HARDY SPACES OF EXACT FORMS ON LIPSCHITZ DOMAINS IN $\mathbb{R}^N$

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Abstract

We introduce Hardy spaces $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ of exact l-forms on domains Ω in $\mathbb{R}^N$. We prove atomic decompositions of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ when Ω is a special Lipschitz domain or a bounded strongly Lipschitz domain in $\mathbb{R}^N$ and use these atomic decompositions to characterize dual spaces of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$. We also establish a "div-curl" type theorem on Ω with an application to coercivity.

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CONTENTS

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1. Introduction

In [LM2], we studied Hardy spaces $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ of exact $l-$forms on $\mathbb{R}^N$. We proved their atomic decompositions: any element in these spaces can be decomposed into a sum of exact atoms. In this paper, we introduce Hardy spaces $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ of exact $l-$forms supported in Lipschitz domains $\bar{\Omega} \subset \mathbb{R}^N$. A natural question to ask is: under what conditions on Ω does it follow that $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ has an exact atomic decomposition, i.e. the atoms are exact and have supports in $\bar{\Omega}$?

We answer this question by proving theorems concerning atomic decompositions of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ when Ω is a special Lipschitz domain or a bounded strongly Lipschitz domain in $\mathbb{R}^N$ (Theorems 3.1 and 4.1). In particular, we show, for $1 \leq l \leq N$, that an $l-$form f on $\mathbb{R}^N$ is in $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ if and only if it has an atomic decomposition

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the atoms a_k are exact $l-$forms and have supports in $\bar{\Omega}$ and $\sum_{k=0}^{\infty} |\lambda_k| < \infty$. A crucial element in the proof is the construction of reflection maps in neighborhoods of Lipschitz boundaries.

As a consequence, applying these atomic decompositions, in Section 5 we characterize the dual spaces of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ as BMO-type spaces. The duality result implies the extension theorems of $BMO_r(\Omega)$ to $BMO(\mathbb{R}^N)$ for special Lipschitz domains and bounded strongly Lipschitz domains.

In Section 6, we establish a "div-curl" type theorem on domains by using the duality relationship obtained in Section 5. It is an extension of the "div-curl" theorem by Coifman, Lions, Meyer and Semmes to the case of domains.

Section 7 is devoted to an application of the "div-curl" type theorem to coercivity properties of quadratic forms, which come from the linearization of polyconvex variational integrals studied in nonlinear elasticity by Ball.

In Appendix A we construct a bilipschitz reflection map defined in a neighborhood of a Lipschitz boundary.

Unless otherwise specified, C denotes a constant independent of functions occurring in the inequalities. Such C may differ at different occurrences.

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2. Definitions and Preliminaries

Before we introduce Hardy spaces of exact forms on domains, we first review some definitions and known results concerning Hardy spaces, BMO and tent spaces, which we use later on.

Unless otherwise specified, domains considered in this paper are special Lipschitz domains and bounded strongly Lipschitz domains. *A special Lipschitz domain* is the domain above the graph of Lipschitz function defined on $\mathbb{R}^{N-1}$. A *strongly Lipschitz domain* is by definition a domain in $\mathbb{R}^N$ whose boundary is covered by a finite number of parts of Lipschitz graphs (up to rotations) at most one of them being infinite.

Definition 2.1. A function $f \in L^1(\Omega)$ belongs to $\mathcal{H}^1_z(\Omega)$ if the zero extension f_z of f to $\mathbb{R}^N$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$. This is a Banach space under the norm

$$\|f\|_{\mathcal{H}^1_z(\Omega)} = \|f_z\|_{\mathcal{H}^1(\mathbb{R}^N)}.$$

For relevant details pertaining to the Hardy spaces on $\mathbb{R}^N$ and on domains Ω of $\mathbb{R}^N$, the reader is referred to [JSW], [M], [CKS], [CDS] (for local Hardy spaces) and [AR].

Definition 2.2. A Lebesgue measurable function a is said to be an $\mathcal{H}^1_z(\Omega)$-*atom* if there exists a cube $Q \subset \Omega$ such that supp $a \subset Q$ and a satisfies the moment condition:

$$\int_Q a(x)\, dx = 0$$

and the size condition:

$$\|a\|_{L^2(Q)} \le |Q|^{-1/2},$$

where $|Q|$ denotes the volume of Q.

