**THEOREM** *Let* $0 \le \omega < \mu \le \pi$, *and let* T *be an operator of type* $\omega$ *which is one–one with dense range. Suppose that* T *and* $T^*$ *satisfy quadratic estimates with respect to functions* $\psi$ *and* $\tilde{\psi}$ *in* $\Psi^+(S^0_\mu)$. *If* $f \in H^\infty(S^0_\mu)$, *then the operator* f(T) *is bounded, and there exists a constant* c *such that*

$$\|f(T)\| \le c\|f\|_\infty$$

*for all* $f \in H_\infty(S^0_\mu)$.

**Proof** Let $\theta$ be any function in $\Psi(S^0_\mu)$ such that $\int_0^\infty \varphi(t)t^{-1}dt = 1$ where $\varphi = \psi\tilde{\psi}\theta$.

For $f \in H_\infty(S^0_\mu)$ and $0 < \epsilon < R < \infty$, define $f_{\epsilon,R} \in \Psi(S^0_\mu)$ by

$$f_{\epsilon,R}(\zeta) = \int_\epsilon^R (f\varphi_t)(\zeta)\,\frac{dt}{t}.$$

We shall use the quadratic estimates to show that

$$\|f_{\epsilon,R}(T)\| \le c\|f\|_\infty$$

for some constant c depending only on T, $\mu$ and $\theta$. The theorem in section 5 can then be applied to give the result. We note that it also gives the formula

$$f(T)u = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_\epsilon^R (f\varphi_t)(T)u\,\frac{dt}{t}$$

for all $u \in \mathcal{X}$.

To prove the bounds on $f_{\epsilon,R}(T)$ we proceed as follows. Let $u,v \in \mathcal{X}$. Then

$$\begin{aligned}
&|\langle f_{\epsilon,R}(T)u,v\rangle| \\
&\quad = \left|\int_\epsilon^R \langle (f\theta_t)(T)\psi_t(T)u\,,\,\tilde{\psi}_t(T^*)v\rangle \frac{dt}{t}\right| \\
&\quad \le \sup_t \|(f\theta_t)(T)\|\left\{\int_0^\infty \|\psi_t(T)u\|^2\,\frac{dt}{t}\right\}^{1/2}\left\{\int_0^\infty \|\tilde{\psi}_t(T^*)v\|^2\,\frac{dt}{t}\right\}^{1/2} \\
&\quad \le q_1q_2 \sup_t \|(f\theta_t)(T)\|\,\|u\|\,\|v\|
\end{aligned}$$



where $q_1$ and $q_2$ are the constants appearing in the quadratic estimates. Recall that $(f\theta_t)(T)$ was defined in section 4 by a contour integral,

$$(f\theta_t)(T) = \frac{1}{2\pi i}\int_\gamma (T-\zeta I)^{-1}f(\zeta)\theta(t\zeta)d\zeta.$$

Therefore

$$\begin{aligned}
\|(f\theta_t)(T)\| &\le \frac{1}{2\pi}\|f\|_\infty \int_\gamma \|(T-\zeta I)^{-1}\|\,|\theta(t\zeta)|\,|d\zeta| \\
&\le \frac{1}{2\pi}\|f\|_\infty \int_\gamma c|\zeta|^{-1}\left\{\frac{ct^s|\zeta|^s}{1+t^{2s}|\zeta|^{2s}}\right\}|d\zeta| \\
&= \kappa\|f\|_\infty
\end{aligned}$$

where $\kappa$ depends on T, $\mu$ and $\theta$ but not f. So

$$|\langle f_{\epsilon,R}(T)u,v\rangle| \le q_1q_2\kappa\|f\|_\infty\|u\|\,\|v\|$$

for all $0 < \epsilon < R < \infty$, $f \in H_\infty(S^0_\mu)$, and $u,v \in \mathcal{X}$.

