OPERATORS WHICH HAVE AN H_∞ FUNCTIONAL CALCULUS

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1. INTRODUCTION

An operator T in a Hilbert space $\mathcal{H}$ is said to be of type ω if the spectrum is contained in the sector $S_\omega = \{\zeta \in \mathbb{C} \mid |\arg\zeta| \le \omega\}$ and the resolvent satisfies a bound of the type $\|(T-\zeta I)^{-1}\| \le C_\mu |\zeta|^{-1}$ for all ζ with $|\arg\zeta| \ge \mu$ and all $\mu > \omega$. Let us suppose for now that T is one-one with dense range.

Such an operator has fractional powers T^s and, if $\omega < \pi/2$, generates an analytic semi-group $\{\exp(-tT)\}$. See [3] for details. However it may or may not happen that it generates a C^0-group $\{T^{is} \mid s \in \mathbb{R}\}$ of bounded operators. It was shown by Yagi that the operators T for which $T^{is} \in \mathcal{L}(\mathcal{H})$ are precisely those for which the domains of the fractional powers of T (and of T^*) are the complex interpolation spaces between $\mathcal{H}$ and $\mathcal{D}(T)$ (and between $\mathcal{H}$ and $\mathcal{D}(T^*)$). They are also precisely those operators for which T and T^* satisfy quadratic estimates [4].

It is shown in this paper that another equivalent property is the existence of an $H_\infty(S^0_\mu)$ functional calculus for $\mu > \omega$ (where S^0_μ denotes the interior of S_μ).

In writing up this paper it seemed useful to have a precise definition of the operators $f(T)$ for functions which are analytic (but not necessarily bounded) on S^0_μ and for operators T which do not necessarily satisfy quadratic estimates. Such a definition is given in section 5, where it is shown in what sense formulae of the form $(fg)(T) = f(T)g(T)$ hold. It appears that the basic properties of the semigroups $\{\exp(-tT)\}$ and of the fractional powers T^s can be derived more simply in this way than usual.

The material in this paper has two heritages. One is operator theory, in which area we use results of Kato, Yagi, and many others; the other is harmonic analysis, where the power of quadratic estimates has been recognized since the Littlewood-Paley

theory appeared and the theory of g-functions was developed by Zygmund and his followers. In particular Stein has explored the relationship of quadratic estimates with multiplier results (which concern the functional calculus of $i^{-1}d/dx$). The motivation for this paper is to better understand the functional calculus of $i^{-1}d/dz|_\gamma$, where γ is a Lipschitz curve in the complex plane, though in fact this material is only briefly alluded to in the last section. This builds upon the work of Calderón, and of Coifman and Meyer.

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2. OPERATORS

Throughout this paper $\mathcal{H}$ denotes a complex Hilbert space. By an *operator* is meant a linear mapping $T : \mathcal{D}(T) \to \mathcal{H}$ where the *domain* $\mathcal{D}(T)$ is a linear subspace of $\mathcal{H}$. The *range* of T is denoted by $\mathcal{R}(T)$ and the *nullspace* by $\mathcal{N}(T)$. The *norm* of T is the (possibly infinite) number

$$\|T\| = \sup\{\|Tu\| \mid u \in \mathcal{D}(T), \|u\| = 1\}.$$

We say that T is *bounded* if $\mathcal{D}(T) = \mathcal{H}$ and $\|T\| < \infty$, and denote the algebra of all bounded operators by $\mathcal{L}(\mathcal{H})$. We call T *densely-defined* if $\mathcal{D}(T)$ is dense in $\mathcal{H}$, and *closed* if its graph $\{(u,Tu) \mid u \in \mathcal{D}(T)\}$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$.

When new operators are constructed from old, the domains are taken to be the largest for which the construction makes sense. For example,

$$\mathcal{D}(S+T) = \mathcal{D}(S) \cap \mathcal{D}(T),$$
$$\mathcal{D}(ST) = \{u \in \mathcal{D}(T) \mid Tu \in \mathcal{D}(S)\},$$

and, if T is one-one,

$$\mathcal{D}(T^{-1}) = \mathcal{R}(T) .$$

We write $S \subset T$ if $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $Su = Tu$ for all $u \in \mathcal{D}(S)$. So $S = T$ if and only if $S \subset T$ and $T \subset S$. Note that

$$\begin{aligned} (ST)U &= S(TU) , \\ S(T+U) &\supset ST + SU , \\ (S+T)U &= SU + TU , \\ S - S &\subset 0 , \end{aligned}$$

and, if S is one-one,

$$S^{-1}S \subset I \text{ and } S S^{-1} \subset I .$$

We remark that if B is bounded and T is closed then the following operators are closed: B, TB, $B^{-1}T$ (if B is one-one) and T^{-1} (if T is one-one).

