Harmonic Analysis of Dirac Operators on Lipschitz Domains

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Abstract. We survey some results concerning Clifford analysis and the L^2 theory of boundary value problems on domains with Lipschitz boundaries. Some novelty is introduced when using Rellich inequalities to invert boundary operators.

Keywords: Clifford analysis, Dirac operator, singular integrals, Rellich inequalities, Maxwell's equations

Math Subject Classifications: 35J55, 35Q60, 45E05

1. Introduction

Clifford analysis has a long history and many applications as the book [2] by Brackx, Delanghe and Sommen testifies. It was introduced into the study of the L^2 boundedness of singular integrals on Lipschitz surfaces in the PhD thesis of Murray [23], written under the supervision of Raphy Coifman. She showed how Clifford analysis could be used to prove the L^2 boundedness of the double layer potential operator on surfaces with small Lipschitz constant, a method extended to all Lipschitz constants by M$^{\mathrm{c}}$Intosh [15]. More direct proofs and related results were then developed in his joint papers with Li, Qian and Semmes [14], [13] and survey paper [16], as well as in work by Gilbert and Murray [9], David, Journé and Semmes [6], Gaudry, Long and Qian [8], Auscher and Tchamitchian [1], Tao [25] and others. Mitrea has made extensive contributions to this theory, and given a good presentation of the field in his book [20].

The L^2 boundedness of the double layer potential operator was in fact proved earlier by Calderón for small Lipschitz constants [3], and by Coifman, M$^{\mathrm{c}}$Intosh and Meyer [5] for all Lipschitz constants. However, as shown in some of the papers mentioned above, the use of Clifford analysis and Dirac operators gives an increased understanding of the topic. This is particularly true when we turn to other related equations such as Maxwell's equation.

There is a long tradition in applying singular integrals to the study of elliptic and parabolic boundary value problems. As well as proving the L^2 boundedness of singular integrals, one needs to show when they are invertible or at least Fredholm. Classically, invertibility is proved using Fredholm theory, but on Lipschitz domains other techniques are needed. Such tools were developed originally by Rellich, Nečas and others, and were specifically used to invert the L^2 double layer potential operator on the boundary of a Lipschitz domain by Verchota [26]. There is much related work on this topic by Dahlberg, Fabes, Jerison, Kenig, Pipher and others.

Rellich inequalities were adapted to the study of Maxwell's equations by Mitrea, Torres and Welland in [22], [21]. Clifford versions were presented by M^cIntosh, Mitrea and Mitrea [17], [18].

In this paper we develop a new way of applying Rellich inequalities to invert boundary operators. This method makes essential use of the full Clifford structure.

We also present an outline of the theory of the L^2 boundedness of the singular Cauchy operator on Lipschitz surfaces and of related singular integral operators. In surveying this material we will in general not make specific references to the papers mentioned above.

We have not attempted to survey research on Clifford analysis and Cauchy integrals on domains satisfying stronger smoothness conditions. See the Introduction by Ryan to [24], and the books of Gürlebeck and Sprössig [10] and [11] for more information and further references.

Acknowledgements. We thank Marius Mitrea for sharing with us his insight and knowledge of this topic.

Much of this material was developed while Hogan and M^cIntosh were at Macquarie University Sydney and Grognard was an Honorary Associate there. While there we received support from the Australian Government through the ARC.

2. Clifford Algebra

The functions we consider in this paper are defined on a subset of $\mathbf{R}^m$, and take their values in the complex Clifford algebra $\mathcal{A} = \mathbf{C}_{(m+1)}$ generated by the basis vectors $e_0, e_1, e_2, \ldots, e_m$ subject to the identification rule

$$e_i e_j + e_j e_i = -2\delta_{ij} \ , \quad 0 \leq i, j \leq m \ .$$

We expand $u \in \mathcal{A}$ as $u = \sum u_S e_S$ where $u_S \in \mathbf{C}$ and $S \subset \{0, 1, \ldots, m\}$. The Euclidean basis element $e_S = e_{j_1} e_{j_2} \ldots e_{j_s}$ if $S = \{j_1, j_2, \ldots, j_s\}$ with $0 \leq j_1 < j_2 < \cdots < j_s \leq m$. In particular, $e_\emptyset = 1$ and $e_{\{j\}} = e_j$.

There is a natural decomposition

$$\mathcal{A} = \wedge^0 \oplus \wedge^1 \oplus \wedge^2 \oplus \cdots \oplus \wedge^{m+1}$$

into linear subspaces $\wedge^p = \{\sum_{\#S=p} u_S e_S\}$ of elements of degree p. We decompose u into its components of degree p as $u = \sum_{p=0}^{m+1} u^{(p)}$.

The algebra $\mathcal{A}$ naturally supports a *Clifford conjugation*, acting as a complex linear anti-automorphism by

$$\overline{u} := \sum_{p=0}^{m+1} (-1)^{p(p+1)/2} u^{(p)} ,$$

and a *complex conjugation* u^{c} which just acts on each component of u in the basis generated by the Euclidean basis vectors by ordinary complex conjugation. Furthermore we have the sesquilinear scalar product $(u, v) = (u\overline{v}^{\mathrm{c}})_\emptyset = \sum u_S v_S^{\mathrm{c}}$, which is used to define the left and right interior product, $\lrcorner$ and $\llcorner$, by

$$(u \lrcorner x, y) := (x, u^{\mathrm{c}} \wedge y) , \quad (x \llcorner u, y) := (x, y \wedge u^{\mathrm{c}}) .$$