These atoms provide a description of $\mathcal{H}^1_z(\Omega)$.

Theorem 2.3. *A function f on Ω belongs to $\mathcal{H}^1_z(\Omega)$ if and only if it has a decomposition*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the a_k's are $\mathcal{H}^1_z(\Omega)$-atoms and $\sum_{k=0}^{\infty} |\lambda_k| < \infty$. Furthermore,

$$\|f\|_{\mathcal{H}^1_z(\Omega)} \sim \inf \left(\sum_{k=0}^{\infty} |\lambda_k| \right),$$

where the infimum is taken over all such decompositions. The constants of the proportionality are independent of f.

Theorem 2.3 is proved by a constructive method in [CKS, Theorem 3.2] on bounded strongly Lipschitz domains for local Hardy spaces. In [C], it is derived from an extension theorem by Jones for BMO [J, Theorem 1] and duality. Another argument can be found in [JSW]. If Ω is unbounded, see [AR, Theorem 1].

Definition 2.4. A function f is said to be in $BMO_r(\Omega)$ if f is locally integrable and

$$\sup_{Q \subset \Omega} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^2\, dx \right)^{1/2} < \infty, \tag{2.1}$$

where the supremum is taken over all cubes Q in the domain Ω.

Remark. The supremum in (2.1) can also be taken over all cubes with sides parallel to the axes or over all balls in Ω [J, Theorem 1].

The duality statement is as follows [C]. See also [JSW] and [AR].

Theorem 2.5. *The dual space of* $\mathcal{H}^1_z(\Omega)$ *is* $BMO_r(\Omega)$.

To define tent spaces we need the following square function

$$S(F)(x) = \left(\int_{\Gamma(x)} |F(y,t)|^2 \, \frac{dydt}{t^{N+1}} \right)^{1/2},$$

where $\Gamma(x) = \{(y,t) : |x-y| < t\}$ denotes the cone with vertex at $x \in \mathbb{R}^N$.

Definition 2.6. The *tent space* $\mathcal{N}^p(\mathbb{R}^{N+1}_+)$, $1 \le p < \infty$, is defined as the space of functions F on $\mathbb{R}^{N+1}_+$ so that $S(F) \in L^p(\mathbb{R}^N)$. The space is equipped with the norm

$$\|F\|_{\mathcal{N}^p(\mathbb{R}^{N+1}_+)} = \|S(F)\|_{L^p(\mathbb{R}^N)}.$$

We now come to the atomic decomposition of $\mathcal{N}^1(\mathbb{R}^{N+1}_+)$. Let us start with the definition of $\mathcal{N}^1(\mathbb{R}^{N+1}_+)$-atoms.

Definition 2.7. An $\mathcal{N}^1(\mathbb{R}^{N+1}_+)$-*atom* is a function α on $\mathbb{R}^{N+1}_+$ supported in a tent $T(B) = \{(x,t) : |x-x_0| \le r-t\}$ over a ball $B = B(x_0, r)$ in $\mathbb{R}^N$ with

$$\int_{T(B)} |\alpha(y,t)|^2 \, \frac{dydt}{t} \le |B|^{-1}.$$

The following atomic decomposition theorem for $\mathcal{N}^1(\mathbb{R}^{N+1}_+)$ was proved by Coifman, Meyer and Stein in [CMS].

Theorem 2.8. *Any* $F \in \mathcal{N}^1(\mathbb{R}^{N+1}_+)$ *can be written as*

$$F = \sum_{k=0}^{\infty} \lambda_k \alpha_k, \tag{2.2}$$

where the α_k*'s are* $\mathcal{N}^1(\mathbb{R}^{N+1}_+)$*-atoms and*

$$\sum_{k=0}^{\infty} |\lambda_k| \le C \|F\|_{\mathcal{N}^1(\mathbb{R}^{N+1}_+)}$$

for some constants C *independent of* F.

Remark. The atoms α_k can be chosen with supp $\alpha_k \subset$ supp F. For if α_k is an atom, so is $\alpha_k \chi_{\text{supp } F}$, where $\chi_{\text{supp } F}$ is the characteristic function on supp F.

Now we introduce Hardy spaces of exact forms on domains Ω (see, for example, Appendix B in [LM2] for information on forms). Let $\mathcal{D}'(\Omega, \wedge^k)$ denote the space of distributions on Ω with values in $\wedge^k$.