We have thus obtained the bounds on $f_{\epsilon,R}$ and hence the result. //

The assumption in the theorem that T has dense range is in fact redundant as it follows from the other hypotheses. In fact if we drop the assumption that T is one–one then we find that $\mathcal{X} = \mathcal{N}(T) \oplus \overline{\mathcal{R}(T)}$, where the symbol $\oplus$ denotes the direct sum in the sense of Banach spaces (and does not imply orthogonality). This is seen as follows. Define $E_+$ by

$$E_+u = \int_0^\infty \varphi_t(T)u\,\frac{dt}{t}, \quad u \in \mathcal{X}.$$

Then $E_+$ is a bounded operator which is zero on $\mathcal{N}(T)$ and the identity on $\mathcal{R}(T)$. So $\mathcal{N}(T) \oplus \overline{\mathcal{R}(T)} \subset \mathcal{X}$. Similarly $\mathcal{N}(T^*) \oplus \overline{\mathcal{R}(T^*)} \subset \mathcal{X}$. So $\mathcal{N}(T) \oplus \overline{\mathcal{R}(T)} = \mathcal{X}$ as required. In general we find that

$$f(T_{\mathcal{R}})E_+u = \int_0^\infty (f\varphi_t)(T)u \,\frac{dt}{t}$$

for $u \in \mathcal{X}$, where $T_{\mathcal{R}}$ is the restriction of $T$ to $\overline{\mathcal{R}(T)}$.

## 8. NECESSITY OF QUADRATIC ESTIMATES

**THEOREM** *Let $T$ be a one–one operator of type $\omega$ with dense range. Write $T = UA$ and $T^* = VB$ where $A$ and $B$ are positive self–adjoint operators and $U$ and $V$ are isomorphisms.*

*The following statements are equivalent:*

(a) *for all $\mu > \omega$ there exist $c_\mu$ such that*

$$\|f(T)\| \le c_\mu \|f\|_\infty \qquad , f \in H_\infty(S^0_\mu)\,;$$

(b) *there exist $\mu > \omega$ and $c$ such that*

$$\|f(T)\| \le c\|f\|_\infty \qquad , f \in H_\infty(S^0_\mu)\,;$$

(c) *$\{T^{is} \mid s \in \mathbb{R}\}$ is a $C^0$ group and, for all $\mu > \omega$, there exist $c_\mu$ such that*

$$\|T^{is}\| \le c_\mu e^{\mu|s|} \qquad , s \in \mathbb{R}\,;$$

(d) *there exists $c$ such that*

$$\|T^{is}\| \le c \qquad , -1 \le s \le 1\,;$$

(e) *for each $\alpha \in (0,1)$, $\mathcal{D}(T^\alpha) = \mathcal{D}(A^\alpha)$, $\mathcal{D}(T^{*\alpha}) = \mathcal{D}(B^\alpha)$ and there exists $c > 0$ such that*

$$c^{-1}\|A^\alpha u\| \le \|T^\alpha u\| \le c\|A^\alpha u\| \qquad , u \in \mathcal{D}(T^\alpha)\,,$$
$$c^{-1}\|B^\alpha u\| \le \|T^{*\alpha} u\| \le c\|B^\alpha u\| \qquad , u \in \mathcal{D}(T^{*\alpha})\,;$$



(f) *there exist $\alpha, \beta \in (0,1)$ and $c$ such that $\mathcal{D}(T^\alpha) \subset \mathcal{D}(A^\alpha)$, $\mathcal{D}(T^{*\beta}) \subset \mathcal{D}(B^\beta)$ and*

$$\|A^\alpha u\| \le c\|T^\alpha u\| \qquad , u \in \mathcal{D}(T^\alpha)$$
$$\|B^\beta u\| \le c\|T^{*\beta} u\| \qquad , u \in \mathcal{D}(T^{*\beta})\,;$$

(g) *for all $\mu > \omega$ and all $\psi \in \Psi(S^0_\mu)$ there exist $q$ such that*

$$\left\{\int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t}\right\}^{1/2} \le q\|u\| \quad \text{and}$$
$$\left\{\int_0^\infty \|\psi(tT^*)u\|^2 \frac{dt}{t}\right\}^{1/2} \le q\|u\| \qquad , u \in \mathcal{X}\,;$$

