SYMBOLIC FUNCTIONAL CALCULUS AND N BODY RESOLVENT ESTIMATES

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Abstract. We construct a functional calculus for symbolic functions of 'scattering pseudodifferential operators' on manifolds with boundary. This is applied to obtain mapping results on the scattering wavefront set of the B operator used by Gérard-Isozaki-Skibsted [4] to obtain N body resolvent estimates. As an application, we construct the plane waves for an N body, short range Schrödinger operator and obtain a scattering wavefront set estimate.

1. Introduction

The aim of this paper is to make the connection between the N body resolvent estimates of Gérard-Isozaki-Skibsted [4] and the scattering wavefront set of Melrose. The N body resolvent estimates of Gérard-Isozaki-Skibsted are estimates on the operator norm of $P\,R(\lambda+i\epsilon)\,Q$ which are uniform as $\epsilon \to 0$, where $R(\lambda)$ is the resolvent family for an N body Schrödinger operator and P and Q are auxiliary operators. This allows the limiting absorption principle to be applied to get estimates for the resolvent on the real axis.

To illustrate the worth of doing this we construct free channel plane waves for N body, short range Schrödinger operators. It is shown in [8] that these plane waves coincide (up to normalization) with those constructed by Herbst and Skibsted [6].

Gérard, Isozaki and Skibsted obtain their estimates using functions of the operator B (see (28)), a first order differential operator on $\mathbb{R}^n$ which roughly looks like $ir\partial_r$ near $r=0$ and $i\partial_r$ near $r=\infty$ (here $r=|z|$, $z\in\mathbb{R}^n$). Since it suffices to consider symbolic functions $f(B)$, $f\in S^m$, we make a study of symbolic functions of scattering pseudodifferential operators (see [13]) which occupies sections two and three of the paper. We include a discussion of the case of closed manifolds in section two, even though the results are well known, since we have a proof for which the generalization to scattering pseudodifferential operators is transparent. Some of our results are contained in the book of Dereziński and Gérard [1], but we prove results in slightly greater generality and discuss the 'classical' behavior of the resulting pseudo-differential operators as well as the effect of microlocal assumptions.

We show that a suitable microlocalization of $f(B)$ is in the scattering calculus. By combining this result with the resolvent estimates of Gérard, Isozaki and Skibsted, we are able to carry out a direct construction of N-body plane waves. That is, we take as first approximation the free plane wave $e^{-i\lambda z\cdot\theta}$ and apply the resolvent to the error.

The resolvent estimates of Gérard, Isozaki and Skibsted effectively exploit only the τ variable (dual to the radial function r) of the scattering cotangent variables.

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We interpret our results as showing that the resolvent estimates of Gérard, Isozaki and Skibsted are 'partially microlocal' in terms of the scattering wavefront set. Partially microlocal estimates are sufficient for many purposes; for example, to define radiation estimates or in obtaining uniqueness theorems ([11], or Proposition 17.8 of [20]). However, it appears to be necessary to obtain fully microlocal estimates in order to get precise results on the singularities of the N-body scattering matrix (as was done in [7] and [18] for $N = 3$).

2. Symbolic functional calculus — Manifolds without boundary

To obtain results about symbolic functions of self adjoint operators, we will follow Helffer-Sjöstrand [9], Gérard [3], Gérard-Isozaki-Skibsted [4], Dereziński-Gérard [1] and Vasy [20] and use the technique of almost analytic extensions. Let S^k be the class of symbols on $\mathbb{R}$, ie functions satisfying the estimates

$$|\partial_t^l f(t)| \leq C_k \langle t \rangle^{k-l}$$

for $k = 0, 1, 2 \ldots$. An almost analytic extension of f is a function $F(\lambda)$, where $\lambda \in \mathbb{C}$ such that $F(\lambda)$ agrees with f on the real axis and satisfies

$$\begin{gathered} \operatorname{supp} F \subset \{\lambda \mid |\operatorname{Im}\lambda| < 1 + |\operatorname{Re}\lambda|\}; \\ |\overline{\partial} F| \leq C_N \, |\operatorname{Im}\lambda|^N \langle \lambda \rangle^{k-1-N} \quad N = 0, 1, 2 \ldots \end{gathered} \tag{1}$$

It is well known that every symbol has an almost analytic extension.

Suppose A is a self-adjoint operator on a Hilbert space H. Helffer and Sjöstrand used the almost analytic extension F of a symbol f to derive an expression for $f(A)$ in terms of the resolvent of A which is very useful for deducing properties of $f(A)$. Assume $-1 \leq k < 0$. Expressing the spectral measure in terms of boundary values of the resolvent and using the estimate (1) for F and Stokes' theorem yields Helffer-Sjöstrand's formula

$$f(A) = \frac{1}{2\pi} \int_{\mathbb{C}} \overline{\partial_\lambda} F(\lambda)(A - \lambda)^{-1} d\lambda d\overline{\lambda}. \tag{2}$$

The purpose of this section is to give a self contained proof that symbolic functions of elliptic pseudodifferential operators (Ψ.d.o.s) of nonnegative order are Ψ.d.o.s, on a closed manifold. This is well known ([2], [17]) but we will give a fairly short proof which generalizes easily to various calculi of Ψ.d.o.s on manifolds with boundary, in particular the scattering calculus.

Theorem 1. *Let M be a closed manifold and suppose that $A \in \Psi^m(M)$. Let a be the principal symbol of A.*

(i) Suppose $m > 0$, and A is self-adjoint and elliptic. Assume $f \in S^k$. Then $f(A)$ is a Ψ.d.o. of order km, and the principal symbol of $f(A)$ is $f(a)$.

(ii) Suppose $m = 0$, and A is self-adjoint (ellipticity is not assumed). Assume f is a smooth function in a neighbourhood of the spectrum of A. Then $f(A)$ is a Ψ.d.o. of order zero, and the principal symbol is $f(a)$.

Remark. In case (ii), by [10], Theorem 18.1.11, A is a bounded operator on L^2, so the spectrum is compact. Since $f(A)$ is determined by f restricted to the spectrum of A, we only need to specify the behaviour of f on a compact set.

Proof. To prove (i):

Initially let us assume that $-1 \leq k < 0$. The extension to general k is then not hard.

The first step is to construct a family of operators $P(\lambda)$ whose total symbol is $(a(x,\xi)-\lambda)^{-1}$. This is an approximation to $(A-\lambda)^{-1}$; later we will will estimate the error $(A-\lambda)^{-1}-P(\lambda)$. Throughout this section, let us fix a quantization map Q from total symbols to Ψ.d.o.s. Thus, Q is specified by a particular choice of finite open cover of M by coordinate patches, together with a particular partition of unity subordinate to this cover and particular coordinate charts for each open set. Then the kernel of the Ψ.d.o. is given by the sum of Weyl quantization of the total symbol expressed in the local coordinates after being localized via the partition of unity. The map Q is a continuous map from the symbol class S^m to the set of Ψ.d.o.s of order m topologized via Hörmander's space of conormal distributions $I^m(M^2;\Delta)$ conormal with respect to the diagonal (see [10], section 18.2).

Since $A\in\Psi^m$, it can be expressed as $A=Q(a)+T$, where a is a symbol of order m on $T^*(M)$ and T has smooth kernel. Define

$$P(\lambda)=Q\big((a-\lambda)^{-1}\big),\quad \operatorname{Im}\lambda\neq 0. \tag{3}$$

Since we are using Weyl quantization, a is real. This allows us to deduce symbol estimates:

Lemma 2. *Let $\|\cdot\|_{\alpha,\beta;m'}$ denote the seminorm on the αth x derivative and βth ξ derivative in the symbol class of order m'. Then the symbol, $(a-\lambda)^{-1}$, of $P(\lambda)$ obeys the estimates*

$$\|(a-\lambda)^{-1}\|_{\alpha,\beta;-m'}\leq C(\alpha,\beta,m,m')\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)^{|\alpha|+|\beta|+1}\langle\lambda\rangle^{m'/m-1} \tag{4}$$

for $0<m'\leq m$.

Proof. First we show

$$|(a-\lambda)^{-1}|\leq C'\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)(\langle\xi\rangle+\langle\lambda\rangle^{1/m})^{-m}. \tag{5}$$

This is equivalent to

$$|(a-\lambda)|\geq C'\left(\frac{|\operatorname{Im}\lambda|}{\langle\lambda\rangle}\right)(\langle\xi\rangle+\langle\lambda\rangle^{1/m})^{m}.$$

We prove this by dividing into three cases. Since A is elliptic, we have $|a|\geq C\langle\xi\rangle^m$ for $\langle\xi\rangle\geq R$. If $\langle\xi\rangle\leq R$ then we can estimate

$$|(a-\lambda)|\geq|\operatorname{Im}\lambda|\geq C|\operatorname{Im}\lambda|\frac{(R+\langle\lambda\rangle^{1/m})^m}{\langle\lambda\rangle}.$$

If $\langle\xi\rangle\geq R$ and $C\langle\xi\rangle^m\geq 2\langle\lambda\rangle$ then we can estimate

$$|(a-\lambda)|\geq\frac{C}{2}\langle\xi\rangle^m\geq\frac{C}{4}(\langle\xi\rangle+\langle\lambda\rangle^{1/m})^m\geq\frac{C}{4}\left(\frac{|\operatorname{Im}\lambda|}{\langle\lambda\rangle}\right)(\langle\xi\rangle+\langle\lambda\rangle^{1/m}).$$

Finally if $\langle\xi\rangle\geq R$ and $C\langle\xi\rangle^m\leq 2\langle\lambda\rangle$ then we can estimate

$$|(a-\lambda)|\geq|\operatorname{Im}\lambda|\geq C'|\operatorname{Im}\lambda|\frac{\big(\langle\lambda\rangle^{1/m}(1+(2C^{-1})^{1/m})\big)^m}{\langle\lambda\rangle}.$$

This establishes (5). To estimate the $(0,0)$ seminorm in $S^{-m'}$ we estimate

$$C'\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)\sup_{\langle\xi\rangle}\langle\xi\rangle^{m'}(\langle\xi\rangle+\langle\lambda\rangle^{1/m})^{-m}.$$

The function $t \mapsto t^{m'}(t + \langle\lambda\rangle^{1/m})^{-m}$ is nonnegative on $[0,\infty)$ and is equal to zero at $t = 0$ and ∞ (since we are assuming $0 < m' \le m$). By elementary calculus there is one critical point, where the value is $C(m,m')\langle\lambda\rangle^{m'/m-1}$. This shows (4) for $\alpha = \beta = 0$.

