

The solution of Kato's conjectures

Pascal AUSCHER^a, Steve HOFMANN^b, Michael LACEY^c, John LEWIS^d,
Alan McINTOSH^e, Philippe TCHAMITCHIAN^f

^a LAMFA, CNRS, FRE 2270, Université de Picardie-Jules-Verne, 33, rue Saint-Leu, 80039 Amiens cedex 1, France

^b Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, États-Unis

^c School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, États-Unis

^d Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, États-Unis

^e Centre for Mathematics and its Applications, ANU, Canberra, ACT 0200, Australie

^f LATP, CNRS, UMR 6632 et faculté des sciences et techniques de St-Jérôme, Université d'Aix-Marseille-III, avenue Escadrille Normandie-Niemen, 13397 Marseille cedex 20, France

E-mail: auscher@mathinfo.u-picardie.fr; hofmann@math.missouri.edu; lacey@math.gatech.edu;
john@ms.uky.edu; alan@maths.anu.edu.au; tchamphi@math.u-3mrs.fr

(Reçu le 7 février 2001, accepté le 12 février 2001)

Abstract. In this Note, we announce the solution of Kato's conjectures about the domain of square roots of differential elliptic operators in the Euclidean space or a Lipschitz domain. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

La solution des conjectures de Kato

Résumé. Nous annonçons, dans cette Note, la solution des conjectures de Kato concernant le domaine de la racine carrée des opérateurs différentiels elliptiques dans l'espace euclidien ou un domaine à bord lipschitzien. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Nous annonçons ici une série de résultats concernant la conjecture de Kato. Les six auteurs de cette Note ont contribué à l'un ou plusieurs des résultats qui suivent et qui seront publiés ultérieurement. La conjecture est complètement résolue pour les opérateurs elliptiques sous forme divergence sur \mathbb{R}^n d'ordre quelconque. Elle est aussi résolue pour les opérateurs d'ordre deux sur un domaine lipschitzien avec conditions au bord de Dirichlet ou de Neumann. Enfin, elle est résolue partiellement dans le cas des systèmes grâce à un résultat de perturbation au voisinage des systèmes auto-adjoints sur \mathbb{R}^n ; la résolution est complète pour tous les systèmes elliptiques. Les énoncés sont les suivants :

THÉORÈME 1. – Pour tout opérateur d'ordre deux $L = -\operatorname{div} A \nabla$ vérifiant (1) sur \mathbb{R}^n , le domaine de sa racine carrée \sqrt{L} coïncide avec l'espace de Sobolev $H^1(\mathbb{R}^n)$ et on a l'équivalence (3) avec des constantes dépendant de la dimension et des constantes dans (1).

Note présentée par Yves MEYER.

Ce résultat était connu uniquement en une dimension [5] et ne connaissait que des réponses partielles en dimensions supérieures. Par un principe de transfert, on peut en déduire le théorème suivant :

THÉORÈME 2. – *Supposons $n \geq 2$. Le domaine de la racine carrée d'un opérateur elliptique d'ordre deux $L = -\operatorname{div} A\nabla$ vérifiant (1) et obéissant aux conditions de Dirichlet ou de Neumann au bord d'un ouvert lipschitzien Ω , est égal à l'espace V sur lequel est défini la forme sesquilinéaire associée à L et on a $\|\sqrt{L} f\|_{L^2(\Omega)} \leq C \|f\|_{H^1(\Omega)}$ pour tout $f \in V$, où C ne dépend que de la dimension, des constantes de (1) et de Ω .*

On remarque que le résultat est déjà connu en une dimension [2]. Passons aux opérateurs d'ordre supérieurs. En adaptant l'argument du théorème 1, on obtient :

THÉORÈME 3. – *Supposons (5) et (6) et soit L l'opérateur elliptique d'ordre supérieur donné par (4). Alors la racine carrée de $L + \kappa$ a pour domaine l'espace de Sobolev $H^m(\mathbb{R}^n)$ et il vient*

$$\|\sqrt{L + \kappa} f\|_2 \leq C (\|\nabla^m f\|_2^2 + \kappa \|f\|_2^2)^{1/2},$$

où C dépend de la dimension, de l'ordre m , des constantes de (5) et (6) mais pas de κ .

