AA1H, Assignment 1

You may talk to others about the problems. But you should write out the solutions yourself, with no one else indicating what to write, and without help from anyone elses notes. If someone else had a major input in your solution, you should indicate this.

The assignment is due in by March 20, 10 am Friday, in the AA1H collection box in the foyer of the Mathematics Department (so this means before the lecture!).

Read Sections 2.1–2.5.

In the following exercises, you should *not* give rigorous proofs from the axioms, *unless* indicated otherwise. You may use all the usual properties of addition, multiplication, subtraction, division, and the usual properties of inequalities. However, you *should* indicate whenever the Completeness Axiom is required.

Exercise 1.

1. Explain why for every integer $n > 0$ one has
$$n = 3m,\ 3m + 1 \text{ or } 3m + 2$$
for some integer m.
2. Show that the square of any integer of the first kind is again of the first kind, but that the square of any integer of the second or third kind is of the second kind.
3. Prove that $\sqrt{3}$ is not rational, by a similar argument to that in Theorem 2.11 (except that now, instead of working with odd and even integers, one works with integers of the first, second, or third kind.)

Exercise 2. Prove from the axioms that *if a, b and c are real numbers, $c \neq 0$, and $ac = bc$, then $a = b$.* HINT: Use an argument similar to that in Theorem 2.2.

Exercise 3. Prove from the axioms that for any number a, $a0 = 0$. HINT: This is just Theorem 2.3.2, and is done on page 18. So all you have to do is to justify the 4 steps in terms of the axioms, rules of logic, or something else already proved from the axioms.

Exercise 4. *An exercise in clear thinking.* For each of the following statments, say if it is true or false, and give a short justification.

1. For every real number m there exists a real number n such that $m < n$.
2. There exists a real number n such that for every real number m, $m < n$.
3. For every human x there exists (or existed) a human y such that x is the child of y.
4. There exists (or existed) a human y such that for every human x, x is the child of y.

Exercise 5. For each of the following sets, decide if they have a least upper bound, and if so say what it is. Do the same for the greatest lower bound.

Also, decide if the least upper bound and greatest lower bound are members of the set.

No explanation is needed, just give the answers.

1. $\{ 1/n : n \in \mathbb{N} \}$.
2. $\{ 1/n : n \in \mathbb{Z} \text{ and } n \neq 0 \}$.
3. $\{ x : x = 0 \text{ or } x = 1/n \text{ for some } n \in \mathbb{N} \}$.
4. $\{ x : 0 \leq x \leq \sqrt{2} \text{ and } x \text{ is rational } \}$.
5. $\{ x : x^2 + x + 1 \geq 0 \}$.
6. $\{ x : x^2 + x - 1 < 0 \}$.
7. $\{ x : x < 0 \text{ and } x^2 + x - 1 < 0 \}$.
8. $\{ 1/n + (-1)^n : n \in \mathbb{N} \}$.

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HINT: In most cases it may help if you write out an alternative description of the set in question.

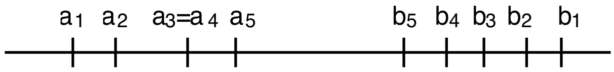
Exercise 6. Consider a sequence of closed intervals

$$I_1 = [a_1, b_1],\ I_2 = [a_2, b_2],\ I_3 = [a_3, b_3],\ \dots\ , I_n = [a_n, b_n],\ \dots$$

which are *nested*, that is

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots \supset [a_n, b_n] \supset \cdots .$$

(By $A \supset B$ we mean that B is a subset of A; that is every member of B is also a member of A. This allows the possibility that A and B have the same members, i.e. $A = B$.)



Being nested is the same as

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots \quad \cdots \le b_n \le \cdots \le b_3 \le b_2 \le b_1.$$

In particular, every a_n is $\le$ every b_m.

Prove there exists a real number x which is in every I_n. HINT: You will need the Completeness Axiom. You should define x to be the least upper bound of a suitable set of real numbers.

Give an example to show that the result is not true if the intervals are (non-empty) *open* intervals (a, b). HINT: Remeber that we do not require $a_1 < a_2 < \dots$, only that $a_1 \le a_2 \le \dots$.