

EXISTENCE AND UNIQUENESS OF ORBITAL MEASURES

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ABSTRACT. We note an elementary proof of the existence and uniqueness of a solution $\mu \in \mathbb{P}(\mathbb{X})$ to the equation $\mu = p\mu_0 + q\widehat{F}\mu$. Here \mathbb{X} is a topological space, $\mathbb{P}(\mathbb{X})$ is the set of Borel measures of unit mass on \mathbb{X} , $\mu_0 \in \mathbb{P}(\mathbb{X})$ is given, $p > 0$, and $q \geq 0$ with $p + q = 1$. The transformation $\widehat{F} : \mathbb{P}(\mathbb{X}) \rightarrow \mathbb{P}(\mathbb{X})$ is defined by $\widehat{F}v = \sum_{n=1}^N p_n v \circ f_n^{-1}$ where $f_n : \mathbb{X} \rightarrow \mathbb{X}$ is continuous, $p_n > 0$ for $n = 1, 2, \dots, N$, N is a finite strictly positive integer, and $\sum_{n=1}^N p_n = 1$. This problem occurs in connection with iterated function systems (IFS).

1. INTRODUCTION

In this note we present an elementary proof of the existence and uniqueness of a solution to the pair of equations

$$(1.1) \quad \mu = p\mu_0 + q\widehat{F}\mu \text{ with } \mu \in \mathbb{P}(\mathbb{X}).$$

Here \mathbb{X} is a topological space, $\mathbb{P}(\mathbb{X})$ is the set of normalized positive Borel measures on \mathbb{X} , $\mu_0 \in \mathbb{P}(\mathbb{X})$ is given, $p > 0$, $q \geq 0$, and $p + q = 1$. The transformation $\widehat{F} : \mathbb{P}(\mathbb{X}) \rightarrow \mathbb{P}(\mathbb{X})$ is defined by

$$(1.2) \quad \widehat{F}v = \sum_{n=1}^N p_n v \circ f_n^{-1} \text{ for all } v \in \mathbb{P}(\mathbb{X}),$$

where $f_n : \mathbb{X} \rightarrow \mathbb{X}$ is continuous, $p_n > 0$ for $n = 1, 2, \dots, N$, N is a finite strictly positive integer and $\sum_{n=1}^N p_n = 1$. Equation (1.1) occurs in connection with iterated function systems (IFS). In [1] extra restrictions are placed on the functions f_n and the space \mathbb{X} to enable a contraction mapping argument to be used, similar to the one used in [3] for the case $p = 0$.

Given $f : \mathbb{X} \rightarrow \mathbb{X}$ we use the same symbol f to denote $f : \mathbb{P}(\mathbb{X}) \rightarrow \mathbb{P}(\mathbb{X})$ where $f(v) = v \circ f^{-1}$ for all $v \in \mathbb{P}(\mathbb{X})$.

Let $\Omega'_{\{1,2,\dots,N\}}$ denote the set of all finite strings of symbols taken from the set $\{1, 2, \dots, N\}$. That is, $\sigma \in \Omega'_{\{1,2,\dots,N\}}$ iff there is $K \in \mathbb{N}$ and $\sigma = \sigma_1\sigma_2\dots\sigma_K$ where $\sigma_k \in \{1, 2, \dots, N\}$ for each $k \in \{1, 2, \dots, K\}$.

Theorem 1. *Let the setting defined above be true. Then the measure $\mu \in \mathbb{P}(\mathbb{X})$ which is defined by*

$$(1.3) \quad \mu = p\mu_0 + \sum_{\sigma \in \Omega'_{\{1,2,\dots,N\}}, |\sigma| \geq 1} pq^{|\sigma|} p_{\sigma_1} p_{\sigma_2} \dots p_{\sigma_{|\sigma|}} f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_{|\sigma|}}(\mu_0).$$

is the unique solution to Equation (1.1).