When $a \in \wedge^1$ and $v \in \mathcal{A}$, the Clifford product av can be decomposed

$$av = -a \lrcorner v + a \wedge v .$$

Though the Clifford algebra is not commutative, we have the identity $(uv)_\emptyset = \sum u_S v_S {e_S}^2 = (vu)_\emptyset$. Another important identity is

$$(uv, w) = (v, \overline{u}^{\mathrm{c}} w) = (u, w\overline{v}^{\mathrm{c}}) .$$

3. Clifford Analysis of $\mathbf{D}_k$

With the Clifford algebra comes the Dirac operator $\mathbf{D} = \sum_{j=1}^m e_j \partial_j$. Since our task is to study elliptic boundary value problems arising as time harmonic solutions $f(x)\exp(-i\omega t)$ to the hyperbolic Dirac operator $-\tilde{e}_0 \partial_0 + \mathbf{D}$, $\tilde{e}_0 := -ie_0$ being the forward time direction, we introduce the *k-Dirac operator*

$$\mathbf{D}_k = \mathbf{D} + ke_0 .$$

We assume that the complex parameter k satisfies $\operatorname{Im} k \geq 0$. The Dirac operator does not preserve homogeneity of degree like the exterior derivative $d = \mathbf{D}\wedge$ and the interior derivative $\delta = -\mathbf{D}\lrcorner$, but it maps functions taking values in the even subalgebra $\mathcal{A}^{\mathrm{even}} := \oplus\wedge^{2p}$ to functions taking values in $\mathcal{A}^{\mathrm{odd}} := \oplus\wedge^{2p+1}$ and vice versa.

The relation between the k-Dirac and the Helmholtz operator is

$$\Delta + k^2 = -(\mathbf{D} + ke_0)^2 .$$

The elliptic operator $\mathbf{D}$ has fundamental solution $E(x) := -\frac{x}{\sigma_{m-1}|x|^m}$. Near $x = 0$, the fundamental solution F_k to $\mathbf{D}_k$ (acting either from left or right) behaves like E. When $\operatorname{Im} k > 0$, F_k has exponential decay at ∞ while when $k \in \mathbf{R} \setminus \{0\}$ it satisfies the decay condition

$$F_k(x)e^{-ik|x|} = c_{m,k}\, |x|^{-\frac{m-1}{2}} \left(\tilde{e}_0 + \frac{x}{|x|} \right) + o\left(|x|^{-\frac{m-1}{2}} \right) \quad \text{as } |x| \to \infty$$

with $c_{m,k} \neq 0$. Note that the leading term is directed along the null cone in hyperbolic space. The amplitude of the gradient has decay $o\left(|x|^{-\frac{m-1}{2}} \right)$. See [17].

An explicit expression for F_k is obtained by applying $-\mathbf{D}_k$ to the fundamental solution of the Helmholtz operator, the Bessel potential $B_k(x)$. When $m = 3$, we have $B_k(x) = -\frac{e^{ik|x|}}{4\pi|x|}$ and readily obtain

$$F_k(x) = \Big(-\frac{x}{|x|^2} + ik(\tilde{e}_0 + \frac{x}{|x|}) \Big) \frac{e^{ik|x|}}{4\pi|x|} \; .$$

In $\mathbf{R}^m$ we will consider Ω^+, being either a bounded domain with strongly Lipschitz boundary Σ, or the region above a Lipschitz graph

$$\Sigma = \{ (x', x_m) \; ; \; x_m = \phi(x') \} \; .$$

Let $\Omega^- := \mathbf{R}^m \setminus (\Omega^+ \cup \Sigma)$. Recall that $\phi : \mathbf{R}^{m-1} \to \mathbf{R}$ being Lipschitz means $|\phi(x') - \phi(y')|/|x' - y'|$ is uniformly bounded, while for a bounded domain, strongly Lipschitz means that for each $y \in \Sigma$ there exists a neighbourhood U_y and a Lipschitz graph Σ_y dividing $\mathbf{R}^m$ into $\Omega_y^\pm$ (with respect to some Euclidean coordinate system) such that $\Sigma \cap U_y = \Sigma_y \cap U_y$ and $\Omega_y^\pm \cap U_y = \Omega^\pm \cap U_y$.

Concerning integral manipulations, the following version of Stokes' theorem, referred to as *the boundary theorem* should be noted.

> *An integral over Σ, where the integrand contains the outward pointing normal n linearly, equals the integral over Ω^+ with n replaced by $\mathbf{D}$.*

In Clifford analysis the most frequently used integrand is gnf, where the boundary theorem tells us that

$$\int_\Sigma g(x)n(x)f(x)\, d\sigma(x) = \int_{\Omega^+} \Big((g\mathbf{D})(x)f(x) + g(x)(\mathbf{D}f)(x) \Big)\, dx \; .$$

Now, consider a *k-monogenic* function f in Ω^+, i.e. a solution to $\mathbf{D}_k f(x) = 0$ there. Applying the identity above with $g(x) = E_k(x) := -F_k(-x)$ (since we need $E_k(\mathbf{D} - ke_0) = \delta_0$), we obtain the reproducing formula

$$f(x) = \int_\Sigma E_k(y - x)n(y)f(y)\, d\sigma(y) \; , \quad x \in \Omega^+. \tag{1}$$

For this to work, we need f to be sufficiently nice up to the boundary.