Definition 2.9. Suppose $1 \le l \le N$. The *Hardy space of exact l-forms supported in* $\bar{\Omega}$ is defined as

$$\begin{aligned}\mathcal{H}^1_{z,d}(\Omega, \wedge^l) = \{f \in & \mathcal{H}^1(\mathbb{R}^N, \wedge^l) : f = dg \text{ for some} \\ & g \in \mathcal{D}'(\mathbb{R}^N, \wedge^{l-1}), \ \ \text{supp } g \subset \overline{\Omega}\}\end{aligned}$$

with the norm

$$\|f\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^l)} = \|f\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^l)}.$$

Remark. When $l = N$, $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ is isomorphic to the Hardy space $\mathcal{H}^1_z(\Omega)$. When $l = N-1$ and $\Omega = \mathbb{R}^N_+$, $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ reduces to the divergence-free Hardy space on $\mathbb{R}^N_+$ which was studied in [LM1].

Definition 2.10. We say that a is an $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-*atom* if

(i) there exists $b \in L^2(\Omega, \wedge^{l-1})$ supported in a cube Q with $4Q \subset \Omega$ such that $a = db$;
(ii) a satisfies the size condition: $\|a\|_{L^2(Q,\wedge^l)} \le |Q|^{-1/2}$.

Remarks. (1) Here we require that supports of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms are away from the boundary of Ω, which is stronger than the usual definition of $\mathcal{H}^1_z(\Omega)$-atoms.

(2) The function b can be chosen to satisfy a size estimate: $\|b\|_{L^2(Q,\wedge^{l-1})} \le C\, l(Q)|Q|^{-1/2}$, where $l(Q)$ denotes the side-length of Q and C is independent of b and Q. This follows from the following lemma, which is a consequence of Theorem 3.3.3 in [Sc].

Lemma 2.11. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Suppose $u = dv \in L^2(\mathbb{R}^N, \wedge^l)$ and $v \in \mathcal{D}'(\mathbb{R}^N, \wedge^{l-1})$ with support in $\bar{\Omega}$. Then there exists $w \in H^1_0(\Omega, \wedge^{l-1})$ and a constant C depending only on Ω such that*

$$u = dw$$

and

$$\|w\|_{H^1(\Omega,\wedge^{l-1})} \le C\|u\|_{L^2(\Omega,\wedge^l)}.$$

If $\Omega = B$, a ball in $\mathbb{R}^N$, we have

$$\|Dw\|_{L^2(B,\wedge^{l-1})} \le C\|u\|_{L^2(B,\wedge^l)}$$

and

$$\|w\|_{L^2(B,\wedge^{l-1})} \le C\ r(B)\|u\|_{L^2(B,\wedge^l)}$$

for a constant C which depends only on the dimension N.

In Lemma 2.11, $H^1_0(\Omega, \wedge^{l-1})$ denotes the space of functions in the Sobolev space $H^1(\Omega, \wedge^{l-1})$ with zero boundary values, $Dw = \left(\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_N}\right)$, $\frac{\partial w}{\partial x_i} = \sum_I \frac{\partial w}{\partial x_i} dx_I$ for $w = \sum_I w_I x_I$ and $r(B)$ is the radius of B. It is easy to see that Lemma 2.11 gives the size estimate of the function b in Remark (2). In fact, suppose Ψ is a bilipschitz function which maps the cube Q to the ball B. Let $\alpha := (\Psi^{-1})^* a$. Applying Lemma 2.11 to α, there exists $\beta \in L^2(B, \wedge^{l-1})$ satisfying

$$\|\beta\|_{L^2(B,\wedge^{l-1})} \le C\ r(B)\|\alpha\|_{L^2(B,\wedge^l)}. \tag{2.3}$$

Denote $b := \Psi^*\beta$. Then (2.3) implies the desired estimate.

3. Atomic Decompositions of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ for Special Lipschitz Domains

In the present section, we prove the atomic decomposition of the space $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ for special Lipschitz domains by using the decomposition of $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ and the reflection property. Suppose $\Omega = \{x : x_N > \Phi(x')\}$, where $\Phi : \mathbb{R}^{N-1} \to \mathbb{R}$ is a Lipschitz function. One of our main theorems is the following:

Theorem 3.1. *Let $1 \leq l \leq N$, and let Ω be a special Lipschitz domain in $\mathbb{R}^N$. An $l-$form f on $\mathbb{R}^N$ is in $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ if and only if it has a decomposition*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the a_k's are $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms and $\sum_{k=0}^{\infty} |\lambda_k| < \infty$.