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Proof. Let $\mathcal{B}(\mathbb{X})$ denote the set of Borel subsets of \mathbb{X} . The series

$$p\mu_0(B) + \sum_{\sigma \in \Omega'_{\{1,2,\dots,N\}}, |\sigma| \geq 1} pq^{|\sigma|} p_{\sigma_1} p_{\sigma_2} \dots p_{\sigma_{|\sigma|}} f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_{|\sigma|}}(\mu_0)(B)$$

is absolutely convergent, uniformly in $B \in \mathcal{B}(\mathbb{X})$, because it consists of non-negative terms and is bounded above, term-by-term, by the absolutely convergent series

$$p + \sum_{\sigma \in \Omega'_{\{1,2,\dots,N\}}, |\sigma| \geq 1} pq^{|\sigma|} p_{\sigma_1} p_{\sigma_2} \dots p_{\sigma_{|\sigma|}} = 1.$$

Hence the value $\mu(B)$ is well-defined for all $B \in \mathcal{B}(\mathbb{X})$ and $\mu : \mathcal{B}(\mathbb{X}) \rightarrow [0, 1]$. Notice that $\mu(\mathbb{X}) = 1$.

Let us define

$$\rho_0 = \mu_0 \text{ and } \rho_n = \sum_{\sigma \in \Omega'_{\{1,2,\dots,N\}}, |\sigma|=n} p_{\sigma_1} p_{\sigma_2} \dots p_{\sigma_{|\sigma|}} f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_{|\sigma|}}(\mu_0) \text{ for } n = 1, 2, \dots$$

Then it is readily verified that $\rho_n \in \mathbb{P}(\mathbb{X})$ and we can rewrite Equation (1.3) as

$$\mu = \sum_{n=0}^{\infty} pq^n \rho_n.$$

Let $\{\mathcal{O}_m \in \mathcal{B}(\mathbb{X}) : m = 1, 2, \dots\}$ be a sequence such that $\bigcup_{m=1}^{\infty} \mathcal{O}_m \in \mathcal{B}(\mathbb{X})$ and

$$\mathcal{O}_{m_1} \cap \mathcal{O}_{m_2} = \emptyset$$

for all $m_1, m_2 \in \mathbb{N}$ with $m_1 \neq m_2$. Then

$$\sum_{m=0}^{\infty} \mu(\mathcal{O}_m) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} pq^n \rho_n(\mathcal{O}_m).$$

Since the series on the right is absolutely convergent, we can interchange the order in which the two summations are evaluated, which yields

$$\sum_{m=1}^{\infty} \mu(\mathcal{O}_m) = \sum_{n=0}^{\infty} pq^n \sum_{m=1}^{\infty} \rho_n(\mathcal{O}_m) = \sum_{n=0}^{\infty} pq^n \rho_n\left(\bigcup_{m=1}^{\infty} \mathcal{O}_m\right) = \mu\left(\bigcup_{m=1}^{\infty} \mathcal{O}_m\right).$$

It follows that μ is indeed a measure on $\mathcal{B}(\mathbb{X})$ and, since $\mu(\mathbb{X}) = 1$, it follows that $\mu \in \mathbb{P}(\mathbb{X})$.

In order to prove the second part of the theorem we note that, since all of the series involved are absolutely convergent, it suffices to demonstrate that the algebra works out correctly, term-by-term. Substituting from Equation (1.3) into Equation

(1.1) we find

$$\begin{aligned}
 & \text{r.h.s. of Equation (1.1)} \\
 = & p\mu_0 + q \sum_{n=1}^N p_n f_n(p\mu_0 + \sum_{\sigma \in \Omega'_{\{1,2,\dots,N\}}, |\sigma| \geq 1} pq^{|\sigma|} p_{\sigma_1} p_{\sigma_2} \dots p_{\sigma_{|\sigma|}} f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_{|\sigma|}}(\mu_0)) \\
 = & p\mu_0 + q \sum_{n=1}^N p_n f_n(p\mu_0 + \sum_{m=1}^N qp_0 p_m f_m(\mu_0) \\
 & + \sum_{\sigma \in \Omega'_{\{1,2,\dots,N\}}, |\sigma| \geq 2} pq^{|\sigma|} p_{\sigma_1} p_{\sigma_2} \dots p_{\sigma_{|\sigma|}} f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_{|\sigma|}}(\mu_0)) \\
 = & p_0 \mu_0 + \sum_{n=1}^N pq p_n f_n(\mu_0) + \sum_{n=1}^N \sum_{m=1}^N pq^2 p_n p_m f_n(f_m(\mu_0)) + \\
 & \sum_{n=1}^N \sum_{\sigma \in \Omega'_{\{1,2,\dots,N\}}, |\sigma| \geq 2} pq^{|\sigma|+1} p_n p_{\sigma_1} p_{\sigma_2} \dots p_{\sigma_{|\sigma|}} f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_{|\sigma|}}(\mu_0) \\
 = & p_0 \mu_0 + \sum_{\sigma \in \Omega'_{\{1,2,\dots,N\}}, |\sigma| \geq 1} p_0 (1-p_0)^{|\sigma|} p_{\sigma_1} p_{\sigma_2} \dots p_{\sigma_{|\sigma|}} f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_{|\sigma|}}(\mu_0) \\
 = & \text{l.h.s. of Equation (1.1)}.
 \end{aligned}$$