A derivative $\partial_x^\alpha \partial_\xi^\beta (a-\lambda)^{-1}$ is a sum of terms of the form

$$\frac{\partial_x^{\gamma_1}\partial_\xi^{\mu_1} a \dots \partial_x^{\gamma_l}\partial_\xi^{\mu_l} a}{(a-\lambda)^{-(l+1)}}$$

where $\sum_{i=1}^l |\gamma_i| = |\alpha|$ and $\sum_{i=1}^l |\mu_i| = |\beta|$. Therefore we get an estimate

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta (a-\lambda)^{-1}| &\le C(\alpha,\beta)\left(\frac{|\operatorname{Im}\lambda|}{\langle\lambda\rangle}\right)^{l+1} (\langle\xi\rangle + \langle\lambda\rangle^{1/m})^{-m(l+1)}\langle\xi\rangle^{ml-|\beta|} \\ &\le C(\alpha,\beta)\left(\frac{|\operatorname{Im}\lambda|}{\langle\lambda\rangle}\right)^{|\alpha|+|\beta|+1} (\langle\xi\rangle + \langle\lambda\rangle^{1/m})^{-m}\langle\xi\rangle^{-|\beta|}. \end{aligned} \tag{6}$$

By the same method as above, we deduce the seminorm estimate (4). □

Consider the integral

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\partial} F(\lambda) Q((a-\lambda)^{-1}) d\lambda d\overline{\lambda}. \tag{7}$$

In the symbol class $S^{-m'}$, (1) and (4) show that the integral is absolutely convergent in each seminorm, provided $0 < m' < -km$. Therefore, in $\Psi^{-m'}$, $-m' > km$, we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\partial} F(\lambda) Q((a-\lambda)^{-1}) d\lambda d\overline{\lambda} &= Q\left(\frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\partial} F(\lambda)(a-\lambda)^{-1} d\lambda d\overline{\lambda}\right) \\ &= Q(f(a)). \end{aligned} \tag{8}$$

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But it is straightforward to check that $f(a)$ is a symbol of order mk, so $Q(f(a))$ is an element of Ψ^{mk}. Thus, replacing $(A-\lambda)^{-1}$ by $P(\lambda)$ gives a Ψ.d.o. of the correct order, with the desired symbol.

Now we estimate the error made in replacing $(A-\lambda)^{-1}$ with $P(\lambda)$. Let

$$(A-\lambda)P(\lambda) = \operatorname{Id} - E(\lambda). \tag{9}$$

By the Weyl symbol calculus, $E(\lambda)$ is a family of Ψ.d.o.s of order -2, and $(A-\lambda)^{-1} - P(\lambda) = (A-\lambda)^{-1}E(\lambda)$, so we can estimate the difference by deriving estimates for $(A-\lambda)^{-1}$ and $E(\lambda)$. First consider $(A-\lambda)^{-1}$.

Lemma 3. *The operator norm of $(A-\lambda)^{-1}$ from H^l to $H^{l+k'm}$, $0 \le k' \le 1$, is bounded by*

$$\|(A-\lambda)^{-1}\|_{H^l \to H^{l+k'm}} \le C\frac{\langle\lambda\rangle^{k'}}{|\operatorname{Im}\lambda|}.$$

Proof. Conjugating by $(1+A^2)^{-l/2}$, which commutes with $(A-\lambda)^{-1}$ and is a continuous bijection between L^2 and H^l, we are reduced to the case $l = 0$.

For $k'=0$ the lemma follows from the self-adjointness of A. For $k'=1$ we can make the following calculation; $\|\cdot\|_{\mathrm{op}}$ denotes operator norm on L^2.

(10)
$$\begin{aligned}\|(A-\lambda)^{-1}\|^2_{L^2\to H^m} &\le C\|(A-\lambda)^{-1}(1+A^2)^{1/2}\|^2_{\mathrm{op}}\\ &= C\|(A-\overline{\lambda})^{-1}(1+A^2)(A-\lambda)^{-1}\|_{\mathrm{op}}\\ &= C\|(A-\overline{\lambda})^{-1}\big((A-\overline{\lambda})(A-\lambda)+\overline{\lambda}(A-\lambda)+\lambda(A-\overline{\lambda})+2|\lambda|^2+1)\big)(A-\lambda)^{-1}\|_{\mathrm{op}}\\ &\le C\|\operatorname{Id}+\overline{\lambda}(A-\lambda)^{-1}+\lambda(A-\overline{\lambda})^{-1}+(2|\lambda|^2+1)(A-\lambda)^{-1}(A-\overline{\lambda})^{-1}\|_{\mathrm{op}}\\ &\le 3C\Big(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\Big)^2.\end{aligned}$$

For other values of k' it follows by interpolation between the extreme cases. ☐

To get estimates on $E(\lambda)$ we follow Seeley [16] and obtain bounds on the operator norm of $E(\lambda)$ acting between Sobolev spaces from the symbol of $E(\lambda)$. Since $E(\lambda)=AP(\lambda)-\lambda P(\lambda)-\operatorname{Id}$, in the local coordinates associated to the quantization Q, the full symbol of $E(\lambda)$ is given by

$$e(x,\xi,\lambda)=e^{i\sigma(D_x,D_\xi;D_y,D_\eta)/2}a(x,\eta)(a(y,\xi)-\lambda)^{-1}\restriction_{x=y,\xi=\eta}-\lambda(a(y,\xi)-\lambda)^{-1}-1,$$

where $\sigma(t,u;v,w)$ is the symplectic form $u\cdot v-t\cdot w$. By [10], Theorem 18.5.4, the symbol $e(x,\xi,\lambda)$ of $E(\lambda)$ has an expansion

$$e(x,\xi,\lambda)\sim\sum_{j\ge1}\big(i\sigma(D_x,D_\xi;D_y,D_\eta)/2\big)^j\frac{1}{j!}a(x,\eta)(a(y,\xi)-\lambda)^{-1}\restriction_{x=y,\xi=\eta},$$

the meaning of the expansion being that the difference between e and the sum up to $j=l-1$ is a symbol of order $-l$, with seminorm estimates

$$\|e(x,\xi,\lambda)-\sum_{j=1}^{l-1}\frac{\big(i\sigma(D_x,D_\xi;D_y,D_\eta)\big)^j}{2^j j!}a(x,\eta)(a(y,\xi)-\lambda)^{-1}\restriction_{x=y,\xi=\eta}\|_{\alpha,\beta,-l}\le C(\lambda),$$

where $C(\lambda)$ is given by a finite sum of products of seminorms of a and $(a-\lambda)^{-1}$. It follows from (4) that we have an estimate

$$\|e(x,\xi,\lambda)\|_{\alpha,\beta;-1}\le C\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)^K \tag{11}$$

where K depends on α and β, and C is now independent of λ.

Lemma 4. *We have*

$$\|E(\lambda)\|_{H^l\to H^{l+1}}\le C\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)^{K(l)}. \tag{12}$$

Proof. By [16], Lemma 1, we have an estimate on the operator norm of an operator K mapping from H^l to H^{l+1} if we have an estimate on the kernel $\tilde K$ acting on the Fourier transform side:

$$|\tilde K(\tau,\eta)|\le C(p)(1+|\tau-\eta|)^{-p}(1+|\eta|)^{-1},\quad p=n+1+|l+1|. \tag{13}$$

Here $\tilde K(\hat f)=\widehat{Kf}$; K is an operator on $\mathbb{R}^n$. Write $E(\lambda)$ as the left quantization of a symbol e_L; it follows from the formula

$$e_L(x,\xi,\lambda)=e^{i\langle D_x,D_\xi\rangle/2}e(x,\xi,\lambda)$$

and Theorem 18.1.7 in Hörmander [10], that e_L satisfies the same estimates (11) as does e (possibly with larger powers of $\langle\lambda\rangle|\operatorname{Im}\lambda|^{-1}$ on the right hand side). With $\hat{}$ denoting Fourier transform in x, we have

$$\tilde{E}(\lambda)(\tau,\eta) = e_L\widehat{(x,\eta,}\lambda)(\tau-\eta).$$

The seminorm estimate (11) for suitable $\alpha = \alpha(p)$ and $\beta = 0$ implies the decay estimate (13) for $\tilde{E}(\lambda)$. $\square$

Combining the two estimates, we see that $(A-\lambda)^{-1} - P(\lambda) = (A-\lambda)^{-1}E(\lambda)$ is bounded as an operator from H^l to $H^{l+k'm+1}$, for $0 \le k' < -k$ with bound

$$C|\operatorname{Im}\lambda|^{-1}\langle\lambda\rangle^{k'}\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)^{K(l)}.$$

Thus in the integral

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\partial}F(\lambda)\big((A-\lambda)^{-1} - P(\lambda)\big)\,d\lambda d\overline{\lambda}, \tag{14}$$

the integrand is bounded as an operator from H^l to $H^{l+1+k'm}$ by $\langle\lambda\rangle^{k-2}\langle\lambda\rangle^{k'} = \langle\lambda\rangle^{-2-(k+k')}$, which is integrable. Thus, the difference between $f(A)$ and $Q(f(a))$ is a bounded operator from L^2 to $H^{1+k'm}$, for $0 \le k' < -k$. This shows that, provided $f(A)$ is actually a Ψ.d.o., then its symbol is equal to the symbol of $Q(f(a))$, which is $f(a)$.