Furthermore,

$$\|f\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^l)} \sim \inf \left(\sum_{k=0}^{\infty} |\lambda_k| \right),$$

where the infimum is taken over all such decompositions. The constants of the proportionality depend only on the dimension N and on the Lipschitz constant of Ω, namely $\|D\Psi\|_\infty$.

Proof. We first prove the "if" part. Suppose that f can be written as a sum of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms. Then $f \in \mathcal{H}^1(\mathbb{R}^N, \wedge^l)$ with

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^l)} \leq C \sum_{k=0}^{\infty} |\lambda_k|.$$

When $N = 1$ we have nothing to do. When $N \geq 2$ we need to prove that the sum $\sum_{k=0}^{\infty} \lambda_k b_k$ is convergent in the sense of distributions on $\mathbb{R}^N$. For then $f = dg$ with $g := \sum_{k=0}^{\infty} \lambda_k b_k \in \mathcal{D}'(\mathbb{R}^N, \wedge^{l-1})$. Since

$$\sum_{k=m}^{n} |\lambda_k| \to 0 \qquad \text{as} \qquad m,\ n \to \infty,$$

we have, for any $\varphi \in C_0^\infty(\mathbb{R}^N, \wedge^{N-l+1})$ with compact support K,

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\Big(\sum_{k=m}^{n}\lambda_k b_k\Big)\wedge\varphi\Big| &\le \sum_{k=m}^{n}|\lambda_k|\Big|\int_{Q_k\cap K} b_k\wedge\varphi\Big|\\ &\le C\sum_{k=m}^{n}|\lambda_k|\|b_k\|_{L^2(Q_k\cap K,\wedge^{l-1})}|Q_k\cap K|^{1/2}\\ &\le C\sum_{k=m}^{n}|\lambda_k| l(Q_k)|Q_k|^{-1/2}|Q_k\cap K|^{1/2}\\ &\le C\sum_{k=m}^{n}|\lambda_k|\min\{1,|\Omega|\}\\ &\le C(1+|\Omega|)\sum_{k=m}^{n}|\lambda_k|\to 0 \qquad \text{as} \ \ m,\ n\to\infty,\end{aligned}$$

where the constant C depends only on φ and we used the size estimate of b_k in Remark (2). The convergence of $\sum_{k=0}^{\infty}\lambda_k b_k$ is proved.

Now we prove the "only if" part. It follows from the proof of Theorem 6.4 in [LM2] that any $f \in \mathcal{H}^1_{z,d}(\Omega,\wedge^l) \subset \mathcal{H}^1_d(\mathbb{R}^N,\wedge^l)$ has a decomposition

$$f = \sum_{k=0}^{\infty}\lambda_k a_k \tag{3.1}$$

with

$$\sum_{k=0}^{\infty}|\lambda_k| \le C\|f\|_{\mathcal{H}^1_d(\mathbb{R}^N,\wedge^l)} = C\|f\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^l)},$$

where the a_k's are $\mathcal{H}^1_d(\mathbb{R}^N,\wedge^l)$-atoms, i.e. there exist $b_k \in L^2(\mathbb{R}^N,\wedge^{l-1})$ supported in cubes $Q_k \subset \mathbb{R}^N$ such that $a_k = db_k$ and $\|a\|_{L^2(Q_k,\wedge^l)} \le |Q_k|^{-1/2}$. .