The *adjoint* of a densely-defined operator T is the operator T^* with largest domain which satisfies

$$\langle Tu,v\rangle = \langle u,T^*v\rangle$$

for all $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(T^*)$. We remark that T^* is closed and $(T^{-1})^* = (T^*)^{-1}$ if T is one-one and has dense range.

The *resolvent set* $\rho(T)$ of T is the set of all $\lambda \in \mathbb{C}$ for which $(T-\lambda I)$ is one-one and $(T-\lambda I)^{-1} \in \mathcal{L}(\mathcal{X})$. The *spectrum* $\sigma(T)$ of T is the complement of $\rho(T)$, together with ∞ if T is unbounded.

3. RATIONAL FUNCTIONS OF T

Suppose T is a closed densely-defined operator in $\mathcal{X}$ with nonempty resolvent set. Then T^n is a closed densely-defined operator for all integers $n \geq 0$. (We take $T^0 = I$.) Moreover, if $m \geq n$, then $\mathcal{D}(T^m)$ is a dense subspace of $\mathcal{D}(T^n)$ under the norm $\|u\|_n = \{\|u\|^2 + \|T^n u\|^2\}^{1/2}$.

If p denotes the polynomial $p(\zeta) = \sum_{k=0}^{m} c_k \zeta^k$, then $p(T)$ is defined by $p(T) = \sum_{k=0}^{m} c_k T^k$. This too is a closed operator with domain $\mathcal{D}(p(T)) = \mathcal{D}(T^m)$, dense in $\mathcal{X}$.

If q denotes a polynomial with no zeros in $\sigma(T)$, and $r(\zeta) = p(\zeta) / q(\zeta)$, then we define $r(T)$ by $r(T) = p(T)(q(T))^{-1}$. This too is a closed densely-defined operator with domain $\mathcal{D}(T^n)$ where $n = \max\{0, \deg p - \deg q\}$.

If r and r_1 are two such rational functions and $\alpha \in \mathbb{C}$, then the following identities hold:

(1) $\alpha(r(T)) + r_1(T) = (\alpha r + r_1)(T)|_{\mathcal{D}(r(T))}$

(2) $r_1(T)r(T) = (r_1 r)(T)|_{\mathcal{D}(r(T))}$

(3) $\sigma(r(T)) = r(\sigma(T))$

(4) $r(T)^* = \bar{r}(T^*)$

where $\bar{r}(\zeta) = \bar{p}(\zeta)/\bar{q}(\zeta)$, $\bar{p}(\zeta) = \sum \bar{c}_k \zeta^k$ and $\bar{q}$ is defined similarly.

Although the preceding paragraphs can be read quickly and appear reasonable, it is actually quite tedious to verify every detail. For example it is easy to see that $r(T)^* \supset \bar{r}(T^*)$, but it takes more work to get the equality. Note that (2) includes the statement

$$\mathcal{D}(r_1(T)r(T)) = \mathcal{D}((r_1 r)(T)) \cap \mathcal{D}(r(T)) .$$

If r has no zeros in $\sigma(T) \cap \mathbb{C}$, it is a consequence of (2) and (4) that

$$(r(T)^*)^{-1} = \bar{r}(T^*)^{-1} = (1/\bar{r})(T^*) = ((1/r)(T))^* = (r(T)^{-1})^* .$$

4. OPERATORS OF TYPE ω

If $0 \le \theta \le \pi$, then

$$S_\theta = \{z \in \mathbb{C} \mid z = 0 \text{ or } |\arg z| \le \theta\}$$

and

$$S_\theta^0 = \{z \in \mathbb{C} \mid z \ne 0 \text{ and } |\arg z| < \theta\}.$$

If $0 \le \omega < \pi$, then an operator T in $\mathcal{X}$ is said to be of *type* ω if T is closed and densely-defined, $\sigma(T) \subset S_\omega \cup \{\infty\}$, and for each $\theta \in (\omega,\pi]$ there exists $c_\theta < \infty$ such that $\|(T-zI)^{-1}\| \le c_\theta |z|^{-1}$ for all non-zero $z \notin S_\theta^0$.