It remains to show that $f(A)$ is actually a pseudodifferential operator. To do this we use a partial Neumann series in $E(\lambda)$ to get better and better approximations to $(A-\lambda)^{-1}$. Let

$$P_N(\lambda) = P(\lambda)(\operatorname{Id} + E(\lambda) + \dots + E(\lambda)^N).$$

Then $(A-\lambda)P_N(\lambda) = \operatorname{Id} - E(\lambda)^{N+1}$. The full Weyl symbol of $P_N(\lambda)$ satisfies the estimates (4), possibly with higher powers of $\langle\lambda\rangle|\operatorname{Im}\lambda|^{-1}$, by (4) for the symbol of $P(\lambda)$, (11), and the symbol calculus. Thus we may form the operator

$$Q_N = \frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\partial}F(\lambda)P_N(\lambda)\,d\lambda d\overline{\lambda}.$$

via a convergent integral in $\Psi^{-m'}$, $-m' > km$ as before. Since the symbol of Q_N is given by the integral of the symbol of $P_N(\lambda)$, Q_N is in Ψ^{km}, and if $N < N'$ then $Q_N - Q_{N'}$ is an operator of order $km - N - 1$.

The error between Q_N and $f(A)$ is given by the integral (14) where $P(\lambda)$ is replaced by $P_N(\lambda)$. Note that $(A-\lambda)^{-1} - P_N(\lambda) = (A-\lambda)^{-1}E(\lambda)^{N+1}$. Iterating (12), we find that

$$\|E(\lambda)^N\|_{H^l \to H^{l+N}} \le C\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)^{K(N,l)}. \tag{15}$$

A similar argument to the one above shows that the difference $f(A) - Q_N$ is bounded as a map between H^l and $H^{l+k'm+N+1}$. Thus, for each N we have a Ψ.d.o. Q_N, such that $Q_N - Q_{N'}$ is of order $\max(mk-N, mk-N')$ and such that $f(A) - Q_N$ is bounded as a map between H^l and $H^{l+k'm+N+1}$. Suppose $N < N'$. It follows that the symbols of Q_N and $Q_{N'}$ agree up to an error in S^{mk-N}. By asymptotic completeness there is a symbol g which agrees with Q_N up to an S^{mk-N}. Let $G = Q(g)$, then by the symbol calculus $f(A) - G$ is bounded as a map H^l to

H^s for every l and s, hence is a smoothing operator. This shows that $f(A)$ is a Ψ.d.o. and completes the proof of part (i) when $-1 \le k < 0$.

To prove the result for other values of k, note that any symbol g may be written

$$g(t) = \left((1+t^2)^{-1/4}\right)^{N_1}(1+t^2)^{N_2}\tilde{g}(t),$$

for nonnegative integers N_1 and N_2 and a symbol $\tilde{g}$ of order between -1 and 0. We have already shown that $(1+A^2)^{-1/4}$ and $\tilde{g}(A)$ are Ψ.d.o.s with the appropriate order and symbol, so by the symbol calculus, the same is true for $g(A)$.

Proof of (ii): This case is somewhat easier, although we do have to check that removing the assumption of ellipticity does not cause problems. Since f can be assumed to be of any order, as only values on a compact interval are relevant, let us decide that $f \in S^{-1}$. We rerun the proof above to the extent possible, pointing out the changes that must be made in this case.

As before, a is real and so $(a-\lambda)^{-1}$ is a symbol for $\operatorname{Im}\lambda \neq 0$. Note that here we gain something from the fact that A is of order 0: precisely in this case, since a is real, $a - \lambda$ is elliptic for $\operatorname{Im}\lambda \neq 0$ regardless of whether a is elliptic.

Lemma 2 must be replaced by

Lemma 5. *The symbol, $(a-\lambda)^{-1}$, of $P(\lambda)$ obeys the estimates*

$$\|(a-\lambda)^{-1}\|_{\alpha,\beta;0} \le C(\alpha,\beta)|\operatorname{Im}\lambda|^{-(|\alpha|+|\beta|+1)}. \tag{16}$$

Then the integral (7) converges in each seminorm for the symbol class S^0. Thus, $Q(f(a))$ is a Ψ.d.o. of order 0.

Lemma 3 must be replaced by

Lemma 6. *For $k \in \mathbb{Z}$, we have the estimate*

$$\|(A-\lambda)^{-1}\|_{H^k \to H^k} \le C\left(|\operatorname{Im}\lambda|^{-1} + |\operatorname{Im}\lambda|^{-|k|-1}\right). \tag{17}$$

Proof. For $k=0$ this follows from self-adjointness of A. To prove (17) for $k=1$, choose $B \in \Psi^1$ such that $B^{-1} \in \Psi^{-1}$. Then

$$\begin{aligned}\|(A-\lambda)^{-1}\|_{H^1\to H^1} &\le C\|B(A-\lambda)^{-1}B^{-1}\|_{\mathrm{op}} \\ &\le C\|(A-\lambda)^{-1} - [(A-\lambda)^{-1}, B]B^{-1}\|_{\mathrm{op}} \\ &\le C\|(A-\lambda)^{-1} + (A-\lambda)^{-1}[A,B](A-\lambda)^{-1}B^{-1}\|_{\mathrm{op}}.\end{aligned}$$

Since $[A,B] \in \Psi^0$ the commutator is a bounded operator on L^2, so the second term is bounded by $C|\operatorname{Im}\lambda|^{-2}$. The same argument works for $k=-1$, exchanging B and B^{-1}. Proceeding inductively in k we prove (17) for every integer k. □

Then the rest of the proof of the theorem goes through. □

From the proof we can deduce some straightforward corollaries.

Corollary 7. *If A and f are both classical, then $f(A)$ is classical.*

Proof. If a and f are classical symbols then it is not hard to check that $f(a)$ is classical. Therefore, Q is classical. Also $P(\lambda)$ is a family of classical operators, so $E(\lambda)$ is too. It follows that the Q_N are all classical, and the symbol of $f(A)$ agrees with that of Q_N up to a symbol of order $mk - 2N$, so it follows that $f(A)$ has classical symbol. □

Corollary 8. *If $A \in \Psi^0$ is self adjoint and classical, and f vanishes on an open set $V \subset \mathbb{R}$, then $f(A)$ is microlocally of order $-\infty$ where $\lim_{t\to\infty} a(x, t\xi) \in V$.*

Proof. Suppose that $\lim_{t\to\infty} a(x_0, t\xi_0) = v_0 \in V$. Choose a smooth function g such that the support of g is contained in V and $g(v_0) = 1$. Then the symbol of $g(A)$ is nonzero on the ray $(x_0, t\xi_0)$ so $g(A)$ is elliptic there. By construction $f(A)g(A) = 0$, so it follows from the calculus of Ψ.d.o.s that $f(A)$ is microlocally of order $-\infty$ in a neighbourhood of this ray. ☐

The proof of the theorem shows that the symbol of $f(A)$ is determined to arbitrary order by local information. Therefore we can expect to get a version of the theorem even when A is not elliptic everywhere.

Corollary 9. *Assume the hypotheses of Theorem 1 (i) hold, except that A is not necessarily elliptic. Suppose that R and R' are zeroth order pseudodifferential operators which are microlocally of order $-\infty$ in a neighbourhood of the characteristic variety of A. Then $Rf(A)R' \in \Psi^{mk}$, with principal symbol $\sigma(R)f(a)\sigma(R')$.*

Proof. Choose a symbol $\chi \in S^0$ such that $\chi - 1$ is of order $-\infty$ in a neighbourhood of $\mathrm{WF}'(R) \cup \mathrm{WF}'(R')$, and χ is of order $-\infty$ in a neighbourhood of $\mathrm{char}(A)$. We construct an approximation where A is elliptic by forming $P_\chi(\lambda) = Q(\chi(a-\lambda)^{-1})$. Then Lemma 2 holds for $P_\chi(\lambda)$. Define $E(\lambda)$ by

$$(A-\lambda)P_\chi(\lambda) = Q(\chi) - E(\lambda).$$

Thus, $E(\lambda) \in \Psi^{-1}$ and satisfies the estimates (11), and hence the mapping estimate of Lemma 4. Define

$$P_{\chi,N}(\lambda) = P_\chi(\lambda)\big(\mathrm{Id} + E(\lambda) + \cdots + E^N(\lambda)\big).$$

Then

$$(A-\lambda)P_{\chi,N}(\lambda) = \mathrm{Id} - E^{N+1}(\lambda) + S_N(\lambda), \tag{18}$$

where $S_N(\lambda) = (Q(\chi) - \mathrm{Id})(\mathrm{Id} + \cdots + E^N)$ satisfies the same seminorm estimates (11) as $E(\lambda)$, and is microlocally of order $-\infty$ near the microlocal support of R and R'. Note that by the composition formula, the symbol of $P_{\chi,N}$ satisfies seminorm estimates

$$\|\sigma(P_{\chi,N}(\lambda))\|_{\alpha,\beta;-m'} \le C(\alpha,\beta,m,m')\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)^{K(\alpha,\beta)} \langle\lambda\rangle^{m'/m-1} \tag{19}$$

for $0 < m' \le m$. Therefore, the integral

$$\frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial} F(\lambda) P_{\chi,N}(\lambda)\, d\lambda d\overline{\lambda}$$

converges absolutely in $\Psi^{m'}$ for $km < m' < 0$. Since the principal symbol is $\chi f(a) \in S^{km}$, we conclude that the integral lies in Ψ^{km}. What remains to show is that

$$Rf(A)R' - R\left(\int \overline{\partial} F(\lambda) P_{\chi,N}(\lambda)\, d\lambda d\overline{\lambda}\right) R'$$

is a Ψ.d.o. with order tending to $-\infty$ as $N \to \infty$. As in the proof above, it suffices to show that the difference is an operator from H^k to H^s with $k \to -\infty$ and $s \to \infty$ as $N \to \infty$.