In order to prove uniqueness, suppose that $v \in \mathbb{P}(\mathbb{X})$ obeys

$$v = p\mu_0 + q(p_1 f_1(v) + p_2 f_2(v) + \dots + p_N f_N(v)).$$

Then, by repeatedly substituting from the left-hand-side into the right-hand-side we find that v can be represented by the same absolutely convergent series as μ , whence $v = \mu$. \square

In [2], Chapter 3, we refer to the unique solution of Equation (1.1) as the **orbital measure** associated with the IFS $\{\mathbb{X}; f_1, f_2, \dots, f_N; p_1, p_2, \dots, p_N\}$, the probabilities p and q , and the **condensation measure** μ_0 . Note that we have *not* required that the underlying space be complete or compact, or that the IFS be contractive or contractive on the average.

Notice that the expressions above could have been written down and handled more succinctly in terms of the operator $\widehat{F} : \mathbb{P}(\mathbb{X}) \rightarrow \mathbb{P}(\mathbb{X})$ defined by Equation (1.2). \widehat{F} acts linearly on the space of linear combinations of Borel measures on \mathbb{X} . Using this notation the series expansion in Equation (1.3) can be written as

$$\begin{aligned}
 \mu &= p(1 - q\widehat{F})^{-1} \mu_0 \\
 &= p \sum_{m=0}^{\infty} q^m (\widehat{F})^m \mu_0.
 \end{aligned}$$

Exercise 1. Let $\mathbb{X} = [0, 1) \subset \mathbb{R}$ with the usual topology. Let $S_{\{f\}}(\mathbb{X})$ be the semi-group generated by the function $f : [0, 1) \rightarrow [0, 1)$ defined by $f(x) = \frac{1}{2} + \frac{x}{2}$. Let $\mu_0 \in \mathbb{P}([0, 1))$ denote a normalized Borel measure all of whose mass is contained in $[0, \frac{1}{2})$. That is, $\mu_0([0, \frac{1}{2})) = 1$, and $\mu_0((\frac{1}{2}, 1)) = 0$. Then the associated orbital measure $\mu \in \mathbb{P}(\mathbb{X})$ satisfies

$$\mu = p\mu_0 + qf(v) = \sum_{n=0}^{\infty} pq^n f^{\circ n}(\mu_0).$$

What happens as $p \rightarrow 0$? Do we get a solution to $v = f(v)$ with $v \in \mathbb{P}(\mathbb{X})$? Show that for each $x \in [0, 1)$ we have

$$\lim_{p_0 \rightarrow 0} \mu([0, x]) = 0.$$

Conclude that we do not obtain, in the limit, a solution to $v = f(v)$ with $v \in \mathbb{P}(\mathbb{X})$. What happens if the interval $[0, 1)$ is replaced by $[0, 1]$?

REFERENCES

- [1] M. F. Barnsley and S. Demko, *Iterated Function Systems and the Global Construction of Fractals*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 399(1985), pp. 243-275.
- [2] M. F. Barnsley, *Superfractals*, Cambridge University Press, Cambridge, to appear.
- [3] J. E. Hutchinson, *Fractals and Self-Similarity*, Indiana. Univ. Math. J., 30 (1981), pp. 713-749.

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