DEFINITION 3.1. *The Cauchy extension of $f \in L^2(\Sigma)$ is*

$$Cf(x) := \int_\Sigma E_k(y-x)n(y)f(y)\,d\sigma(y)\ , \quad x \in \Omega^+ \cup \Omega^-\ .$$

Let $C^\pm f := Cf|_{\Omega^\pm}$ and in the graph case write $C_\tau f(y) := Cf(y+\tau e_m)$, $\tau \in \mathbf{R}\setminus\{0\}$. The Hardy projections P^+f and P^-f are the boundary values of C^+f and of $-C^-f$ respectively in the $L^2(\Sigma)$ sense. The ranges of the projections are called Hardy spaces and will be denoted $P^\pm L^2$. The principal value Cauchy integral is

$$C_\Sigma f(x) = 2\,\mathrm{p.v.}\int_\Sigma E_k(y-x)n(y)f(y)\,d\sigma(y)\ , \quad x \in \Sigma\ .$$

We spend the rest of this and the next section investigating the properties of C, outlining the proof that $P^\pm$ are well defined and bounded complementary projections. Assume here that Σ is a graph and $k=0$. The case of a bounded domain can be obtained from this, as can the case of a general k since C_Σ^k is a compact perturbation of C_Σ.

PROPOSITION 3.2. *For any $f \in L^2(\Sigma)$*

$$f = \lim_{\tau\to 0^+}(C_\tau f - C_{-\tau} f)$$

both in L^2 and pointwise a.e. Indeed, the difference kernel $K_\tau(x,y) := (E(y-(x+\tau e_m)) - E(y-(x-\tau e_m)))n(y)$ is an approximate unit.

Proof. Use the estimate $|K_\tau(x,y)| \lesssim \tau/|y-(x+\tau e_m)|^m$ and the fact that $\int_\Sigma K_\tau(x,y)\,d\sigma(y) = 1$ by the boundary theorem.

COROLLARY 3.3. *For any $f \in L^2(\Sigma)$ and $t>0$, we have*

$$\begin{aligned} C^-(C_t f) &= 0 \\ C^+(C_t f)(y+\tau e_m) &= (C^+f)(y+(\tau+t)e_m)\ , \quad \tau>0,\ y\in\Sigma\ . \end{aligned}$$

Similar statements holds for $t<0$.

Proof. Having the estimate

$$|E(y-(x+(t+\tau)e_m)) - E(y-(x+te_m))| \lesssim \frac{\tau}{|y-(x+te_m)|^m}\ ,$$

it follows that $C_t f = \lim_{\tau\to 0^+} C_\tau(C_t f)$. From the proposition we get $\lim_{\tau\to 0^-} C_\tau(C_t f) = 0$. But $C^-(C_t f)$ being monogenic and zero on a hypersurface implies $C^-(C_t f) = 0$ in all Ω^-.

For the second identity, observe that the two sides are equal when $\tau = 0$, which implies equality for all $\tau > 0$ as above.

4. L^2 Boundedness of Hardy Projections

The difficult thing with the Hardy projections is to show that $L^2 = P^+L^2 \oplus P^-L^2$. By Proposition 3.2 it is enough to show that we have convergence in L^2 for $C_\tau f$ as $\tau \to 0^+$. For this we need uniform L^2 bounds on $C_\tau f$. For $\Sigma = \mathbf{R}^{m-1}$, this follows easily from Fourier theory, since $\hat{E} \in L^\infty$. For a general Lipschitz surface Σ one can proceed as in [14], where the two dimensional case in [5] is generalised to $\mathbf{R}^m$ following [4]. Below we outline very briefly the main steps of the proof in [14] of the needed uniform L^2 estimates. The key ingredient is the following square function estimate, where

$$\|G\|_+^2 := \int_{\Omega^+} |G(x)|^2 \mathrm{dist}(x,\Sigma)\, dx \ .$$

PROPOSITION 4.1. *If G is monogenic in Ω^+ and continuous up to Σ with boundary trace g and satisfies estimates $|G(x)| \leq C_G/(1+|x|)^{m-1}$ and $|\nabla G(x)| \leq C_G/(1+|x|)^m$, $x \in \Omega^+$ (e.g. if $g = C_\tau f$ and $f \in L^2(\Sigma)$ has compact support), then*

$$\|g\|_{L^2(\Sigma)} \lesssim \|\nabla G\|_+ \ ,$$

independently of the constant C_G.

With the analogous theorem for right monogenic functions in Ω^-, together with Schur estimates, we prove the following lemma.

LEMMA 4.2. *Let $H \in C_0^\infty(\Omega^+;\mathcal{A})$ and*

$$S_{\tau,j}H(y) := \int_{\Omega^+} \overline{H(x)} \partial_j E(y-(x+\tau e_m))\mathrm{dist}(x,\Sigma)\, dx, \quad y \in \Sigma \ .$$

Then $\|S_{\tau,j}H\|_{L^2(\Sigma)} \lesssim \|H\|_+$, with constant independent of $\tau > 0$.

Now, combining the proposition and the lemma, using duality and Fubini's theorem, we obtain the desired estimate

$$\begin{aligned}
\|C_\tau f\|_{L^2(\Sigma)} &\lesssim \|\nabla C(C_\tau f)\|_+ \\
&\lesssim \sum_{j=1}^m \sup_{\|H_j\|_+ \leq 1} \int_{\Omega^+} \Big(\frac{\partial F}{\partial x_j}(x+\tau e_m), H_j(x)\Big) \mathrm{dist}(x,\Sigma)\, dx \\
&\lesssim \sum_{j=1}^m \sup_{\|H_j\|_+ \leq 1} \int_\Sigma \Big(S_{\tau,j}H_j(y), \overline{f}(y)\overline{n}(y)\Big)\, d\sigma(y) \\
&\lesssim \|f\|_{L^2(\Sigma)} \qquad (2)
\end{aligned}$$

for all compactly supported $f \in L^2(\Sigma)$, with a constant independent of f and τ. By Fatou's lemma this can be extended to all $f \in L^2(\Sigma)$.