To do this, we need to consider the action of $(A-\lambda)$ on the right as well as the left, so let

$$P_{\chi,N}(\lambda)(A-\lambda) = \mathrm{Id} - E^N(\lambda) + S_N(\lambda) + F_N(\lambda). \tag{20}$$

Then $F_N(\lambda)$ also satisfies the estimates of Lemma 4. Combining (18) and (20) we find that

$$(A-\lambda)F_N(\lambda) = [E^N(\lambda) - S_N(\lambda), A].$$

It follows that, microlocally where A is elliptic, $F_N(\lambda) \in \Psi^{-N-1}$ and satisfies seminorm estimates (where we multiply to R to microlocalize)

$$\|\sigma(RF_N(\lambda))\|_{\alpha,\beta;-N-1} \leq C(\alpha,\beta,N)\left(\frac{\langle\lambda\rangle}{|\operatorname{Im}\lambda|}\right)^{K(\alpha,\beta)}. \tag{21}$$

Therefore

$$\begin{aligned} R\left((A-\lambda)^{-1} - P_{\chi,N}(\lambda)\right)R' &= R\,(A-\lambda)^{-1}\big(E^N(\lambda) - S_N(\lambda)\big)R' \\ &= R\big(E^N(\lambda) - S_N(\lambda) - F_N(\lambda)\big)(A-\lambda)^{-1}R'. \end{aligned}$$

By Seeley's lemma (Lemma 4), the seminorm estimates on E^N, S_N and F_N give corresponding operator estimates. The first expression for the error is a map from H^{-N-1} to L^2, with a bound $C|\operatorname{Im}\lambda|^{-1}(\langle\lambda\rangle/|\operatorname{Im}\lambda|)^k$, and the second expression is a map from L^2 to H^{N+1} with a similar bound. Interpolating, we find that the error is a map from $H^{-(N+1)/2}$ to $H^{(N+1)/2}$ with a similar bound. Integrating and using (1) with $N=k$, we find that the difference between $Rf(A)R' - R$ and our approximation using $P_{\chi,N}$ is a bounded map from $H^{-(N+1)/2}$ to $H^{(N+1)/2}$. This completes the proof of the corollary. $\square$

3. Manifolds with boundary

In this section we obtain the analogue of Theorem 1 in the scattering calculus. The scattering calculus is a generalization of a certain pseudodifferential calculus on $\mathbb{R}^n$, which we shall describe first.

Consider the set of symbols $a(z,\xi)$ on $\mathbb{R}^n$, z, $\xi \in \mathbb{R}^n$, satisfying estimates

$$|\partial_z^\alpha \partial_\xi^\beta a(z,\xi)| \leq C\langle z\rangle^{-l-|\alpha|}\langle\xi\rangle^{m-|\beta|}. \tag{22}$$

Thus a is symbolic in both z and ξ. Then we say that a is a symbol of order (m,l), and write $a \in S^{m,l}(\mathbb{R}^n)$. The infimum of possible constants C in (22) is a seminorm which will be denoted $\|\cdot\|_{\alpha,\beta;m,l}$.

We will study 'scattering pseudodifferential operators' defined by quantization of these symbols. Thus, we will say that an operator $A : \mathcal{S} \to \mathcal{S}'$ is a scattering pseudodifferential operator of order (m,l), and write $A \in \Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$, if

$$A = A(z,z')|dzdz'|^{\frac12}, \quad A(z,z') = \frac{1}{(2\pi)^n}\int e^{i(z-z')\cdot\xi}a(\frac{z+z'}{2},\xi)\,d\xi.$$

These symbols are conormal in z as $|z| \to \infty$, and as a result the corresponding Ψ.d.o.s have good properties as $|z|, |z'| \to \infty$. As noticed by Melrose [13], this allows the definition of a 'boundary symbol' which lives at spatial infinity, hence a 'scattering wavefront set' at infinity for functions on $\mathbb{R}^n$ which is a powerful tool for studying scattering theory.

To analyze these operators we compactify at spatial infinity. Thus let $X^n = \overline{\mathbb{R}^n}$ be the radial compactification of $\mathbb{R}^n$ to an n-ball, with $x = 1/|z|$ as boundary defining function. The boundary is S^{n-1}; we will denote the point $z/|z| \in S^{n-1}$ by

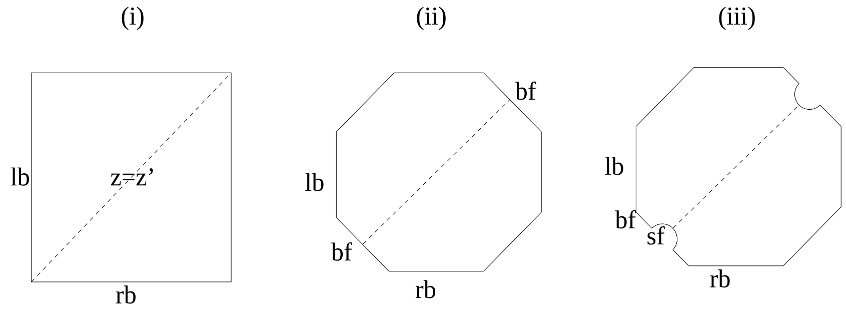

FIGURE 1. Blowing up to produce the double scattering space

$\hat{z}$ and use $y \in \mathbb{R}^{n-1}$ for local coordinates in S^{n-1}. The kernel of A is a distribution on $(X^n)^2$ conormal, in some sense, to the diagonal Δ. However the singularity structure of the kernel at spatial infinity is unclear since on $(X^n)^2$ the diagonal is not a product-type submanifold at infinity (see 1(i) in the figure), and the natural coordinates, $z-z'$, transverse to the diagonal are not smooth near infinity. (We use coordinates z, z' on $(X^n)^2$ where the unprimed coordinates are on the first factor and the primed coordinates are on the second; similarly we use $x, x'; y, y'$.) To remedy this we perform two blowup operations on $(X^n)^2$. We will use the notation

$$[Y; S_1, \dots S_k]$$

to denote the manifold with corners obtained by blowing up the manifold with corners Y around the product submanifold S_1, then blowing up around the lift of the submanifold S_2 (which is required to be a product submanifold), etc.

First we construct $(X^n)^2_b = [(X^n)^2; (\partial X^n)^2]$ illustrated in figure 1(ii). This is blowing up the submanifold $\{x = x' = 0\}$, giving a new face we shall denote bf refer to simply as the 'b-face' since it occurs in the b-calculus [12]. The original faces we shall call lb and rb ('left boundary' and 'right boundary') at $\{x = 0\}$ and $\{x' = 0\}$ respectively. We have new coordinates y, y' and x/x' in the interior of the b-face and the lift of the diagonal is now defined by $\Delta = \{x/x' = 1, y = y'\}$ showing that it is now a product submanifold, meeting bf transversally and disjoint from lb and rb, as in figure 1(ii). To create a space where $z - z'$ are smooth coordinates when $|z - z'| \le 1$ say, up to the boundary we need to blow up the boundary of Δ. This space we call the double scattering space and denote $(X^n)^2_{\rm sc}$:

$$(X^n)^2_{\rm sc} = [(X^n)^2; (\partial X^n)^2; \partial\Delta]. \tag{23}$$

See figure 1(iii). This means blowing up $\{x/x' = 1, y = y', x = 0\}$ in $(X^n)^2_b$, creating a new face sf ('scattering face') with coordinates $((y - y')/x, (x - x')/xx', y)$ in the interior of the face. Since

$$z - z' = \frac{\hat{z}}{x} - \frac{\hat{z}'}{x'} = \frac{\hat{z} - \hat{z}'}{x} + \hat{z}\frac{x - x'}{xx'},$$

we see that $z - z'$ is smooth in the interior of sf. In fact sf has the structure $X^n \times S^{n-1}$, with coordinate $z - z'$ for the first factor and y for the second.