If $0 < \mu \le \pi$, then

$$H_\infty(S_\mu^0) = \{f : S_\mu^0 \to \mathbb{C} \mid f \text{ is analytic and } \|f\|_\infty < \infty\}$$

where $\|f\|_\infty = \sup\{|f(z)| \mid z \in S_\mu^0\}$, and

$$\Psi(S_\mu^0) = \left\{f \in H_\infty(S_\mu^0) \mid \exists\, s > 0,\, c \ge 0 \text{ such that } |f(z)| \le \frac{c|z|^s}{1+|z|^{2s}} \text{ for all } z \in S_\mu^0\right\}.$$

It is straightforward to define $\psi(T)$ if $\psi \in \Psi(S_\mu^0)$ and T is of type ω, where $0 \le \omega < \mu \le \pi$. We proceed as follows.

Let $\omega < \theta < \mu$ and let γ be the contour defined by the function

$$g(t) = \begin{cases} -t\, e^{-i\theta}, & -\infty < t \le 0 \\ t\, e^{i\theta}, & 0 \le t < \infty. \end{cases}$$

Define $\psi(T) \in \mathcal{L}(\mathcal{X})$ by

$$\psi(T) = \frac{1}{2\pi i}\int_\gamma (T-\zeta I)^{-1}\psi(\zeta)\,d\zeta.$$

This integral is absolutely convergent in the norm topology on $\mathcal{L}(\mathcal{X})$. It is not difficult to show that the definition is independent of $\theta \in (\omega,\mu)$, and that, if ψ is a rational function, then this definition is consistent with the previous one. We can also show

that, if ψ_1 is also in $\Psi(S_\mu^0)$ and $\alpha \in \mathbb{C}$, then

$$\alpha(\psi(T)) + \psi_1(T) = (\alpha\psi+\psi_1)(T) \tag{1}$$

$$\psi_1(T)\psi(T) = (\psi_1\psi)(T) \tag{2}$$

$$\sigma(\psi(T)) = \psi(\sigma(T)) \tag{3}$$

$$\psi(T)^* = \overline{\psi}(T^*). \tag{4}$$

Moreover, if r is a rational function which is bounded on S_μ and $\psi \in \Psi(S_\mu^0)$, then $r\psi \in \Psi(S_\mu^0)$ and

$$r(T)\psi(T) = (r\psi)(T) = \psi(T)r(T).$$

The operator $(r+\psi)(T)$ can be defined without ambiguity by

$$(r+\psi)(T) = r(T) + \psi(T).$$

We conclude this section with a convergence theorem.

THEOREM *Let* T *be an operator of type* ω *where* $0 \le \omega < \mu \le \pi$. *Let* (ψ_α) *be a net in* $\Psi(S_\mu^0)$ *such that* $\|\psi_\alpha\|_\infty \to 0$.

(a) *If there exist* c *and* $s > 0$ *such that* $|\psi_\alpha(\zeta)| \le c|\zeta|^s(1+|\zeta|^{2s})^{-1}$ *for all* $\zeta \in S_\mu^0$ *and all* α, *then* $\|\psi_\alpha(T)\| \to 0$.

(b) *If there exist* c, M *and* $s > 0$ *such that* $|\psi_\alpha(\zeta)| \le c|\zeta|^s$ *for all* $|\zeta| \le 1$ *and all* α *and* $\|\psi_\alpha(T)\| \le M$ *for all* α, *and if* $u \in \mathcal{X}$, *then* $\psi_\alpha(T)u \to 0$.

(c) *If there exists* M *such that* $\|\psi_\alpha(T)\| \le M$ *for all* α *and if* $u \in \overline{\mathcal{R}(T)}$, *then* $\psi_\alpha(T)u \to 0$.

Proof. To prove (a), use the definition of $\psi_\alpha(T)$ and break up the integral into three parts corresponding to $|\zeta| < \delta$, $\delta \le |\zeta| \le \Delta$, and $|\zeta| > \Delta$ for δ sufficiently small and Δ sufficiently large.

To prove (b) apply part (a) to the functions φ_α defined by $\varphi_\alpha(\zeta) = (1+\zeta)^{-1}\psi_\alpha(\zeta)$ to see that $\psi_\alpha(T)u \to 0$ for all $u \in \mathcal{D}(T)$. Then use the uniform boundedness to obtain the result.

Part (c) has a similar proof.

