

AA1H, ASSIGNMENT 2

You may talk to others about the problems. But you should write out the solutions yourself, with no one else indicating what to write, and without help from anyone else's notes. If someone else had a major input in your solution, you should indicate this.

The assignment is due in by March 27, Friday 10 am, in the AA1H collection box in the foyer of the Mathematics Department (so this means before the lecture!).

Read Sections 2.6, 3.1, and 3.2 to the beginning of Definition 3.13. Also read Section 1.5 of [Adams] to the end of Example 3.

In the following exercises, you should *not* give rigorous proofs from the axioms, *unless* indicated otherwise. You may use all the usual properties of addition, multiplication, subtraction, division, and the usual properties of inequalities. However, you *should* indicate whenever the Completeness Axiom is required.

Exercise 1. Let $f(x) = 1/(1+x)$ (so by our conventions, the domain of f is $\{x : x \neq -1\}$).

1. What is the domain of the function g given by $g(x) = f(f(x))$? Give as simple an expression as you can for $g(x)$ on its domain.
2. What is the domain of the function h given by $h(x) = f(1/x)$? Give as simple an expression as you can for $h(x)$ on its domain.

Exercise 2. A function f is a *polynomial function of degree n* if there exist numbers a_0, a_1, \dots, a_n such that

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n.$$

1. If x_1, \dots, x_n are distinct numbers, find a polynomial function of degree $n-1$ which is 1 at x_i and 0 at x_j for $j \neq i$. *HINT:* The product of all $(x-x_j)$ for $j \neq i$, is 0 at x_j if $j \neq i$. This product is usually denoted by

$$\prod_{\substack{j=1 \\ j \neq i}}^n (x-x_j).$$

2. Now find a polynomial function of degree $n-1$ such that $f(x_i) = b_i$ for $i = 1, \dots, n$, where b_1, \dots, b_n are given numbers. (You should use the functions f_i from part 1. The formula you will obtain is called the *Lagrange interpolation formula*.)

Exercise 3. With Example 3 on page 85 of [Adams] as a guide, use the formal definition of a limit to prove that

$$\lim_{x \rightarrow 3} x^2 = 9.$$

Exercise 4. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ |x| & \text{if } x \text{ is rational} \end{cases}$$

(Draw a diagram. Note that if $-1 < x < 1$ then $x^2 < |x|$.)

Use the formal definition of a limit to prove that

$$\lim_{x \rightarrow 0} f(x) = 0.$$

Exercise 5. Lay, Q32, Q34 page 12.

Exercise 6. Lay, Q26 p24. *First* find the reduced row echelon form. *Then* answer the question.

Exercise 7. Read and understand the solution to Lay, Q31 p24. Then do Lay, Q30 p24.

Exercise 8. ★¹

This is for extra credit only. You may also take an extra week to try it.

Let A and B be two sets (not necessarily different).

1. Suppose $f : A \rightarrow B$ and $g : B \rightarrow A$. Suppose also that $g \circ f(a) = a$ for all $a \in A$ (i.e. $g(f(a)) = a$ for all $a \in A$).

(a) Prove that

$$\text{if } a_1 \neq a_2 \text{ then } f(a_1) \neq f(a_2).$$

We say that the function f is *one to one*.

(*HINT: Suppose P and Q are two statements. Then “ P implies Q ” means that “if P is true then Q is true”. This is equivalent to “if Q is false then P is false”.*)

We say that

$$P \Rightarrow Q \text{ is equivalent to } \neg Q \Rightarrow \neg P.$$

Thus one possible way of establishing the required result is to prove that if $f(a_1) = f(a_2)$ then $a_1 = a_2$.

(b) Prove that

every $a \in A$ can be written in the form $g(b)$ for some $b \in B$.

In other words, the range of g is A . We say that g is *onto*.

2. Suppose $f : A \rightarrow B$ and f is one to one. Prove that there is a function $g : B \rightarrow A$ with the property that $g \circ f(a) = a$ for all $a \in A$.

3. Suppose $g : B \rightarrow A$ and g is onto. Prove that there is a function $f : A \rightarrow B$ with the property that $g \circ f(a) = a$ for all $a \in A$.

¹★ denotes a question that is even more challenging than the others, and is for extra credit. But you should still try it, or at least think about it in the bus!

But in any case, read and study the HINT. It is a very important logical point.