To understand this space better it is helpful to construct it in various different ways. Another way is to start with X^{2n}, the compactification of $\mathbb{R}^{2n}$ with coordinates (z, z'), and then blow up the submanifolds $\{z = 0, x' = 0\}$ and $\{z' = 0, x = 0\}$. This produces something identical to $(X^n)^2_b$, although it is now lb and rb which

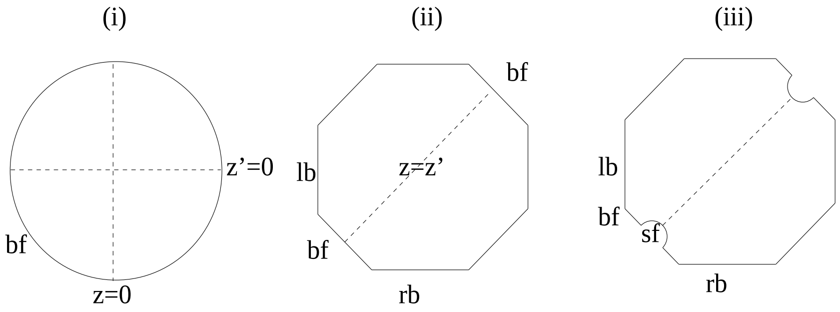

FIGURE 2. Another way to produce the scattering double space

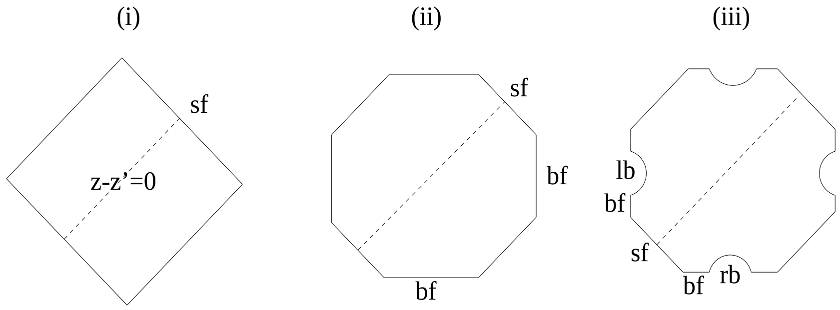

FIGURE 3. A third way to produce the scattering double space

are the blown-up faces and bf which is the original face. We then do the second blowup as before. See figure 2(i) – 2(iii).

A third way is to start with $(X^n)^2$ but with coordinates $z - z'$ and $z + z'$ on the two factors, and then blow up the corner. This blowup produces bf. If then the submanifolds which are the intersections of the boundary with $\{z = 0\}$ and $\{z' = 0\}$, then we get the faces lb and rb. The scattering face sf is present from the beginning. See figure 3(i) – 3(iii). (Actually these three spaces are only identical for $n \geq 2$; when $n = 1$ they differ by some off-diagonal blowups, due to the fact that the boundary of $\overline{\mathbb{R}^n}$ is not connected when $n = 1$. Unfortunately this is the only case we can illustrate!)

On the space $(X^n)^2_{\mathrm{sc}}$ we can give an intrinsic description of the kernels of scattering Ψ.d.o.s. Namely, an operator $A : \mathcal{S} \to \mathcal{S}'$ is a scattering pseudodifferential operator of order (m, l) if and only if its kernel is conormal of order m at the diagonal, conormal of order l at sf and rapidly vanishing at lb, rb and bf.

To simplify discussion of the symbol in this calculus, let us restrict attention to operators with classical symbols, that is, with complete asymptotic expansions in both z and ξ. Suppose A is of order $(m, 0)$. Write $z = |z|\hat{z} = \hat{z}/x$. Then the symbol of A, $a(x, \hat{z}, \xi)$, has a limit $a(0, \hat{z}, \xi)$ as $x \to 0$. It can be determined from the kernel of A by restricting the kernel to $y = y_0$ on sf and taking inverse Fourier transform in the variable $z - z'$. It is a fundamental observation of Melrose that $a(0, \hat{z}, \xi)$ is a completely well defined function on $T^*(\mathbb{R}^n)$ extended and then restricted to the sphere at infinity; it is not just well defined modulo symbols of order $m-1$. This is called the (principal) boundary symbol of A, $\sigma_b(A)(\hat{z},\xi)$. We also have the usual interior symbol, which is $a(z,\xi)$ modulo symbols of order $m-1$ in ξ. The pair consisting of the interior symbol and the boundary symbol (which satisfy a compatibility condition at $|z|=|\xi|=\infty$) constitute the 'joint symbol' of A. (In [13] this is packaged a little differently; the total symbol is regarded as the function $\langle z\rangle^l\langle\xi\rangle^{-m}a(z,\xi)$ restricted to the boundary of $X^n_z \times \overline{\mathbb{R}^n}_\xi$; the restriction to z-infinity is the boundary symbol and the restriction to ξ-infinity is the interior symbol. Compatibility is simply equality at the intersection of the two boundaries.) In the general, not necessarily classical, case we can think of the joint symbol of $A \in \Psi^{m,l}$ as being $a(z,\xi)$ modulo symbols in $S^{m-1,l-1}$. The joint symbol of A determines A up to operators one order lower in both indices; in fact there is a short exact sequence:

$$0 \to \Psi_{\mathrm{sc}}^{m-1,l-1} \to \Psi_{\mathrm{sc}}^{m,l} \to S^{m,l}/S^{m-1,l-1} \to 0. \tag{24}$$

The joint symbol is also multiplicative. This can be seen by considering the asymptotic expansion for the operation on symbols corresponding to composition of operators. Since we shall need this formula in the proof of the theorem in this section, let us take the time to consider it.

Lemma 10. *If $A_1 \in \Psi^{m_1,l_1}$ as symbol a_1 and $A_2 \in \Psi^{m_2,l_2}$ has symbol a_2, then $B = A_1A_2$ has symbol*

$$b(z,\xi) = e^{i\sigma(D_w,D_\xi;D_z,D_\eta)/2}a_1(z,\eta)a_2(w,\xi)\restriction_{z=w,\xi=\eta} \in S^{m_1+m_2,l_1+l_2}$$

and has an asymptotic expansion

$$b(z,\xi) \sim \sum_{j\geq 1}\big(i\sigma(D_w,D_\xi;D_z,D_\eta)/2\big)^j\frac{1}{j!}a_1(z,\eta)a_2(w,\xi)\restriction_{x=y,\xi=\eta}.$$

The meaning of this is that

$$b - \sum_{j<k}\big(i\sigma(D_w,D_\xi;D_z,D_\eta)/2\big)^j\frac{1}{j!}a_1(z,\eta)a_2(w,\xi)\restriction_{x=y,\xi=\eta}$$

is in $S^{m_1+m_2-k,l_1+l_2-k}$, and each seminorm in the symbol class $S^{m_1+m_2-k,l_1+l_2-k}$ is estimated by a finite sum of products of seminorms of a_1 in S^{m_1,l_1} and a_2 in S^{m_2,l_2}.

Proof. This result is contained in [10], chapter 18, and can be seen in at least two different ways. First, it is a special case on Theorem 18.5.4 where the metric g is

$$g = \langle z\rangle^{-2}dz^2 + \langle\xi\rangle^{-2}d\xi^2.$$

The second way, which is more elementary since it does not rely on the very general machinery of sections 18.4 and 18.5 of [10] is to run through the proof of Theorem 18.1.7 including the decay in the z-derivatives in the estimates (one must use cutoffs in z as well as ξ). □

Thus, we see $b = a_1a_2$ modulo $S^{m_1+m_2-1,l_1+l_2-1}$, and so the principal symbol is multiplicative.

Other elements of the pseudodifferential calculus on closed manifolds generalize to the scattering case. The scattering Ψ.d.o.s are bounded maps on Sobolev spaces; in fact, scattering Ψ.d.o.s of order (m,l) map the weighted Sobolev space $H^{s,k} = \langle z\rangle^{-k}H^s$ to $H^{s-m,k+l}$ [13]. And the usual construction to get a pseudodifferential parametrix to an elliptic operator carries over, provided the operator is 'totally elliptic'.

An operator is said to be totally elliptic if $|a(z,\xi)| \geq C\langle z\rangle^{-l}\langle \xi\rangle^m$ when $\langle z\rangle \geq R$ or $\langle \xi\rangle \geq R$. This is equivalent to the condition that both the interior and the boundary symbol are invertible as symbols. For example, the operator $\Delta - \lambda^2$ on $\mathbb{R}^n$ has interior principal symbol $|\xi|^2$ and boundary symbol $|\xi|^2 - \lambda^2$, so the boundary symbol is invertible if and only if $\operatorname{Im}\lambda \neq 0$. Thus, the Laplacian is *not* totally elliptic in the scattering calculus.

If a, the symbol of A, is a totally elliptic symbol then a^{-1} is defined when ξ is large or when z is large and can be extended to a symbol b of order $(-m,-l)$. Let B be the operator with symbol b. By the symbol calculus just described, $AB - \mathrm{Id}$ is an operator of order $(-1,1)$, hence a bounded operator from L^2 to $H^{1,1}$, hence a compact operator on L^2. Thus B is a parametrix for A, and it may be modified via a Neumann series to an inverse modulo $\Psi^{-\infty,\infty}$ which are operators with Schwartz kernels in $\mathcal{S}(\mathbb{R}^{2n})$.

Now we turn to scattering pseudodifferential operators in general. These may be defined on any manifold with boundary, X. The idea is to use the calculus just described as a local model at infinity (i.e. at the boundary), but to refrain from assuming that the global structure of infinity is as in $\mathbb{R}^n$ (i.e. that the boundary is S^{n-1}).

We may define scattering pseudodifferential operators on X by describing their kernels. To do this we form the double scattering space X^2_{sc} analogously to above:

$$X^2_{\mathrm{sc}} = [X^2; (\partial X)^2; \partial\Delta].$$

The picture is much the same as with $X = X^n$ above: we have boundary hypersurfaces lb, rb, bf and sf and the lifted diagonal is a product submanifold transverse to sf and disjoint from the other boundaries. The class $\Psi^{m,l}(X)$ of scattering pseudodifferential operators of order (m,l) is defined to be the class of operators with kernels on X^2_{sc} which are a scattering half-density times a distribution conormal to the diagonal of order m and conormal to sf, of order l, and rapidly vanishing together with all derivatives at lb, rb, bf. The scattering half density is a density of the form

$$\left|\frac{dx dx' dy dy'}{x^{n+1}x'^{n+1}}\right|^{\frac{1}{2}};$$

this is modelled on the Euclidean case since $dzdz' = (dxdx'd\hat{z}d\hat{z}')/x^{n+1}x'^{n+1}$.