(h) *there exist $\mu > \omega$, $\psi \in \Psi^+(S^0_\mu)$, $\tilde{\psi} \in \Psi^+(S^0_\mu)$ and $q$ such that*

$$\left\{\int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t}\right\}^{1/2} \le q\|u\| \quad \text{and}$$
$$\left\{\int_0^\infty \|\tilde{\psi}(tT^*)u\|^2 \frac{dt}{t}\right\}^{1/2} \le q\|u\| \qquad , u \in \mathcal{X}\,.$$

**Proof** We shall verify the implications (a) ⇒ (c) ⇒ (e) ⇒ (g) ⇒ (a). The cycle (b) ⇒ (d) (e) ⇒ (f) ⇒ (h) ⇒ (b) is proved similarly.

(a) ⇒ (c) : The bounds in (c) are obtained by applying part (a) to $f_{(s)}(\zeta) = \zeta^{is}$. The theorem in section 5 can then be applied to see that $f_{(s)}(T)u \to u$ as $s \to 0$ for all $u \in \mathcal{X}$.

(c) ⇒ (e) : This is a result on complex interpolation which can be proved as usual by applying the maximum modulus theorem on the strip $\{z \in \mathbb{C} \mid 0 \le \mathit{Re}\, z \le 1\}$. For example, the first inequality can be verified for $u \in \mathcal{R}(T(I+T)^{-3})$ by applying it to the function

$$f(z) = e^{z^2} A^z T^{-z} u\,.$$

All the technicalities needed to do so have been derived in sections 5 and 6. For example the continuity of $f$ can be proved using the theorem in section 5 as can the analyticity of $f$ on the open strip. Further details are left to the reader.

(e) ⇒ (g) : For $\alpha \in (0,1)$, let $\psi_{(\alpha)}(\zeta) = \zeta^{1-\alpha}(1+\zeta)^{-1}$. We first show that T satisfies a quadratic estimate with respect to $\psi_{(\alpha)}$. There exist bounded operators U and $W_\alpha$ such that

$$TA^{-1} = U|_{\mathcal{R}(A)} \quad \text{and} \quad A^{\alpha}T^{-\alpha} = W_\alpha|_{\mathcal{R}(T^\alpha)}\,.$$

It can easily be computed that

$$\psi_{(\alpha)}(T) = \{T(I+T)^{-1} + (I+T)^{-1}U\}\psi_{(\alpha)}(A)W_\alpha\,,$$

so

$$\begin{aligned}&\int_0^\infty \|\psi_{(\alpha)}(tT)u\|^2 \frac{dt}{t}\\ &\leq c\int_0^\infty \|\psi_{(\alpha)}(tA)W_\alpha u\|^2 \frac{dt}{t}\\ &\leq c'\|W_\alpha u\|^2 \leq q_\alpha\|u\|^2 \qquad , u \in \mathcal{H}\,,\end{aligned}$$

as required. We used the fact that the positive operator A satisfies a quadratic estimate.

Now let $\psi \in \Psi(S^0_\mu)$. There exist $\theta, \varphi \in \Psi(S^0_\mu)$ and $\alpha, \beta \in (0,1)$ such that

$$\psi(T) = \theta(T)\psi_{(\alpha)}(T) + \varphi(T)\psi_{(\beta)}(T)\,.$$

On returning to the definition of $\psi(tT)$ we find that

$$\|\psi(tT)\| \leq \kappa\{\|\psi_{(\alpha)}(tT)\| + \|\psi_{(\beta)}(tT)\|\}$$

where $\kappa$ depends on $\theta$ and $\varphi$. It now follows from the quadratic estimates for $\psi_{(\alpha)}$ and $\psi_{(\beta)}$ that T satisfies a quadratic estimate with respect to $\psi$.

The dual estimate is proved similarly.

(g) ⇒ (a) : This was proved in the previous section. //

The above theorem, with the exception of parts (a) and (b), is essentially due to Yagi [4], though various parts of it were known previously. The implication (c) ⇒ (e), for example, is taken from the proof of the Heinz–Kato theorem.



## 9. OPERATORS SATISFYING QUADRATIC ESTIMATES

We have already seen that positive self-adjoint operators satisfy quadratic estimates. So do normal operators with spectra in a sector, and also maximal accretive operators.