5. MORE GENERAL FUNCTIONS OF T

In this section $0 \le \omega < \mu \le \pi$ and T is an operator in $\mathcal{X}$ which is not only of type ω but also one-one with dense range. Let

$\mathcal{F}(S_\mu^0) = \{f : S_\mu^0 \to \mathbb{C} \mid f$ is analytic and $|f(z)| \le c(|z|^k + |z|^{-k})$ for some k and $c\}$.

For $f \in \mathcal{F}(S_\mu^0)$ with $|f(z)| \le c(|z|^k + |z|^{-k})$ define $f(T)$ by

$$f(T) = (\psi(T))^{-1}(f\psi)(T)$$

where $\psi(\zeta) = \left[\frac{\zeta}{(1+\zeta)^2}\right]^{k+1}$. The operator $(f\psi)(T) \in \mathcal{L}(\mathcal{X})$ was defined in section 4, while $\psi(T) \in \mathcal{L}(\mathcal{X})$ was defined in section 3. We remark that the operator $\psi(T)$ is one-one with dense range. So $f(T)$ is a closed operator which is densely-defined because its domain includes $\mathcal{R}(\psi(T))$ as is seen by noting that

$$\begin{aligned} f(T)\psi(T) &= \psi(T)^{-1}(f\psi)(T)\psi(T) \\ &= \psi(T)^{-1}\psi(T)(f\psi)(T) \\ &= (f\psi)(T). \end{aligned}$$

It is not difficult to show that this definition is consistent with those of sections 3 and 4. Moreover if $f, f_1 \in \mathcal{F}(S_\mu^0)$ and $\alpha \in \mathbb{C}$, then

(1) $$\alpha(f(T)) + f_1(T) = (\alpha f + f_1)(T)|_{\mathcal{D}(f(T))}$$

(2) $$f_1(T)f(T) = (f_1 f)(T)|_{\mathcal{D}(f(T))}$$

(3) $$f(T)^* = \bar{f}(T^*).$$

This time however there is no spectral mapping theorem. The problem is that $f(T)$ may be unbounded even if f is bounded.

The following can be said about bounds, as is seen by applying (2) above. Suppose $f, g \in \mathcal{F}(S_\mu^0)$ and $g = hf$ for some $h \in \mathcal{F}(S_\mu^0)$ for which $h(T) \in \mathcal{L}(\mathcal{X})$. (e.g. $h = \psi + r$ where $\psi \in \Psi(S_\mu^0)$ and r is a bounded rational function.) Then $\mathcal{D}(g(T)) \supset \mathcal{D}(f(T))$ and $\|g(T)u\| \le c\|f(T)u\|$ for all $u \in \mathcal{D}(f(T))$ and some $c \in \mathbb{R}$.

We conclude this section, like the last, with a convergence theorem.

THEOREM *Let $0 \le \omega < \mu \le \pi$. Let T be an operator of type ω which is one-one with dense range. Let (f_α) be a uniformly bounded net in $H_\infty(S_\mu^0)$, let $f \in H_\infty(S_\mu^0)$, and suppose, for some $M < \infty$, that*

(a) $$\|f_\alpha(T)\| \le M,$$

and

(b) *for each $0 < \delta < \Delta < \infty$, $\sup\{|f_\alpha(\zeta) - f(\zeta)| \mid \zeta \in S_\mu^0$ and $\delta \le |\zeta| \le \Delta\} \to 0$.*

Then $f(T) \in \mathcal{L}(\mathcal{X})$ and $f_\alpha(T)u \to f(T)u$ for all $u \in \mathcal{X}$. So $\|f(T)\| \le M$.

Proof Let $\psi(\zeta) = \zeta(1+\zeta)^{-2}$. Apply part (c) of the earlier theorem to see that $f_\alpha(T)\psi(T)u = (f_\alpha\psi)(T)u \to (f\psi)(T)u = f(T)\psi(T)u$ for all $u \in \mathcal{X}$. As $\psi(T)$ has dense range, $f(T) \in \mathcal{L}(\mathcal{X})$ and $\|f(T)\| \le M$. Now use the uniform boundedness to see that $f_\alpha(T)u \to f(T)u$ for all $u \in \mathcal{X}$.

6. COMPLEX POWERS OF T

We continue to assume that T is an operator of type ω which is one-one with dense range. In this case T^{-1} is also of type ω.