What this means is, when localizing the kernel of A to a coordinate neighbourhood and expressing as an oscillatory integral, there are two possibilities. Either the coordinate patch is contained in the interior of X or it intersects the boundary. In the first case, the localized kernel has the form of a standard Ψ.d.o. on $\mathbb{R}^n$ with compact support in both factors, or it has the form, by changing variable to $z = (1/x, y/x)$ and $z' = (1/x', y'/x')$, of a Ψ.d.o. of the class described in the first part of this section. Thus any properties of the scattering calculus on $\mathbb{R}^n$ which are defined 'locally at infinity' (i.e. may be defined by localizing z and z' to small conic open sets) carry over to the scattering calculus on X. In particular, the boundary symbol and the joint symbol carry over, as does the exact sequence (24), multiplicitivity of the joint symbol, boundedness between Sobolev spaces and construction of parametrices for totally elliptic operators.

The main result of this section is that Theorem 1 generalizes to the scattering calculus. We will not prove this in full generality, but prove the case which seems the most interesting; that is, for operators of order $(m,0)$ with $m>0$. Notice that the truth of the theorem is suggested by the fact that if a is a scattering symbol of order $(m,0)$, with elliptic interior symbol, then $f(a)$ is a symbol of order $(km,0)$.

Theorem 11. *Let X be a manifold with boundary. Suppose that A is a scattering pseudodifferential operator of order $(m,0)$, $m>0$ on X which is self-adjoint and has interior symbol elliptic uniformly up to the boundary. (Note that total ellipticity is not assumed, though the assumptions imply that the boundary symbol is nonzero when $|\xi|$ is large.) Let a be the symbol of A. If $f\in S^k(\mathbb{R})$, then $f(A)$ is a Ψ.d.o. of order $(mk,0)$ which principal joint symbol $f(a)$.*

Proof. The proof follows exactly the same lines as in the previous section. The main difference is that when localizing to a coordinate patch meeting the boundary, one gets a scattering Ψ.d.o. supported in a conic open set in $\mathbb{R}^n$. Thus we must adapt the proof in this situation.

Lemma 2 holds if we replace the the $\|\cdot\|_{\alpha,\beta,m'}$ seminorm by the $\|\cdot\|_{\alpha,\beta,m',0}$ seminorm for $0<m'\leq m$. The proof is almost identical. Thus, $Q(f(a))\in\Psi^{-m',0}$, $-m'>km$ is defined by a convergent integral. Consider $E(\lambda)$ defined as above. Then $E(\lambda)\in\Psi_{\mathrm{sc}}^{-1,1}$ and we still get the estimate (11) for the seminorms in the symbol class $(-1,1)$ due to Lemma 10. Also Lemma 4 holds; to see this first note that the condition (13) may be weakened to the condition

$$|\tilde{K}(\tau,\eta)|\leq C(p)(1+|\tau-\eta|)^{-p}(1+|\eta|)^{-1},\quad |\tau-\eta|\geq 1, \tag{25}$$

$$|\tilde{K}(\tau,\eta)|\leq (|\tau-\eta|)^{-n+\epsilon}(1+|\eta|)^{-1},\quad |\tau-\eta|\leq 1. \tag{26}$$

This follows from the seminorm estimates (11) in the symbol class $(-1,1)$. The rest of the proof goes through without change. □

Using an argument identical in structure to that of the proof of Corollary 9, we can show

Corollary 12. *Assume that A satisfies the hypotheses of Theorem 11, except that the interior symbol is not necessarily elliptic. Suppose that R and R' are scattering pseudodifferential operators of order $(0,0)$ which are microlocally of order $-\infty$ in a neighbourhood of the interior characteristic variety of A (that is, where the interior principal symbol is not elliptic). Then $Rf(A)R'$ is a scattering pseudodifferential operator of order $(mk,0)$ with principal symbol $\sigma(R)f(a)\sigma(R')$.*

We will use this corollary below in order to get a handle on terms which come up while working with the N-body resolvent estimates of Gérard, Isozaki and Skibsted.

4. N-body resolvent estimates

A generalized N body Schrödinger operator is a differential operator H on $\mathbb{R}^D$ which takes the form

$$H=\Delta+V,\quad V=\sum_{k=1}^{M}\pi_k^*(V_k), \tag{27}$$

where π_k is a projection to a subspace X^k of dimension d_k, $1\leq d_k\leq D-1$, and V_k is a function on X^k. It is necessary to assume that the 'two-body potentials' V_k have some decay properties on X^k. We will consider two different decay conditions in this paper. If the V_k are of Schwartz class on X^k, we will say that the two-body potentials are 'very short range'. If they satisfy the weaker condition

$$|D^\alpha_{z'} V_k(z')| \leq C_\alpha \langle z' \rangle^{-1-\epsilon-|\alpha|} \quad \forall \alpha,$$

for some $\epsilon > 0$, then they will be said to be 'short range'. The orthogonal complement of X^k is denoted X_k and called the k-th collision plane. When $\mathbb{R}^D$ is compactified, the intersection of X_k with the boundary will be denoted C_k and called the kth cluster set. The union $C = \cup C_k$ will be called the cluster set. For a very short range operator, its significance is that it is the part of configuration space at infinity where the potential is nontrivial. For a general short range operator, it is the part of configuration space where H fails to be a scattering differential operator (ie, a differential operator that lies in the scattering calculus described in the previous section).

Gérard, Isozaki and Skibsted used the operator

$$B = \frac{1}{2i} \sum_k \Big(\frac{z_j}{\langle z \rangle} \nabla_j + \nabla_j \frac{z_j}{\langle z \rangle} \Big) \tag{28}$$

in conjunction with Mourre-type commutator estimates to obtain N body resolvent estimates. They analyzed certain functions $f(B)$, where f is a classical symbol (for example, f which are zero for $t < t_0$ and one for $t > t_1$). In the next section we will make a slight modification to their estimates in order to be able to use the scattering calculus. The strategy is to introduce suitable microlocalizing operators R, R' so that Corollary 12 applies to $Rf(B)R'$ and so that introducing these terms does not disturb the estimates too much. This will then be used to give a 'direct' construction of N body plane waves. The details are written down for very short range operators, although as we remark at the end, it can easily be modified to treat short range operators. First, however, we state a version of the limiting absorption principle for convenient reference which follows easily from the results of [4, 5].

The estimates of Gérard-Isozaki-Skibsted needed are the following.

Theorem 13 (Gérard-Isozaki-Skibsted)**.** *Assume that $\phi \in C_c^\infty(\mathbb{R})$ satisfies $\phi(\lambda) = 1$ and has support sufficiently close to λ, $F_+ \in S^0$, supported in $(-\lambda + \delta, \infty)$, $F_- \in S^0$, supported in $(-\infty, -\lambda + \delta)$, $t > 1$ and $m > -1/2$. Then*

$$\| X^{-m-t} \phi(H) R(\lambda^2 + i\epsilon) F_+(B) X^m \| \leq C, \tag{29}$$

$$\| X^m F_-(B) \phi(H) R(\lambda^2 + i\epsilon) X^{-m-t} \| \leq C, \tag{30}$$

and for any $s \in \mathbb{R}$,

$$\| X^s F_-(B) \phi(H) R(\lambda^2 + i\epsilon) F_+(B) X^s f \| \leq C, \tag{31}$$

uniformly as $\epsilon \to 0$.

For the proof, see [5], Theorem 2.10. We can actually drop the factor $\phi(H)$ in the estimates, since the difference involves the factor $(1-\phi)(H)R(\lambda^2 + i\epsilon) = \psi(H)$ for some $\psi \in \mathcal{F}^{-1}$. Performing commutations with X in (29) or (30) and with $F_-(B)$ in (31) and using Lemma 2.4 of [5] we obtain the same estimates for $\psi(H)$.

These results can be turned very easily into a version of the limiting absorption principle and a uniqueness theorem for tempered eigenfunctions of H which is microlocal in the sense of the $F_+(B)$ operators of Gérard, Isozaki and Skibsted. We let F_+, F_- be as above, $F_+ \equiv 1$ on $(-\lambda + 2\delta, \infty)$, $F_- \equiv 1$ on $(-\infty, -\lambda +$

$\delta/2)$. We write $\tilde{F}_- = 1 - F_+$ which is thus supported in $(-\infty, -\lambda + 2\delta]$, so it is similar to F_- but has slightly larger support. Also recall the notation $H^{r,s}$ for the weighted Sobolev space $\langle z\rangle^{-s}H^r = \langle z\rangle^{-s}(1+\Delta)^{-r/2}L^2$ on $\mathbb{R}^D$. It follows from the commutator calculus of Gérard-Isozaki-Skibsted that $F_\pm(B)$ are bounded operators on $H^{0,p}$ for all p. The uniqueness statement of the theorem is not optimal; the optimal version would only require $s' > -1/2$, but it is sufficient for our purposes.