Recall that *the non-tangential maximal function* of $F : \Omega^+ \to \mathcal{A}$ is

$$\mathcal{N}F(y) := \sup_{x \in y+\Gamma} |F(x)| , \quad y \in \Sigma ,$$

where Γ is an open infinite cone with apex at 0 and axis along e_m. The angle of the cone is chosen small enough that $y+\Gamma \subset \Omega^+$ for all $y \in \Sigma$.

PROPOSITION 4.3. *The estimate* $\|\mathcal{N}C^\pm f\|_{L^2(\Sigma)} \lesssim \|f\|_{L^2(\Sigma)}$ *holds for all* $f \in L^2(\Sigma)$.

Proof. Use Corollary 3.3 to obtain

$$\begin{aligned} \|\mathcal{N}(C^+(C_t f - C_{-t} f))\| &= \|\mathcal{N}(C^+(C_t f))\| \\ &\lesssim \|\mathcal{M}(C_t f)\| \lesssim \|C_t f\| \lesssim \|f\| . \end{aligned}$$

The bound of the non-tangential maximal function $\mathcal{N}$ by the Hardy–Littlewood maximal function $\mathcal{M}$ comes from the fact that we can estimate K_τ in Proposition 3.2 by $\tau/|y-(x+\tau e_m)|^m$.

To pass to the limit with t, observe that $\mathcal{N}(C^+(C_t f - C_{-t} f))$ is increasing as $t \to 0^+$ and apply the monotone convergence theorem.

Note that this proposition not only gives control of Cf near Σ but also shows that $C_\tau f \to 0$ in L^2 as $\tau \to \pm\infty$.

THEOREM 4.4. *For any* $f \in L^2(\Sigma)$*, boundary values to* $C^\pm f$ *exist both in* $L^2(\Sigma)$ *and pointwise non-tangentially a.e., so* $\{P^\pm\}$ *are complementary projections in* $L^2(\Sigma)$*, i.e. bounded operators satisfying*

$$(P^\pm)^2 = P^\pm , \quad P^+P^- = 0 = P^-P^+ \quad \textit{and} \quad P^+ + P^- = I .$$

Thus we have a topological splitting $L^2(\Sigma) = P^+L^2 \oplus P^-L^2$*. We also have* $P^+ - P^- = C_\Sigma$ *and the Plemelj–Sokhotski jump formulae* $P^\pm = \frac{1}{2}(I \pm C_\Sigma)$*.*

Proof. Let $f_t = C_t f - C_{-t} f$ and observe that for $C^\pm f_t$ we have boundary values both in L^2 and non-tangentially a.e. Write

$$\|C_\tau f - C_\sigma f\| \leq \|C_\tau(f - f_t)\| + \|C_\tau(f_t) - C_\sigma(f_t)\| + \|C_\sigma(f_t - f)\| .$$

Choosing t small and using (2) we can make the first and last term small, and then choosing τ and σ small gives the middle term small. Using the non-tangential maximal function one can similarly prove that pointwise non-tangential boundary values exist a.e.

Finally a remark on the ranges of $C^\pm$. We have seen that C^+f is a monogenic function in Ω^+ with $\mathcal{N}C^+f \in L^2(\Sigma)$. The converse is also true, as can be proved with a limiting argument from the reproducing formula (1). A similar result holds in the case of a bounded domain if we use truncated cones for the boundary behaviour and an appropriate radiation condition at infinity depending on k. See [17].

5. Duality

A bilinear pairing $\langle \mathcal{K}, \mathcal{H} \rangle$ between two Hilbert spaces is called a duality if the estimates

$$|\langle k, h\rangle| \lesssim \|k\|\|h\| , \quad \|h\| \lesssim \sup_{\|k\|=1} |\langle k, h\rangle| \quad \text{and} \quad \|k\| \lesssim \sup_{\|h\|=1} |\langle k, h\rangle|$$

hold. Denote the adjoint operator of $T : \mathcal{H} \to \mathcal{H}$ relative to this duality by $T' : \mathcal{K} \to \mathcal{K}$.

The relevant duality here is not the $L^2(\Sigma)$ scalar product but rather the weighted duality

$$\langle g, f\rangle := \int_\Sigma (g(y)n(y)f(y))_\emptyset \, d\sigma(y) \ .$$

Here $g \in \overleftarrow{L}^2(\Sigma) = \mathcal{K}$ and $f \in \overrightarrow{L}^2(\Sigma) = \mathcal{H}$, the two spaces being just two identical copies of $L^2(\Sigma)$. Up to now we have made use of $\mathbf{D}_k = \mathbf{D} + ke_0$ acting from the left with Cauchy extensions C, using E_k in the kernel, and Hardy projections $P^\pm$. All this takes place in $\overrightarrow{L}^2$. To emphasise this we sometimes overline these operators, e.g. $\mathbf{D}_k = \overrightarrow{\mathbf{D}}_k$. We may equally well do the same thing in $\overleftarrow{L}^2$. Here we have a Dirac operator $\overleftarrow{\mathbf{D}}_k = \mathbf{D} + k(-e_0)$ acting from the right with Cauchy extension

$$\overleftarrow{C} f(x) = \int_\Sigma f(y)n(y)\overleftarrow{E}_k(y-x)\, d\sigma(y) \ , \quad x \in \Omega^+ \cup \Omega^- \ ,$$

using $\overleftarrow{E}_k(x) := F_k(x) = -\overrightarrow{E}_k(-x)$, The boundary values $I \pm \overleftarrow{C}_\Sigma$ define Hardy projections $\{\overleftarrow{P}^\pm\}$. All the theory in Sections 3 and 4 goes through for these operators in $\overleftarrow{L}^2$, mutatis mutandis.