Let $f_\lambda(\zeta) = \zeta^\lambda$ for $\lambda \in \mathbb{C}$. For each λ, $f_\lambda \in \mathcal{F}(S_\mu^0)$, so we can define a closed densely-defined operator T^λ by $T^\lambda = f_\lambda(T)$. This seems to be an efficient way to define T^λ, for not only is it included as part of a general functional calculus, but also the following facts follow from the results of section 5 without further ado:

$$(1)\qquad T^{\lambda}T^{\mu} = T^{\lambda+\mu}\big|_{\mathcal{D}(T^{\mu})}$$

$$(2)\qquad T^{-\lambda} = (T^{\lambda})^{-1} = (T^{-1})^{\lambda}$$

$$(3)\qquad (T^{*})^{\lambda} = (T^{\bar{\lambda}})^{*}$$

$$(4)\qquad \mathcal{D}(T^{\mu}) \supset \mathcal{D}(T^{\lambda}) = \mathcal{D}((I+T)^{\lambda})$$

if $0 \le Re(\mu) < Re(\lambda)$, and

$$\|T^{\mu}u\| \le c(\|T^{\lambda}u\| + \|u\|)$$

and

$$c^{-1}(\|T^{\lambda}u\| + \|u\|) \le \|(I+T)^{\lambda}u\| \le c(\|T^{\lambda}u\|+\|u\|)$$

for $u \in \mathcal{D}(T^{\lambda})$.

The formulae usually used to define T^{λ} can now be derived using the theorem in section 4. For example, to show that, if $0 < s < 1$, then

$$T^{s}u = \beta_s \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_{\epsilon}^{R} t^{-s}(I+tT)^{-1}Tu\, dt$$

for all $u \in \mathcal{D}(T)$, we apply that theorem to the net $\psi_{\epsilon,R}$ defined by

$$\psi_{\epsilon,R}(\zeta) = \beta_s \int_{\epsilon}^{R} t^{-s}(1+t\zeta)^{-1}dt\ \zeta(1+\zeta)^{-1} .$$

(Here $\beta_s^{-1} = \int_0^{\infty} t^{-s}(1+t)^{-1}dt.$)

What if we drop the assumption that T is one-one with dense range? We can proceed as follows. Let

$$\mathcal{F}_0(S_{\mu}^0) = \{f \in \mathcal{F}(S_{\mu}^0) \mid \exists\ f(0) \in \mathbb{C} \text{ such that } |f(\zeta) - f(0)| \le c|\zeta|^s \text{ for } |\zeta| \le 1 \text{ and some } s > 0\} .$$

For $f \in \mathcal{F}_0(S_{\mu}^0)$ define $f(T)$ by

$$f(T) = (\theta(T))^{-1}(f\theta)(T)$$

where $\theta(\zeta) = (1+\zeta)^{-k-1}$ and k is large enough that $|f(\zeta)| \le c|\zeta|^k$ for $|\zeta| \ge 1$. Then $(f\theta)(T) = f(0)(1+T)^{-1} + g(T) \in \mathcal{L}(\mathcal{X})$ because $g \in \Psi$, where $g(\zeta) = (1+\zeta)^{-k-1}f(\zeta) - (1+\zeta)^{-1}f(0)$. Also $\theta(T)$ is a bounded one-one operator with dense range. So $f(T)$ is a closed densely-defined operator. Now proceed as before and we find that operators T^{λ} can be defined which satisfy properties (1), (3), (4) provided $Re\,\lambda > 0$.

7. QUADRATIC ESTIMATES

In the theory developed in section 5, there is no guarantee that $f(T) \in \mathcal{L}(\mathcal{X})$ when f is bounded. Indeed this is not always the case. However it is if T and T^{*} satisfy quadratic estimates.

Let T be an operator of type ω where $0 \le \omega < \mu \le \pi$ and let $\psi \in \Psi(S_{\mu}^0)$. To say that T satisfies a quadratic estimate with respect to ψ means that

$$\left\{ \int_0^{\infty} \|\psi(tT)u\|^2 \frac{dt}{t} \right\}^{1/2} \le q\|u\|$$

for some constant q and all $u \in \mathcal{X}$.

Such an estimate holds for example if T is positive self-adjoint with $q = \left\{ \int_0^{\infty} |\psi(t)|^2 t^{-1}dt \right\}^{1/2}$. It also holds in a lot of other interesting cases.

Let us use the notations

$$\Psi^{+}(S_{\mu}^0) = \{\psi \in \Psi(S_{\mu}^0) \mid \psi(t) > 0 \text{ for all } t \in (0,\infty)\}$$

and

$$\psi_t(\zeta) = \psi(t\zeta) \ , \quad 0 < t < \infty .$$