One could point to a large number of instances where estimates of one type or another of those listed in section 8 have been used by people working in partial differential equations or harmonic analysis. Yagi has used some of this material to show that certain classes of elliptic operators with smooth coefficients satisfy quadratic estimates. (See the references at the end of his paper in this volume.) Thus such operators have an $H_\infty$-functional calculus.

How about the operator S in $L_2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ defined by

$$(Su)(x) = -g(x)^{-1}u''(x)$$

where $g \in L_\infty(\mathbb{R})$ and $Re\, g(x) \geq \kappa > 0$ for all $x \in \mathbb{R}$ ? This can be handled via the following result.

**THEOREM** *Let* $T = W^{-1}A$ *where* A *is a positive self–adjoint operator and* W *is a bounded operator satisfying* $Re(Wu,u) \geq \kappa\|u\|^2$ *for some* $\kappa > 0$ *and all* $u \in \mathcal{H}$. *Then* T *and* $T^*$ *are one–one operators of type* $\omega < \pi/2$ *which satisfy quadratic estimates if the following condition* (C) *is satisfied:*

(C) *There exist constants* c *and* m *such that*

$$\left\{\int_0^\infty \|Q_t(BP_t)^k u\|^2 \frac{dt}{t}\right\}^{1/2} \leq c(1+k^m)\|B\|^k\|u\|$$

*and*

$$\left\{\int_0^\infty \|Q_t(B^*P_t)^k u\|^2 \frac{dt}{t}\right\}^{1/2} \leq c(1+k^m)\|B\|^k\|u\|$$

*for all* $u \in \mathcal{X}$ *and* $k = 1,2,\ldots,$ *where*

$$P_t = (I+t^2A^2)^{-1}\,,\ Q_t = tA(I+t^2A^2)^{-1} \ \textit{and}\ B = I - \lambda W$$

$$\textit{for some}\ \lambda \in (0, 2\kappa\|W\|^{-2})\,.$$

**Proof** It is straightforward to check that $\|B\| < 1$, so $T = \lambda(I-B)^{-1}A$. Let $\tau = t\lambda$. Then

$$(I+itT)^{-1} = R_\tau \sum_{k\neq 0}^{\infty} (BR_\tau)^k (I-B)$$

where

$$R_\tau = (I+i\tau A)^{-1} = P_\tau - iQ_\tau$$

and the series converges because $\|R_\tau\| \le 1$ and $\|B\| < 1$.

It is not difficult to show that $T$ is a one-one operator of type $\omega$ for some $\omega < \pi$.

Let $\psi(\zeta) = \zeta(1+\zeta^2)^{-1}$. Then

$$\begin{aligned}\psi(tT) &= \tfrac{i}{2}\left\{(I+itT)^{-1} - (I-itT)^{-1}\right\}\\ &= \tfrac{i}{2}\left\{R_\tau \sum_{k=0}^{\infty} (BR_\tau)^k - R_{-\tau}\sum_{k=0}^{\infty}(BR_{-\tau})^k\right\}(I-B)\\ &= \tfrac{1}{2}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\left\{(R_\tau B)^{k-s}Q_\tau(BP_\tau)^s + \right.\\ &\qquad\qquad \left. + (R_{-\tau}B)^{k-s}Q_\tau(BP_\tau)^s\right\}(I-B)\,.\end{aligned}$$

Hence

$$\begin{aligned}&\left\{\int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t}\right\}^{1/2}\\ &\le \sum_{k=0}^{\infty}\sum_{s=0}^{k} \|B\|^{k-s}\left\{\int_0^\infty \|Q_\tau(BP_\tau)^s(I-B)u\|^2\frac{d\tau}{\tau}\right\}^{1/2}\\ &\le \sum_{k=0}^{\infty}\sum_{s=0}^{k}\|B\|^k c(1+s^m)\|I-B\|\,\|u\|\\ &= q\|u\| \qquad\qquad , u \in \mathcal{X}\,,\end{aligned}$$

as required. The dual estimate is proved similarly. //



To apply this theorem in the case when $W$ denotes multiplication by the $L_\infty$ function $g$, and $A = D^2$ where $D = -i\frac{d}{dx}$, we need to check that the condition (C) is satisfied. However this is a consequence of the similar estimates proved in [1] where $P_t$ and $Q_t$ were defined in terms of $D$ rather than $A$.