Pour les systèmes, on n'obtient par une extension de l'argument de [1] que le cas perturbatif du cas auto-adjoint ; l'argument du théorème 3 s'adapte et permet d'enlever l'hypothèse sur les coefficients. Plus précisément :

THÉORÈME 4. – *Supposons que L soit un système elliptique $N \times N$ d'ordre $2m$, $m \in \mathbb{N}^*$, sur \mathbb{R}^n , $n \geq 1$, de la forme (4) au sens où (5) et (6) généralisés aux systèmes sont valables. Il existe $\varepsilon > 0$ dépendant de la dimension et des constantes m , λ , Λ dans (5) et (6) tel que si*

$$\|a_{\alpha\beta} - a_{\beta\alpha}^*\|_\infty \leq \varepsilon,$$

alors la racine carrée de $L + \kappa$ a pour domaine l'espace de Sobolev $H^m(\mathbb{R}^n, \mathbb{C}^N)$ et on a

$$\|\sqrt{L + \kappa} f\|_2 \leq C (\|\nabla^m f\|_2^2 + \kappa \|f\|_2^2)^{1/2},$$

où C dépend de ε mais pas de κ .

Remarque 1. – Tous ces énoncés sont pour des opérateurs homogènes. Une conséquence d'un résultat général de [3] est que si L est l'un des opérateurs précédents et si P est une perturbation de L par des termes d'ordre inférieur sous forme divergence, alors le domaine $\sqrt{L + P + s}$ est égal à celui de $\sqrt{L + \kappa}$, où $s > 0$ est tel que $L + P + s$ soit maximal-acréatif sur L^2 .

Remarque 2. – Nous signalons l'existence de résultats L^p pour les racines carrées d'opérateurs du second ordre sur \mathbb{R}^n vérifiant (1) et (2) qui étendent ceux obtenus dans [3] sous une hypothèse plus forte : en effet, on a

$$\|L^{1/2} f\|_{\mathcal{H}^1} \leq C_1 \|\nabla f\|_{\mathcal{H}^1}.$$

Ici, \mathcal{H}^1 est l'espace de Hardy classique sur \mathbb{R}^n . Donc, par interpolation,

$$\|L^{1/2} f\|_p \leq C_p \|\nabla f\|_p,$$

pour tout $p \in (1, 2]$. La constante C_p ne dépend que de n , λ , Λ et p . Voir [6] pour des résultats presque optimaux concernant l'inégalité L^p inverse.

1. Statements of the results and history

We announce a series of results including the solution of Kato's conjecture. The six authors of this Note have contributed to the proof of one or several theorems presented here and full details will be presented elsewhere.

Let $A = A(x)$ be an $n \times n$ matrix of complex, L^∞ coefficients, defined on \mathbb{R}^n , and satisfying the ellipticity (or “accretivity”) condition:

$$\lambda|\xi|^2 \leq \Re A\xi \cdot \xi^* \quad \text{and} \quad |A\xi \cdot \zeta^*| \leq \Lambda|\xi||\zeta|, \quad (1)$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \leq \Lambda < \infty$. Here, $u \cdot v^*$ is the usual inner product in \mathbb{C}^n so that $A\xi \cdot \xi^* \equiv \sum_{j,k} a_{j,k}(x)\xi_k \bar{\xi}_j$. We define a second order divergence form operator

$$Lf \equiv -\operatorname{div}(A\nabla f), \quad (2)$$

which we interpret in the usual weak sense via a sesquilinear form $\int_{\mathbb{R}^n} A\nabla f \cdot \bar{\nabla g}$ for f, g in the Sobolev space $H^1(\mathbb{R}^n)$.

The accretivity condition (1) enables one to define a square root $L^{1/2} \equiv \sqrt{L}$ (see [11]), and a fundamental question posed by T. Kato is to determine whether one can solve the “square root problem”, i.e., establish the estimate

$$\|\sqrt{L}f\|_2 \sim \|\nabla f\|_2, \quad (3)$$

where \sim is the equivalence in the sense of norms, with constants C depending only on n, λ and Λ , and $\|v\|_2 = (\int_{\mathbb{R}^n} |f(x)|_H^2 dx)^{1/2}$ denotes the usual norm for functions on \mathbb{R}^n valued in a Hilbert space H . We announce that this question has an affirmative answer.