LEMMA 5.1. *The dualities* $(\overrightarrow{P}^\pm)' = \overleftarrow{P}^\mp$ *hold.*

Proof. Since $P^\pm = \frac{1}{2}(I \pm C_\Sigma)$ it is enough to prove $(\overrightarrow{C}_\Sigma)' = -\overleftarrow{C}_\Sigma$. Formally this follows from the calculation

$$\begin{aligned}
\langle g, \overrightarrow{C}_\Sigma f\rangle &= \int_\Sigma \Big(g(x)n(x) \int_\Sigma \overrightarrow{E}_k(y-x)n(y)f(y)\, d\sigma(y)\Big)_\emptyset d\sigma(x) \\
&= \int_\Sigma \Big(\int_\Sigma g(x)n(x)(-\overleftarrow{E}_k(x-y)) d\sigma(x)\ n(y)f(y)\Big)_\emptyset d\sigma(y) \\
&= \langle -\overleftarrow{C}_\Sigma g, f\rangle \ .
\end{aligned}$$

6. Splittings of $L^2(\Sigma)$ and Rellich estimates

We now discuss the elliptic boundary value problem for $\mathbf{D}_k$ which was hinted at before. For an elliptic operator of order d one should impose $d/2$ boundary conditions to get a well-posed problem. For $\mathbf{D}_k$ this means half a boundary condition should do.

DEFINITION 6.1. *Let $Q^\pm : L^2(\Sigma) \to L^2(\Sigma)$ be the projections*

$$Q^\pm f := \tfrac{1}{2}(f \pm nfn) \ .$$

We denote the range of $Q^\pm$ by $Q^\pm L^2$ and set $\overleftarrow{Q}^\pm = \overrightarrow{Q}^\pm := Q^\pm$.

In case $f \in \mathcal{A}^{\text{odd}}$, then $Q^+ f$ is the tangential part of f while $Q^- f$ is the normal part of f, the situation being reversed when $f \in \mathcal{A}^{\text{even}}$. Note that $(\overrightarrow{Q}^\pm)' = \overleftarrow{Q}^\pm$.

The two pairs of complementary projections $\{P^\pm\}$ and $\{Q^\pm\}$ induce the two splittings

$$L^2(\Sigma) = P^+L^2 \oplus P^-L^2 = Q^+L^2 \oplus Q^-L^2 \ .$$

It is interesting to investigate the geometric relation between them.

DEFINITION 6.2. *Assume $\{A^\pm\}$ is a pair of complementary projections on a Hilbert space $\mathcal{H}$. We say that a bounded projection B is transversal to $\{A^\pm\}$ if $A^+ : B(\mathcal{H}) \to A^+(\mathcal{H})$ and $A^- : B(\mathcal{H}) \to A^-(\mathcal{H})$ are both isomorphisms.*

If the restricted projections are Fredholm with index zero rather than isomorphisms, we say that B is 0-transversal to $\{A^\pm\}$.

If B is transversal to $\{A^\pm\}$, then in particular

$$\|A^+x\| \approx \|x\| \approx \|A^-x\| \ , \quad x \in B(\mathcal{H}) \ .$$

The case that we have in mind is the following fact concerning the comparability of the normal and tangential part of a k-monogenic function.

THEOREM 6.3. *Let Ω^+ be a bounded open subset of $\mathbf{R}^m$ with strongly Lipschitz boundary. Then there exists a discrete set $S \subset \mathbf{R}$ such that $P^\pm$ is transversal to $\{Q^\pm\}$ for $k \notin S$, and is 0-transversal to $\{Q^\pm\}$ when $k \in S$.*

In case of a Lipschitz graph Σ and $k = 0$, $P^\pm$ is transversal to $\{Q^\pm\}$.

All these statements also hold when the P's and Q's switch roles.

This is a way of stating the well-posedness of the boundary value problem consisting in finding a function F in Ω^+ with boundary trace $f \in L^2(\Sigma)$ such that

$$\begin{cases} \mathbf{D}_k F &= 0 \quad \text{in } \Omega^+ \\ Q^+ f &= g \in Q^+ L^2 \quad \text{on } \Sigma \ . \end{cases} \tag{3}$$

By Theorem 6.3, if $k \notin S$, there exists a unique solution $f \in P^+L^2$ such that $Q^+f = g$ and thus a unique solution F to (3). If $k \in S$ the result still holds modulo finite dimensions.

Note that the null space of $Q^+|_{P^+L^2}$ is $P^+L^2 \cap Q^-L^2$, it being $\{0\}$ when $k = 0$ and Σ is a graph since Ω^+ then has trivial topology. When $k = 0$ and Ω^+ is bounded, the dimension of this intersection depends on the topology of the domain. See [19] for the classical boundary integral operators.