We thus have that the operator $S$ defined before the theorem satisfies square function estimates, along with $S^*$. So, by the results in section 8, $\mathcal{D}(S^{1/2}) = \mathcal{D}((D^2)^{1/2}) = H^1(\mathbb{R})$ and $\|f(S)\| \le c\|f\|_\infty$ for $f \in H_\infty(S^0_{\pi/2})$. As pointed out in Meyer's lecture notes in Madrid, these facts can be used as follows. Say we want to solve the elliptic boundary value problem:

$$\begin{cases} g(x)\dfrac{\partial^2 u}{\partial t^2}(x,t) + \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0\ , & x \in \mathbb{R}\,, t > 0\,,\\ \dfrac{\partial u}{\partial t}\ (x,0) = g(x) & , t > 0\,.\end{cases}$$

Then we find that the solution

$$u(\cdot,t) = -e^{-t\sqrt{S}}\,S^{-1/2}g$$

is defined for all $g \in H^1(\mathbb{R})$ and satisfies

$$\sup_{t>0}\left\{\int |u(x,t)|^2 dx\right\}^{1/2} \le c\left\{\int |\tfrac{dg}{dx}(x)|^2 dx\right\}^{1/2}.$$

## 10. DOUBLE SECTORS

It is just as interesting, if not more so, to consider operators $T$ with spectra in a double sector

$$\$_\omega = \left\{\zeta \in \mathbb{C} \;\middle|\; \zeta \in S_\omega \ \text{ or } \ -\zeta \in S_\omega\right\}$$

for $0 \le \omega < \pi/2$. We can again show that if $T$ and $T^*$ satisfy quadratic estimates then $f(T)$ is defined and satisfies $\|f(T)\| \le c_\mu\|f\|_\infty$ for all $f \in H_\infty(\$^0_\mu)$ where $\mu > \omega$.

In particular this applies when $\mathcal{X} = L_2(\gamma)$, $\gamma = \{s+ig(s) \mid s \in \mathbb{R}\}$, $g$ is a Lipschitz function, and $T = D_\gamma = \frac{1}{i}\frac{d}{dz}\Big|_\gamma$. Then $T$ and $T^*$ satisfy quadratic estimates, so $T$ has an $H_\infty$-functional calculus. In particular $\mathrm{sgn}(T) \in \mathcal{L}(\mathcal{X})$ where $\mathrm{sgn}\,\zeta = +1$ if $Re\,\zeta > 0$ and $\mathrm{sgn}\,\zeta = -1$ if $Re\,\zeta < 1$. The operator $\mathrm{sgn}(T)$ is none other than the Cauchy singular integral operator on $\gamma$. See [1] and [2], where the case of Lipschitz surfaces is treated too.

This paper is already too long, so details will be left as a challenge to the reader.

## REFERENCES


1. R. Coifman, A. McIntosh, Y. Meyer, *L'intégral de Cauchy définit un opérateur borné sur $L^2$ pour les courbes lipschitziennes*, Annals of Mathematics, **116** (1982) 361–387.

2. Alan McIntosh, *Clifford algebras and the higher dimensional Cauchy integral*, Proceedings of the XVII semester, "Approximation theory and function spaces", Banach Center, Warsaw, Poland, 1986.

3. Hiroki Tanabe, *Equations of Evolution*, Pitman, London, S.F., Melbourne, 1979.

4. Atsushi Yagi, *Coincidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs*, C.R. Acad. Sc. Paris, 299, Série I, 1984, 173–176.



School of Mathematics and Physics
Macquarie University NSW 2109
AUSTRALIA



Reprinted (with minor corrections) from *Miniconference on Operator Theory and Partial Differential Equations* (Macquarie University, Sept 8–10, 1986) Proceedings of the Centre for Mathematical Analysis, Australian National University, 14(1986) 210–231.