THEOREM 1. – *For any second operator $-\operatorname{div} A\nabla$ satisfying (1) the domain of \sqrt{L} coincides with $H^1(\mathbb{R}^n)$ and (3) holds with constants depending only on n, λ and Λ .*

Previously, the square root problem had been solved completely only in one dimension [5], where it is essentially equivalent to the problem of proving the L^2 boundedness of the Cauchy integral operator on a Lipschitz curve.

In the case $n > 1$, four of us [1], had previously proved a restricted version of Theorem 1, also essentially conjectured by Kato in [11]. The restricted version treats the case that A is close, in the L^∞ norm, to a real symmetric matrix of bounded, measurable coefficients. Kato had posed this version of the problem on account of its applicability to the perturbation theory for hyperbolic equations [13].

Prior to the latter result, Theorem 1 had been proved in higher dimensions in the case that A is close to a constant matrix, in either the L^∞ norm [4,7,9,10], or the BMO norm [7,3].

Next, consider a strongly Lipschitz domain Ω of \mathbb{R}^n and define an elliptic operator L from the sesquilinear form $\int_{\Omega} A\nabla f \cdot \bar{\nabla g}$ for f, g in a closed subspace V of the Sobolev space $H^1(\Omega)$ that contains $H_0^1(\Omega)$. The matrix A satisfies (1) on Ω . When $V = H_0^1(\Omega)$, L is the Dirichlet operator corresponding to the coefficients A and when $V = H^1(\Omega)$, L is the Neumann operator. These are the only cases considered here.

THEOREM 2. – *If $n \geq 2$, the domain of the square root of an elliptic second order operator subject to Dirichlet or Neumann boundary condition on a strongly Lipschitz domain is equal to V and one has $\|\sqrt{L}f\|_{L^2(\Omega)} \leq C\|f\|_{H^1(\Omega)}$ for all $f \in V$, with C depending only on n, λ, Λ and Ω .*

Remark 1. – When $n = 1$, this theorem is true and proved in [2]. Actually, all possible boundary conditions can be considered.

It appears that the method to prove Theorem 1 is can be adapted to elliptic operators of any order. We use standard notation for partials and multiindices. Consider an homogeneous elliptic operator of order $2m$, $m \in \mathbb{N}^*$, having a representation of the form:

$$Lf = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha\beta} \partial^\beta f), \quad (4)$$

where the coefficients $a_{\alpha\beta}$ are complex-valued L^∞ functions on \mathbb{R}^n satisfying

$$\left| \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x) \partial^\beta f(x) \partial^\alpha \bar{g}(x) dx \right| \leq \Lambda \|\nabla^m f\|_2 \|\nabla^m g\|_2 \quad (5)$$

and the Gårding inequality

$$\Re \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x) \partial^\beta f(x) \partial^\alpha \bar{f}(x) dx \geq \lambda \|\nabla^m f\|_2^2 - \kappa \|f\|_2^2, \quad (6)$$

for some $\lambda > 0$, $\kappa \geq 0$ and $\Lambda < +\infty$ independent of f, g in the Sobolev space $H^m(\mathbb{R}^n)$. Here, $\nabla^m f = (\partial^\alpha f)_{|\alpha|=m}$. We have the following result which, in fact, covers Theorem 1.

THEOREM 3. – *Under the hypotheses (5) and (6), the square root of the operator $L + \kappa$ with L given by (4) has domain equal to the Sobolev space $H^m(\mathbb{R}^n)$ and we have*

$$\|\sqrt{L + \kappa} f\|_2 \leq C (\|\nabla^m f\|_2^2 + \kappa \|f\|_2^2)^{1/2} \quad (7)$$

with C depending only on n, m, λ and Λ but not on κ .