Theorem 14. *Suppose that $u \in H^{0,m}$ and $\tilde{F}_-(B)u \in H^{0,s}$ for some $m \in \mathbb{R}$, $s > 1/2$. Then $v_\epsilon = R(\lambda^2 + i\epsilon)u$ converges in $\mathcal{S}'$ to some $v \in H^{0,p}$ as $\epsilon \to 0$ for $p < \min(m-1, -1/2)$; we write $v = R(\lambda^2 + i0)u$ as usual. Moreover, v satisfies $(H - \lambda^2)v = u$, $F_-(B)v \in H^{0,s-t}$ for any $t > 1$; in particular, $F_-(B)v \in H^{0,s'}$ for some $s' > 1/2$ if $s > 3/2$. Furthermore, if $s > 3/2$ then v is the unique element of $\mathcal{S}'$ with the property that $(H - \lambda^2)v = u$ and $F_-(B)v \in H^{0,s'}$ for some $s' > 1/2$.*

Proof. First we decompose u as $u = F_+(B)u + \tilde{F}_-(B)u$. The assumption on $\tilde{F}_-(B)u$, and (29), together with $u = X^m X^{-m}u$, $X^{-m}u \in L^2$, imply that $\|X^{-p}(1+\Delta)v_\epsilon\|_{L^2}$ is bounded as $\epsilon \to 0$ for $p < \min(m-1, -1/2)$, i.e. that v_ϵ is bounded in $H^{2,p}$. Since the inclusion $H^{2,p} \to H^{0,p'}$ is compact for any $p' < p$, we conclude that if $\{\epsilon_j\}_{j=1}^\infty$ is any sequence converging to 0, it has a subsequence $\{\epsilon_{k_j}\}_{j=1}^\infty$ such that $v_{\epsilon_{k_j}}$ converges to some v in $H^{0,p'}$ as $j \to \infty$. Since the sequence $v_{\epsilon_{k_j}}$ is bounded in $H^{0,p}$, we can actually conclude that $v \in H^{0,p}$ and $v_{\epsilon_{k_j}} \to v$ weakly in $H^{0,p}$.

Writing $v_\epsilon = R(\lambda^2 + i\epsilon)(F_+(B) + \tilde{F}_-(B))u$, membership of $F_-(B)v$ in $H^{0,s'}$ follows immediately from (30) and (31). Moreover,

$$(H - \lambda^2)v = \lim_{j\to\infty}(H-\lambda^2)R(\lambda^2 + i\epsilon_{k_j})u = \lim_{\epsilon\to 0}(u + i\epsilon_{k_j}R(\lambda^2 + i\epsilon_{k_j})u) = u. \tag{32}$$

Now suppose that $s > 3/2$ and $v' \in \mathcal{S}'$ also satisfies $F_-(B)v \in H^{0,s'}$, $s' > 1/2$, and $(H-\lambda^2)v' = u$. Let $\tilde{v} = v - v'$, so $(H-\lambda^2)\tilde{v} = 0$ and $F_-(B)\tilde{v} \in H^{0,s'}$. Since $\tilde{v} \in \mathcal{S}'$, it lies in some weighted Sobolev space $H^{r,m}$, but since $(1+\Delta)\tilde{v} = (1+\lambda^2 - V)\tilde{v}$, and $(1 + \lambda^2 + V)$ is a bounded operator on every $H^{r,m}$, it follows iteratively that $\tilde{v} \in H^{0,m}$; indeed, $\tilde{v} \in H^{\infty,m}$. Thus, by the definition of distributional derivatives, for any $\phi \in \mathcal{S}$ we have $\langle \tilde{v}, (H-\lambda^2)\phi\rangle = 0$. Now apply this with $\phi_\epsilon = R(\lambda^2 - i\epsilon)f$, $f \in \mathcal{S}$. For $\epsilon > 0$

$$\begin{aligned} 0 &= \langle \tilde{v}, (H-\lambda^2)R(\lambda^2 - i\epsilon)f\rangle = \langle \tilde{v}, f - i\epsilon R(\lambda^2 - i\epsilon)f\rangle \\ &= \langle \tilde{v}, f\rangle + \langle i\epsilon R(\lambda^2 + i\epsilon)\tilde{v}, f\rangle. \end{aligned} \tag{33}$$

By the first part of the proof, $R(\lambda^2 + i\epsilon)\tilde{v}$ remains bounded in $H^{0,p}$ as $\epsilon \to 0$, so $\epsilon R(\lambda^2 + i\epsilon)\tilde{v}$ converges to zero in $H^{0,p}$. Therefore, the right hand side converges to $\langle \tilde{v}, f\rangle$. Thus, $\langle \tilde{v}, f\rangle = 0$ for all $f \in \mathcal{S}$, so $\tilde{v} = 0$. Hence, $v = v'$ proving the desired uniqueness.

This shows that every convergent subsequence $v_{\epsilon_{k_j}}$ has the same limit v. It follows that the whole family v_ϵ actually converges to v, so we can unambiguously write $v = R(\lambda^2 + i0)u$. □

Remark. The same argument (but considering $\operatorname{Im}\epsilon \neq 0$ explicitly) shows that $R(\lambda^2 + i0)u$, u as above, is a continuous function of λ with values in $\mathcal{S}'$, or in $H^{0,p'}$, or weakly continuous with values in $H^{0,p}$. Also, elliptic regularity (cf. the argument above) shows that $u \in H^{l,m}$ implies $v \in H^{l+2,p}$ for $l \geq 0$.

Remark. The reason for the non-optimal uniqueness statement is that in (33) we use an estimate for $R(\lambda^2+i\epsilon)$ instead of taking advantage of the factor of ϵ and estimating $\epsilon R(\lambda^2+i\epsilon)$.

Remark. Another proof of this theorem would go via the standard version of the limiting absorption principle, [15]. One would show, using Theorem 13 above, that for $f \in \mathcal{S}$ the pairing $\langle v_\epsilon, f\rangle = \langle u, R(\lambda^2 - i\epsilon)f\rangle$ is well defined in the limit $\epsilon \to 0$.

5. Plane waves in N-body scattering

In this section we shall construct plane waves for N-body Schrödinger operators. First, suppose that H is very short range (ie, all two-body potential functions V_k Schwartz). In this case, the plane wave $u_{\theta,\lambda}$ is by definition the unique function satisfying $(H-\lambda^2)u_{\theta,\lambda} = 0$, $u_{\theta,\lambda} = e^{-i\lambda z\cdot\theta} + u'_{\theta,\lambda}$ with $u'_{\theta,\lambda}$ satisfying $F_-(B)u'_{\theta,\lambda} \in H^{0,s}$ for F_- as in the previous section and for some $s > 1/2$.

Let us begin by giving a heuristic argument why $u'_{\theta,\lambda}$ should exist. The error term upon taking $e^{-iz\cdot\theta}$ as approximate plane wave is $Ve^{-iz\cdot\theta}$. This satisfies the wavefront set estimate

$$^{sc}\mathrm{WF}(Ve^{-i\lambda z\cdot\theta}) \subset \{\tau \le \lambda - \delta\}, \quad 1 - \frac{\delta}{\lambda} = \max\{\theta\cdot\hat{z} \mid \hat{z} \in C\}.$$

Suppose for a moment that $F_+(B)$ were a scattering Ψ.d.o. with symbol $F_+(-\tau)$. For suitably chosen F_+, therefore, the function $Ve^{-i\lambda z\cdot\theta}$ would be almost the same as $F_+(B)Ve^{-i\lambda z\cdot\theta}$; in fact, we would have $Ve^{-i\lambda z\cdot\theta} - F_+(B)Ve^{-i\lambda z\cdot\theta}$ rapidly decreasing.

Thus, by Theorem 14, $R(\lambda^2+i0)Ve^{-i\lambda z\cdot\theta}$ would be defined and satisfy $F_-(B)R(\lambda^2+i0)Ve^{-i\lambda z\cdot\theta} \in H^{0,p}$ for all p, which would make it clear that the plane wave is not identically zero. In the rest of this section we make these arguments watertight. This is done by 'making' $F_+(B)$ a scattering Ψ.d.o., via Corollary 12. Of course this will introduce error terms which we must control.

Theorem 15. *Suppose $\theta \in S^{N-1}\setminus C$. Then $u'_{\theta,\lambda} = R(\lambda^2+i0)Ve^{-i\lambda z\cdot\theta}$ is defined via* (29) *and limiting absorption, and satisfies*

$$(H-\lambda^2)(e^{-i\lambda z\cdot\theta} + u'_{\theta,\lambda}) = 0, \quad {}^{sc}\mathit{WF}(u'_{\theta,\lambda}) \subset \{\tau \le \lambda - \delta\}. \tag{34}$$

The scattering wavefront estimate should be interpreted as allowing points of the form $(\hat{z}, \infty(0,\mu))$ in the scattering wavefront set but disallowing points of the form $(\hat{z}, \infty(\tau,\mu))$ for $\tau > 0$.