We begin to sketch the proof of Theorem 6.3 by presenting three abstract lemmata to make the logic more transparent. First we recall the following well–known result relating a priori estimates with the semi–Fredholm property, i.e. having finite dimensional nullspace and closed range.

LEMMA 6.4. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces, $T : \mathcal{X} \to \mathcal{Y}$ be bounded, and $K : \mathcal{X} \to \mathcal{Z}$ be compact. Assume the a priori estimate*

$$\|x\|_{\mathcal{X}} \lesssim \|Tx\|_{\mathcal{Y}} + \|Kx\|_{\mathcal{Z}} \ , \quad x \in \mathcal{X} \ .$$

Then T is a semi-Fredholm operator.

LEMMA 6.5. *Assume a Hilbert space $\mathcal{H}$ splits as*

$$\mathcal{H} = A^+(\mathcal{H}) \oplus A^-(\mathcal{H}) = B^+(\mathcal{H}) \oplus B^-(\mathcal{H})$$

with respect to two pairs of complementary projections $\{A^\pm\}$ and $\{B^\pm\}$. Then a priori estimates for the four restricted projections $A^\pm|_{B^\pm(\mathcal{H})}$ imply estimates for the other four $B^\pm|_{A^\pm(\mathcal{H})}$. If the first four estimates are strict, i.e. the compact terms are zero, then so are the other four.

Proof. Assume we have a priori estimates

$$\begin{aligned} \|u\| &\lesssim \|A^\pm u\| + \|K^\pm u\| \ , \quad u \in B^+(\mathcal{H}) \\ \|u\| &\lesssim \|A^\pm u\| + \|L^\pm u\| \ , \quad u \in B^-(\mathcal{H}) \end{aligned}$$

where $K^\pm$ and $L^\pm$ are compact operators. Then by decomposing $A^+(\mathcal{H}) \ni u = B^+u + B^-u$ and observing that $A^-B^+u + A^-B^-u = 0$ we get

$$\begin{aligned} \|u\| &\leq \|B^+u\| + \|B^-u\| \lesssim \|B^+u\| + \|A^-B^-u\| + \|L^-B^-u\| \\ &= \|B^+u\| + \|A^-B^+u\| + \|L^-B^-u\| \lesssim \|B^+u\| + \|L^-B^-u\|. \end{aligned}$$

Last we give a duality lemma for pairs of projections. Here we suppose that there is a duality $\langle \mathcal{K}, \mathcal{H} \rangle$ between $\mathcal{H}$ and another Hilbert space $\mathcal{K}$.

LEMMA 6.6. *Assume $\mathcal{H}$ splits in two ways as in the previous lemma. Then $\mathcal{K} = (A^+)'(\mathcal{K}) \oplus (A^-)'(\mathcal{K}) = (B^+)'(\mathcal{K}) \oplus (B^-)'(\mathcal{K})$. If*

$$\begin{aligned} A^+ &: B^+(\mathcal{H}) \longrightarrow A^+(\mathcal{H}) \\ (A^-)' &: (B^-)'(\mathcal{K}) \longrightarrow (A^-)'(\mathcal{K}) \end{aligned}$$

both satisfy a priori estimates, then they are Fredholm operators.

If the estimates are strict, then they are both isomorphisms.

More generally, if there exists an isomorphism $j : A^-(\mathcal{H}) \cap B^+(\mathcal{H}) \to (A^+)'(\mathcal{K}) \cap (B^-)'(\mathcal{K})$, then A^+ and $(A^-)'$ both have index 0.

Proof. Observe that $\langle \mathcal{K}, \mathcal{H} \rangle$ restricts to a (non-degenerate) duality

$$\langle (A^+)'(\mathcal{K}) \cap (B^-)'(\mathcal{K}),\ A^+(\mathcal{H}) \ominus A^+B^+(\mathcal{H}) \rangle \ .$$

This shows that the dimension of the cokernel of $A^+|_{B^+(\mathcal{H})}$ equals that of the kernel of $(A^-)'|_{(B^-)'(\mathcal{K})}$. An isomorphism j shows that the latter dimension equals the dimension of the kernel of $A^+|_{B^+(\mathcal{H})}$. Thus it has index zero. Similarly for $(A^-)'|_{(B^-)'(\mathcal{K})}$.

The application we have in mind is $A^\pm = Q^\pm$ and $B^\pm = P^\pm$ and the duality is $\langle \overleftarrow{L}^2, \overrightarrow{L}^2 \rangle$ as above with $j(f) = \overline{f} e_0$. Note that j maps $\overrightarrow{P}^\pm L^2$ to $\overleftarrow{P}^\pm L^2$ and $\overrightarrow{Q}^\pm L^2$ to $\overleftarrow{Q}^\mp L^2$ and thus satisfies the condition of Lemma 6.6.

We can now apply the three lemmata if we prove the Rellich-type a priori estimates for $Q^\pm|_{P^\pm L^2}$.

PROPOSITION 6.7. *Let Ω^+ be bounded. If $f \in P^+L^2$, then*

$$\|f\| \lesssim \|Q^\pm f\| + (|k|+1)\|Cf\|_{L^2(U \cap \Omega^+)} \ ,$$

where U denotes a neighbourhood of Σ with compact closure. The operator C is compact from $L^2(\Sigma)$ to $L^2(U \cap \Omega^+)$.

The same estimate holds if $f \in P^- L^2$ and $U \cap \Omega^-$ replaces $U \cap \Omega^+$.