Remark 2. – When $m \geq 2$, the Gårding inequality (6) classically implies

$$\Re \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \xi^\beta \overline{\xi^\alpha} \geq \lambda \sum_{|\alpha|=m} |\xi^\alpha|^2, \quad \text{a.e.} \quad (8)$$

It is classical that (8) implies (6) in the case of uniformly continuous coefficients. The same conclusion holds for VMO coefficients (P. Tchamitchian, unpublished). In both cases, the conclusion that the domain of $\sqrt{L + \kappa}$ is $H^m(\mathbb{R}^n)$ was known earlier (from [7] and [3]) but not, of course, the precise behavior of the constants in (7) with respect to κ , which can be 0.

We have restricted ourselves to scalar operators in Theorem 3. One may wonder about the situation for elliptic systems, that is, in the equations (4), (5) and (6), the coefficients $a_{\alpha\beta}$ become $N \times N$ -matrices and the functions f, g take their values in \mathbb{C}^N . The ellipticity is granted by the Gårding inequality. In this case also, one can consider the square root of L . It turns out that there is an algebraic obstacle to adapt the method of proof of Theorem 3 to this situation. However, the method used in [1] to prove the analyticity of square roots near the self-adjoint case, can be generalized so as to obtain the following result. We have recently overcome the obstacle and Theorem 4 holds without (9).

THEOREM 4. – *Assume that L is a vector-valued operator as in (4) satisfying (5) and (6) generalized to systems. There is an $\varepsilon > 0$ depending only on n, m, λ and Λ such that if*

$$\|a_{\alpha\beta} - a_{\beta\alpha}^*\|_\infty \leq \varepsilon, \quad (9)$$

The solution of Kato's conjectures

then the square root of $L + \kappa$ has domain equal to the Sobolev space $H^m(\mathbb{R}^n, \mathbb{C}^N)$ and we have

$$\|\sqrt{L + \kappa} f\|_2 \leq C(\|\nabla^m f\|_2^2 + \kappa \|f\|_2^2)^{1/2}, \quad (10)$$

with C depending only on ε but not on κ .

The equation (9) means that L is near a self-adjoint system as $a_{\beta\alpha}^*$ are the matrix-valued coefficients of L^* .

Remark 3. – In all of these results, the operators (or systems) have been considered homogeneous, that is without lower order terms. It turns out that, and this is a consequence of a general principle proved in [3], if L is one of the operator in the above statements and P is a perturbation with lower order terms in divergence form, then the domain of $\sqrt{L + P + s}$ is equal to that of $\sqrt{L + \kappa}$, where $s > 0$ is chosen so that $L + P + s$ is a maximal-accretive operator on L^2 .

Remark 4. – We mention also the existence of L^p results for second order operators as in (1) and (2) generalizing those of [3] obtained under an additional hypothesis: we have

$$\|L^{1/2} f\|_{\mathcal{H}^1} \leq C_1 \|\nabla f\|_{\mathcal{H}^1}.$$

Here, \mathcal{H}^1 is the classical Hardy space on \mathbb{R}^n . Hence, by interpolation,

$$\|L^{1/2} f\|_p \leq C_p \|\nabla f\|_p,$$

for all $p \in (1, 2]$. The constant C_p depends only on n, λ, Λ and p . See [6] for nearly optimal results concerning the reverse L^p -inequality.

2. Sketch of proof of Theorem 1

There are several steps. Using a theorem by Lions [12], one has to prove only the inequality $\|\sqrt{L} f\|_2 \leq C \|\nabla f\|_2$. Then, one reduces this to a quadratic estimate

$$\int_0^{+\infty} \|(1 + t^2 L)^{-1} t L f\|_2^2 \frac{dt}{t} \leq C \int_{\mathbb{R}^n} |\nabla f|^2,$$

via either functional calculus for maximal accretive operators or standard almost-orthogonality arguments of Littlewood–Paley theory. The next step is to observe that off-diagonal estimates in some average sense allow to reduce the proof to a Carleson measure estimate