Proof. First we show that $F_+(B)Ve^{-i\lambda z\cdot\theta}$ is the same as $Ve^{-i\lambda z\cdot\theta}$ up to an error in $H^{0,\infty}$. Choose F_+ so that F_+ is supported in $(-\lambda+\delta/2,\infty)$ and is identically equal to 1 on $(-\lambda+3\delta/4,\infty)$. By Corollary 12, if ϕ and $\phi' \in C_c^\infty(\mathbb{R})$ are identically equal to 1 on $[0,\lambda^2]$, then $\phi'(\Delta)F_+(B)\phi(\Delta)$ is a scattering pseudodifferential operator of order $(-\infty,0)$, with symbol $\phi\phi'(\tau^2+|\mu|^2)F_+(-\tau)$. Moreover, the difference between Id and $\phi'(\Delta)F_+(B)\phi(\Delta)$ is microlocally of order $(-\infty,\infty)$ where $\phi(\tau^2+|\mu|^2)$ and $F_+(-\tau)$ are identically equal to one. Choose ϕ, ϕ' so that $1-\phi'$ is zero on a neighbourhood of the support of ϕ. We have

$$\begin{aligned} F_+(B)Ve^{-i\lambda z\cdot\theta} &= VF_+(B)e^{-i\lambda z\cdot\theta} - [V,F_+(B)]e^{-i\lambda z\cdot\theta} \\ &= \Big(V\phi'(\Delta)F_+(B)\phi(\Delta) + V(1-\phi')(\Delta)F_+(B)\phi(\Delta) - [V,F_+(B)]\Big)e^{-i\lambda z\cdot\theta} \\ &= \Big(V\phi'(\Delta)F_+(B)\phi(\Delta) - V(1-\phi')(\Delta)[\phi(\Delta),F_+(B)] - [V,F_+(B)]\Big)e^{-i\lambda z\cdot\theta} \\ &= \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned} \tag{35}$$

The key term is of course (I). By Corollary 12, (I) is equal to $Ve^{-i\lambda z\cdot\theta} + g$, with $g \in \mathcal{S}$. The other terms are error terms and we would like to estimate them. We use the commutator expansion:

$$\begin{aligned} [A, f(B)] &= \sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j!} \operatorname{ad}_j(A,B) f^{(j)}(B) + R_n \\ R_n &= -\frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial} F(\lambda)(B-\lambda)^{-1} \operatorname{ad}_n(A,B)(B-\lambda)^{-n} d\lambda d\overline{\lambda}. \end{aligned} \tag{36}$$

Here $\operatorname{ad}_n(\cdot, B)$ denotes the operation of commutation with B n times. We will use this expansion to show that terms (II) and (III) are in $H^{0,l}$ for every l.

For term (II), after substituting (36) for the commutator, each term in the sum is an operator of order $(-\infty,\infty)$ after left-multiplication with $(1-\phi')(\Delta)$ so we need only consider R_n. Note that $\operatorname{ad}_n(\phi(\Delta), B)$ maps $H^{0,k}$ to $H^{0,k+n}$ since it is a scattering Ψ.d.o. of order $(-\infty,-n)$. As for $(B-\lambda)^{-1}$, we have

Lemma 16. *The operator $(B-\lambda)^{-1}$ maps $H^{0,k}$ to $H^{0,k}$ with operator norm*

$$\le C\big(|\operatorname{Im}\lambda|^{-1} + |\operatorname{Im}\lambda|^{-|k|-1}\big). \tag{37}$$

Proof. Identical in structure to the proof of Lemma 6. ☐

Using (1) and Lemma 16, we see that R_n maps $H^{0,k}$ to $H^{0,k+n}$. It follows that (II) is in $H^{0,l}$ for every l.

Term (III) can be treated the same way. Expanding

$$[V, F_+(B)] = \sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j!} \operatorname{ad}_j(V,B) F_+^{(j)}(B) + R_n,$$

we can write $F_+^{(j)}(B)e^{-i\lambda z\cdot\theta} = (\phi'(\Delta) + (1-\phi')(\Delta))F_+^{(j)}(B)\phi(\Delta)e^{-i\lambda z\cdot\theta}$ and treat this term as above; notice that in this case $F_+^{(j)}$ vanishes on $(-\lambda + 3\delta/4, \infty)$ so $\phi'(\Delta)F_+^{(j)}(B)\phi(\Delta)e^{-i\lambda z\cdot\theta}$ is rapidly decreasing with all derivatives in a conic neighbourhood of the collision planes, while $\operatorname{ad}_j(V,B)$ is a multiplication operator by a function which is rapidly decreasing outside any conic neighbourhood of the collision planes. Thus each term in the sum is rapidly decreasing. The remainder term is now

$$R_n = -\frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial} F(\lambda)(B-\lambda)^{-1} \operatorname{ad}_n(V,B)(B-\lambda)^{-n} d\lambda d\overline{\lambda}.$$

It is not hard to show that $\operatorname{ad}_n(V,B)$ maps $H^{0,k}$ to $H^{0,k+n}$ for all k. Using this, (37), and (1), we see again that R_n maps $H^{0,k}$ to $H^{0,k+n}$. Hence (III) is also in $H^{0,l}$ for every l.

Thus, $Ve^{-i\lambda z\cdot\theta} = F_+(B)X^m h + g$, with $h \in L^2$ and $g \in H^{0,\infty}$. By Theorem 14, $v = R(\lambda^2 + i0)(Ve^{-i\lambda z\cdot\theta})$ is defined and it satisfies $(H-\lambda^2)v = Ve^{-i\lambda z\cdot\theta}$, $F_-(B)v \in H^{0,\infty}$.

We now prove the scattering wavefront estimate (34) for $u'_{\theta,\lambda} = v$. Write $v = v_1 + v_2$, where $v_1 = R(\lambda^2 + i0)F_+(B)X^m h$ and $v_2 = R(\lambda^2 + i0)g$. Suppose that $q = (\hat{z}, \tau, \mu) \in {}^{sc}\mathrm{WF}(u'_{\theta,\lambda})$ is such that either $\tau^2 + |\mu|^2 < \infty$ and $\tau \geq \lambda - \delta$, or $\tau^2 + |\mu|^2 = \infty$ and $\tau/|\mu| > 0$. Then one can find $T, T' \in \Psi^{0,0}_{\mathrm{sc}}$ which are elliptic at q, microlocally of order $(-\infty, \infty)$ near the interior characteristic variety of B, and $T' - 1$ microlocally of order $(-\infty, \infty)$ near the microlocal support of T. Then, by Corollary 12, $TF_-(B)T'$ is a scattering Ψ.d.o. of order $(0,0)$ elliptic at q. We will show that $TF_-(B)T'u'_{\theta,\lambda}$ is in $H^{k,l}$ for every k,l which shows that $q \notin {}^{sc}\mathrm{WF}(u'_{\theta,\lambda})$.

Since TT' is a bounded operator on every Sobolev space $H^{k,l}$, (31) for v_1 and (30) for v_2 shows that

$$TT'F_-(B)u'_{\theta,\lambda} \in \langle z\rangle^{-l}L^2 \quad \text{for all } l \geq 0.$$

The left hand side is equal to

$$\begin{aligned} &TF_-(B)T'u'_{\theta,\lambda} - T[T', F_-(B)]u'_{\theta,\lambda} \\ &= TF_-(B)T'u'_{\theta,\lambda} - \sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j!} T\,\mathrm{ad}_j(T', B) f^{(j)}(B) + TR_n u'_{\theta,\lambda}, \\ &R_n = -\frac{1}{2\pi}\int_{\mathbb{C}} \overline{\partial} F(\lambda)(B-\lambda)^{-1}\,\mathrm{ad}_n(T', B)(B-\lambda)^{-n}\,d\lambda d\overline{\lambda}. \end{aligned}$$

The first term is our desired term, so we need to show that the other terms lie in $H^{k,l}$ for all (k,l).

The terms in the sum are all in $H^{k,l}$ for arbitrary (k,l) since $T\,\mathrm{ad}_j(T', B)$ are operators of order $(-\infty, \infty)$. A similar argument to that for term (II) above shows that the operator in the integral maps from $H^{0,l}$ to $H^{0,l-n}$ for all l.

This shows that $TF_-(B)T'u'_{\theta,\lambda}$ lies in $H^{0,l}$ for all l. For fixed l, we show by induction on k that $TF_-(B)T'u'_{\theta,\lambda}$ lies in $H^{k,l}$ for all k. Assume true for a particular value of k. Then

$$\begin{aligned} (1+\Delta)TF_-(B)T'u'_{\theta,\lambda} &= TF_-(B)T'(1+\Delta)u'_{\theta,\lambda} + [\Delta, TF_-(B)T']u'_{\theta,\lambda} \\ &= TF_-(B)T'\Big((1+\lambda^2)u'_{\theta,\lambda} + Vu'_{\theta,\lambda} + Ve^{-i\lambda z\cdot\theta}\Big) + [\Delta, TF_-(B)T']u'_{\theta,\lambda}. \end{aligned}$$

The first term is in $H^{k,l}$ and the second in $H^{k-1,l}$. Since $(1+\Delta)$ is of order $(2,0)$ and totally elliptic, this implies that $TF_-(B)T'u'_{\theta,\lambda}$ lies in $H^{k+1,l}$. By induction it follows that $TF_-(B)T'u'_{\theta,\lambda}$ lies in $H^{k,l}$ for all (k,l), proving that $q \notin {}^{sc}\mathrm{WF}(u'_{\theta,\lambda})$. This completes the proof of Theorem 15. □

Remark. The construction may be modified to go through when the potentials are short range, as opposed to very short range. The key point is to start with an approximation u_0 to the plane wave such that the error after applying $H - \lambda^2$ has scattering wavefront contained in $\{\tau < \lambda - \delta\}$ for some $\delta > 0$. In the very short range case, this is very easily achieved by taking $u_0 = e^{-i\lambda z\cdot\theta}$. In the short range case, via a standard iterative method (see eg [14], section 1) one can find a function u_0 with an expansion near $\hat{z} = \theta$ of the form

$$e^{-i\lambda z\cdot\theta}(1 + w(z)),$$

with $w(z)$ conormal at $|z| = \infty$ and $w = O(|z|^{-\epsilon})$ for some $\epsilon > 0$, such that $(H - \lambda^2)u_0$ is rapidly decreasing in a conic neighbourhood of the ray $t\theta$. Then u_0 satisfies ${}^{sc}\mathrm{WF}((H-\lambda^2)u_0) \subset \{\tau < \lambda - \delta\}$ for $\delta > 0$, and the rest of the construction goes through without change.

Remark. It is shown in [8] that the plane wave $u_{\omega,\lambda}$ is equal to a constant times the generalized eigenfunction $f(\omega; z)$ constructed in [6]. These were shown in [6] to form the Schwartz kernel for the free channel distorted Fourier transform of H.

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