Proof. Following [17] and [18], the proof uses the commutation properties of $\mathcal{A}$ and the boundary theorem. Take $\theta \in C^1_0(U; \wedge^1_{\mathbf{R}})$ with the property that $(n, \theta) \geq c > 0$ on Σ. We get for $f \in P^\pm L^2$

$$\begin{aligned} |f|^2(n,\theta) &= \tfrac{1}{2}(f, f(n\theta + \theta n)) = -\tfrac{1}{2}\Big((f\theta, fn) + (fn, f\theta)\Big) \\ &= -\mathrm{Re}\,(fn, f\theta) = -\mathrm{Re}\,(2(Q^\pm f)n, f\theta) \mp \mathrm{Re}\,(nf, f\theta) \ . \end{aligned}$$

Integrating over Σ, using the boundary theorem on the expression $(nf, f\theta)$ in the second term and writing $F = Cf$, we obtain

$$\begin{aligned} \|f\|^2 \lesssim\ & \|Q^\pm f\|\|f\| \\ & + \left| \int_{\Omega^\pm} \Big((-ke_0 F, F\theta) - (F, -ke_0 F\theta) - (F, \mathbf{D}F\dot{\theta}) \Big)\, dx \right| , \end{aligned}$$

where the dot indicates that $\mathbf{D}$ only acts there. Using the inequality $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ on the first term with suitable ϵ we obtain the estimate.

That the perturbation term is compact can be shown by using Schur's test with suitable exponents.

Proof of Theorem 6.3:

Bounded Σ, $\mathrm{Im}\, k \ge 0$: Applying Proposition 6.7, Lemmata 6.5 and 6.6 gives that $P^\pm$ is 0-transversal to $\{Q^\pm\}$ and vice versa.

Bounded Σ, $\mathrm{Im}\, k > 0$: Here we can use the boundary theorem on $(nf, e_0 f)$ to eliminate the compact term. Calculating, we get

$$\begin{aligned} \int_\Sigma (nf, e_0 f) &= \int_{\Omega^+} \Big((-ke_0 F, e_0 F) - (F, (-e_0)(-ke_0 F)) \Big) \\ &= (-2i)\mathrm{Im}\, k \int_{\Omega^+} |F|^2 \end{aligned}$$

and $(nf, e_0 f) = 2i\mathrm{Im}\,(Q^\pm f, e_0 nf)$. After some algebra this yields the strict a priori estimate

$$\|f\| \lesssim \frac{|k|+1}{\mathrm{Im}\, k} \|Q^\pm f\| ,$$

which shows that $P^\pm$ is transversal to $\{Q^\pm\}$ and vice versa via Lemmata 6.5 and 6.6.

Bounded Σ, discreteness of S: This follows from analytic Fredholm theory applied to $Q^\pm P^\pm : Q^\pm \to Q^\pm$.

Graph Σ, $k = 0$: The proof of Proposition 6.7 here works with $\theta = e_m$ and gives a strict a priori estimate directly. This implies the transversality as above.

7. Harmonic functions

Rellich's original estimate was formulated as the comparable size in $L^2(\Sigma)$ of the normal and tangential derivatives of a harmonic function

$u : \Omega^+ \to \mathbf{R}$. This result follows from the integral identity arising in the proof of Proposition 6.7 by regarding u as a map into $\wedge^0$, letting

$$F = \mathbf{D}u : \Omega^+ \to \wedge^1 \ ,$$

and noting that F is monogenic, that $Q^+ f$ is the tangential derivative of u, and that $Q^- f$ is the normal derivative of u.

Many integral identities under the name of Rellich in the literature can be derived from the results of Section 6 in a similar way. Applications also include the known estimates [26] for the acoustic double-layer potential operator

$$K_\Sigma \phi(x) := 2\,\text{p.v.} \int_\Sigma (E_k(y-x), n(y))\phi(y)\, d\sigma(y) = (C_\Sigma \phi)_\emptyset(x) \ , \quad x \in \Sigma \ ,$$

where $\phi \in L^2(\Sigma; \mathbf{C})$.

PROPOSITION 7.1. *The operators $I \pm K_\Sigma$ are bounded Fredholm maps on $L^2(\Sigma; \mathbf{C})$ with index 0. When $\operatorname{Im} k > 0$ they are isomorphisms.*

Proof. Identify $\mathbf{C}$ with $\wedge^0 \subset \mathcal{A}$. Straightforward calculations give

$$(I \pm K_\Sigma^*)\psi = -2nQ^- P^{\pm}_{(-k^c)}(n\psi) \ , \quad \psi \in L^2(\Sigma; \wedge^0) \ .$$

Thus Theorem 6.3 gives a priori estimates for $I \pm K_\Sigma^*$. Furthermore, as in [7], it can be shown that $\lambda I \pm K_\Sigma^*$ is also semi-Fredholm for $|\lambda| \geq 1$, $\lambda \in \mathbf{R}$. It is an isomorphism when $|\lambda| > \|K_\Sigma^*\|$, so by general perturbation theory the index is 0.