$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\gamma_t(x)|^2 \frac{dx dt}{t} < \infty,$$

where the supremum is taken over all cubes Q of \mathbb{R}^n with sides parallel to the axes, $|Q|$ and $\ell(Q)$ being respectively the measure and the sidelength of such a cube. Here

$$\gamma_t = (1 + t^2 L)^{-1} t L \varphi$$

with $\varphi(x) = x$ for $x \in \mathbb{R}^n$. The last step is to prove this estimate by a T(b) argument, in the spirit of the T(b) theorems for singular integrals, in establishing through a stopping-time argument that for some small $\varepsilon > 0$,

$$\sup \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\gamma_t(x)|^2 \frac{dx dt}{t} \leq C \sum_{w \in W} \sup \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\gamma_t(x) \cdot (S_t^Q \nabla f_{Q,w}^\varepsilon)(x)|^2 \frac{dx dt}{t}$$

and next using some technical estimates that

$$\int_Q \int_0^{\ell(Q)} |\gamma_t(x) \cdot (S_t^Q \nabla f_{Q,w}^\varepsilon)(x)|^2 \frac{dx dt}{t} \leq C|Q|,$$

where C depends only on ε, n, λ and Λ , the suprema being taken over all cubes Q . Here,

$$f_{Q,w}^\varepsilon = (1 + (\varepsilon \ell(Q))^2 L)^{-1} ((\varphi - \varphi(x_Q)) \cdot w^*),$$

where x_Q is the center of Q , w is a unit vector in \mathbb{C}^n . The set W above is a finite set of such unit vectors whose cardinality depends only on ε and n . Also, $S_t^Q f(x) = \frac{1}{|Q'|} \int_{Q'} f(y) dy$ for x in the sub-dyadic cube Q' of Q and $\ell(Q')/2 < t \leq \ell(Q')$. The idea behind the use of $f_{Q,w}^\varepsilon$ is that its gradient is close in the mean to $\nabla \varphi \cdot w^* = w^*$ in the sense that

$$\left| \int_Q 1 - (\nabla f_{Q,w}^\varepsilon(x) \cdot w) dx \right| \leq C \varepsilon^{1/2} |Q|.$$

This is the crucial inequality in the stopping-time argument.

References

- [1] Auscher P., Hofmann S., Lewis J., Tchamitchian Ph., Extrapolation of Carleson measures and the analyticity of Kato's square root operator, *Acta Math.* (à paraître).
- [2] Auscher P., Tchamitchian Ph., Conjecture de Kato sur les ouverts de \mathbb{R} , *Rev. Mat. Iberoamericana* 8 (1992) 149–199.
- [3] Auscher P., Tchamitchian Ph., Square Root Problem for Divergence Operators and Related Topics, Astérisque, Vol. 249, Soc. Math. de France, 1998.
- [4] Coifman R., Deng D., Meyer Y., Domaine de la racine carrée de certains opérateurs différentiels accréatifs, *Ann. Inst. Fourier* 33 (1983) 123–134.
- [5] Coifman R., McIntosh A., Meyer Y., L'intégrale de Cauchy définit un opérateur borné sur $L^2(\mathbb{R})$ pour les courbes lipschitziennes, *Ann. Math.* 116 (1982) 361–387.
- [6] Duong X.T., McIntosh A., The L^p boundedness of Riesz transforms associated with divergence forms operators, in: Workshop on Analysis and Applications, Brisbane, 1997, Proceedings of the Centre for Mathematical Analysis, Vol. 37, ANU, Canberra, 1999, pp. 15–25.
- [7] Escauriaza L. (communication personnelle).
- [8] Fabes E., Jerison D., Kenig C., Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure, *Ann. Math.* 119 (1984) 121–141.
- [9] Fabes E., Jerison D., Kenig C., Multilinear square functions and partial differential equations, *Amer. J. Math.* 107 (1985) 1325–1367.
- [10] Journé J.-L., Remarks on the square root problem, *Publ. Math.* 35 (1991) 299–321.
- [11] Kato T., Fractional powers of dissipative operators, *J. Math. Soc. Japan* 13 (1961) 246–274.
- [12] Lions J.-L., Espaces d'interpolation et domaines de puissances fractionnaires, *J. Math. Soc. Japan* 14 (1962) 233–241.
- [13] McIntosh A., Square roots of operators and applications to hyperbolic PDE, in: Proceedings of the Miniconference on Operator Theory and PDE, CMA, The Australian National University, Canberra, 1983.