilon - \delta$ definition of a limit to prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. HINT: Suppose $\varepsilon > 0$ is any positive number, and find a formula for an N (depending on ε) such that $|\frac{n}{n+1} - 1| < \varepsilon$ whenever $n \geq N$.
2. Similarly prove that $\lim_{n\to\infty} \frac{n+3}{n^3+4} = 1$
3. Prove that $\lim_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = 0$. You may use any of the theorems about limits from the notes. HINT: You should first try to prove $\lim_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[8]{n^2} = 0$.
4. What can be said about the sequence (a_n) if it converges and all the a_n are integers?

Exercise 3.

1. Prove that if $0 < a < 2$ then $a < \sqrt{2a} < 2$.
2. Prove that the sequence
$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ldots$$
converges.
3. Find the limit. HINT: Note that if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2L}$ by Theorem 5.3.

Exercise 4. This question gives a general method for finding a formula for the nth term of certain types of sequences.

The Fibonacci sequence is defined by
$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2} \text{ if } n > 2.$$

1. Forgetting for the moment the first two conditions $a_1 = 1$ and $a_2 = 1$, for which r is $a_n = r^n$ a solution of $a_n = a_{n-1} + a_{n-2}$ for all $n > 2$?
2. Prove that if $a_n = f(n)$ and $a_n = g(n)$ are both solutions of $a_n = a_{n-1} + a_{n-2}$ for all $n > 2$, then so is $a_n = \alpha f(n) + \beta g(n)$ for any real numbers α and β.
3. Hence find a formula for the nth term a_n of the Fibonacci sequence. (The answer is on page 17 of the Notes. Note that you are *not* supposed to prove this formula by Induction. The idea here is to actually find the formula.)