8. Maxwell's Equation

We conclude by briefly describing how the above theory can be applied to Maxwell's equations

$$\begin{aligned} \mathbf{D} \wedge B &= 0 \\ \partial_0 B + \mathbf{D} \wedge E &= 0 \\ \partial_0 D + \mathbf{D} \lrcorner H &= -J \\ \mathbf{D} \lrcorner D &= \rho \ , \end{aligned}$$

where $E = \epsilon^{-1} D = E_1 e_1 + E_2 e_2 + E_3 e_3$ and $B = \mu H = B_1 e_{23} + B_2 e_{31} + B_3 e_{12}$. From bottom to top these are Gauss' law, Ampère–Maxwell's law, Faraday's law and the magnetic Gauss' law and they take scalar, vector, bivector and trivector values respectively. Taking energy as unit, the physically natural quantity to work with is the electromagnetic field

$$f(t,x) = \epsilon^{1/2}\tilde{e}_0 E + \mu^{-1/2} B \ .$$

As in [12] we observe that Maxwell's equations can be written entirely with Clifford algebra (with time and space dependent ϵ and μ) as

$$(-\tfrac{1}{c}\tilde{e}_0\partial_0 + \mathbf{D}) f(t,x) + Rf(t,x) = j \ ,$$

where the speed of propagation is $c(t,x) := (\epsilon(t,x)\mu(t,x))^{-1/2}$. The zero order term is

$$Rf := T(f)(-\tfrac{1}{c}\tilde{e}_0\partial_0 + \mathbf{D})\ln(\epsilon^{-1/2}) + S(f)(-\tfrac{1}{c}\tilde{e}_0\partial_0 + \mathbf{D})\ln(\mu^{1/2}) \ ,$$

where $T(f) := \frac{1}{2}(f - \tilde{e}_0 f \tilde{e}_0) = \epsilon^{1/2}\tilde{e}_0 E$ and $S(f) := \frac{1}{2}(f + \tilde{e}_0 f \tilde{e}_0) = \mu^{-1/2}B$, and the "four-current" is $j := \tilde{e}_0\epsilon^{-1/2}\rho + \mu^{1/2}J$.

Here we will just consider the case when the coefficients are constant on Ω^+ or Ω^- and $j = 0$ there. Then Maxwell's equation takes the form

$$(-\tfrac{1}{c}\tilde{e}_0\partial_0 + \mathbf{D}) f(t,x) = 0 \ ,$$

so that time harmonic solutions $f(t,x) = e^{-i\omega t} f(x)$ are k-monogenic, where $k = \omega/c$ is the wave number.

In solving the boundary value problem (3) in $\mathbf{R}^3$ for Maxwell's equation, the Rellich estimates do not completely solve the problem since we have a constraint on f requiring that it should take values only in $\wedge^2$. Let us see what necessary conditions on $g := Q^+ f = n(n \lrcorner f)$ we have when $f \in P^+ L^2$ and $C^+ f$ takes values in $\wedge^2$. Of course $g \in Q^+ L^2(\Sigma; \wedge^2)$, but we also get, after an integration by parts, that

$$\begin{aligned} 0 \ &= \ (C^+ f)_\emptyset(x) = \int_\Sigma (E_k(y-x) n(y) f(y))_\emptyset \, d\sigma(y) \\ &= \ \int_\Sigma B_k(y-x)(-\delta_\partial(n \lrcorner g)(y) + k e_0 \lrcorner (n \lrcorner g)(y)) \, d\sigma(y), \end{aligned}$$

where $B_k(x) = -\frac{e^{ik|x|}}{4\pi|x|}$. Here we have defined the operator δ_∂ in $L^2(\Sigma)$ via duality by

$$\int_\Sigma (\delta_\partial(n \lrcorner g), \Phi|_\Sigma) = \int_\Sigma (n \lrcorner g, (d\Phi)|_\Sigma)$$

for all C^1 functions Φ in a neighbourhood of Σ. Note that this operator only is well-defined for tangential functions and as the notation suggests it is closely related to the usual interior derivative δ_Σ. Now, varying x over Ω^+, we conclude that $\delta_\partial(n \lrcorner g) = k e_0 \lrcorner (n \lrcorner g)$, at least if $\operatorname{Im} k > 0$.

THEOREM 8.1. *If* $\operatorname{Im} k > 0$, $g \in L^2(\Sigma, \wedge^2)$, $n \wedge g = 0$ *and* $\delta_\partial(n \lrcorner g) = ke_0 \lrcorner (n \lrcorner g)$, *then the boundary value problem* (3) *in* $\mathbf{R}^3$ *has a unique solution*

$$F : \Omega^+ \longrightarrow \wedge^2 \ .$$

Moreover, its boundary trace f *satisfies* $\|f\|_{L^2(\Sigma)} \lesssim \|g\|_{L^2(\Sigma)}$.

Similar results are true modulo finite dimensions when k *is real.*

Proof. If $\operatorname{Im} k > 0$, Theorem 6.3 gives a unique $F : \Omega^+ \to \mathcal{A}$ with normal boundary trace g. To show that F maps into $\wedge^2$, decompose $f = f^{(0)} + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)}$, where $f^{(p)} \in \wedge^p$. Since $\mathbf{D}_k$ switches $\mathcal{A}^{\text{even}}$ and $\mathcal{A}^{\text{odd}}$ it follows that $f^{\text{odd}} = f^{(1)} + f^{(3)} \in P^+L^2$. Since $Q^+ f^{\text{odd}} = 0$ we get $f^{(1)} = f^{(3)} = 0$.

By the differentiability condition on g, $\frac{1}{2}(I + K_\Sigma) f^{(0)} = f^{(0)} - (P^+ g)_\emptyset = f^{(0)}$, and thus an application of Proposition 7.1 gives $f^{(0)} = 0$. Furthermore $f^{(4)} = 0$ simply because Q^+ is injective on $L^2(\Sigma; \wedge^4)$. Thus $f = f^{(2)} \in P^+ L^2$, and this implies that $F : \Omega^+ \longrightarrow \wedge^2$.

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