Define the reflection

$$r : \Big(x', x_N\Big) \to \Big(x', 2\Phi(x') - x_N\Big) \quad \text{in} \ \ \mathbb{R}^N,$$

where $x' = (x_1,\cdots,x_{N-1})$. Since supp $f \subset \overline{\Omega}$ then supp $r^*f \subset \overline{\Omega^c}$, where r^*f is the pull-back of f [T, page 64]. Combining this with (3.1) and noting that $r^*(db_k) = d(r^*b_k)$, we have

$$\begin{aligned} f|_\Omega &= (f - r^*f)|_\Omega\\ &= \sum_{k=0}^{\infty}\lambda_k(a_k - r^*a_k)|_\Omega\\ &= \sum_{k=0}^{\infty}\lambda_k d(b_k - r^*b_k)|_\Omega. \end{aligned} \tag{3.2}$$

For $k = 0,1,\cdots$, let

$$\alpha_k = \begin{cases} a_k - r^*a_k & \text{in } \Omega;\\ 0 & \text{in } \overline{\Omega^c}\end{cases}$$

and

$$\beta_k = \begin{cases} b_k - r^* b_k & \text{in } \Omega; \\ 0 & \text{in } \overline{\Omega^c}. \end{cases}$$

Then (3.2) gives

$$f = \sum_{k=0}^{\infty} \lambda_k \alpha_k. \tag{3.3}$$

We now show that α_k and β_k satisfy the following conditions:

(1) there exists a cube $Q'_k \subset \mathbb{R}^N$ such that supp $\beta_k \subset Q'_k$;

(2) $\alpha_k = d\beta_k$ in $\mathbb{R}^N$;

(3) $\|\alpha_k\|_{L^2(Q'_k, \wedge^l)} \leq C|Q'_k|^{-1/2}$, where the constant C is independent of α_k and Q'_k.

We first prove (2). It suffices to show that

$$\int_{\mathbb{R}^N} (\alpha_k - d\beta_k) \wedge w = 0$$

for all $w \in C_0^\infty(\mathbb{R}^N, \wedge^{N-l})$. In fact, from Stokes' theorem we have

$$\begin{aligned} \int_{\mathbb{R}^N} \alpha_k \wedge w &= \int_\Omega (a_k - r^* a_k) \wedge w \\ &= \int_\Omega d(b_k - r^* b_k) \wedge w \\ &= (-1)^{l+1} \int_\Omega (b_k - r^* b_k) \wedge dw + \int_{\partial\Omega} n \vee \big(n \wedge (b_k - r^* b_k) \wedge w\big) \\ &= (-1)^{l+1} \int_{\mathbb{R}^N} \beta_k \wedge dw \\ &= \int_{\mathbb{R}^N} d\beta_k \wedge w \end{aligned}$$

for all $w \in C_0^\infty(\mathbb{R}^N, \wedge^{N-l})$, where we used the fact that for any form $b \in \mathcal{D}'(\mathbb{R}^N, \wedge^k)$,

$$n \wedge (b - r^* b)|_{\partial\Omega} = 0. \tag{3.4}$$

In fact, from (9.6) of [T, page 362], (3.4) is equivalent to

$$i^*(b - r^* b) = 0, \tag{3.5}$$

where $i : x \in \partial\Omega \to x \in \mathbb{R}^N$ is the inclusion. Note that r is the identity map on $\partial\Omega$. Then (3.5) follows from the following

$$i^*(b - r^* b) = i^*(b) - (r \circ i)^*(b) = 0.$$

We now prove (1) and (3). The proof is divided into three cases according to the location of the support Q_k of a_k.

Case (a): $Q_k \subset \Omega$. Set $Q'_k = Q_k$, then $\alpha_k = a_k$ and $\beta_k = b_k$.

Case (b): $Q_k \subset \Omega^c$. Then $\alpha_k = -r^* a_k$ and $\beta_k = -r^* b_k$. Let Q'_k be the smallest cube containing $r(Q_k)$ and let $a_k = \sum_{1\le i_1<\cdots<i_l\le N} a_k^{i_1,\cdots,i_l}(x)\, dx_{i_1}\wedge\cdots\wedge dx_{i_l}$, $y = r(x)$. We have

$$\begin{aligned}\int_{Q'_k} |r^* a_k(x)|^2\, dx &= \int_{Q_k} \Big| \sum_{1\le i_1<\cdots<i_l\le N} a_k^{i_1,\cdots,i_l}(y) \frac{\partial(y_{i_1},\cdots,y_{i_l})}{\partial(x_{i_1},\cdots,x_{i_l})}\Big|^2 |\det(Dr(y))|\, dy \\ &\sim \|a_k\|^2_{L^2(Q_k,\wedge^l)}, \end{aligned} \tag{3.6}$$

where the implicit constants depend only on N and $\|D\Psi\|_\infty$. Combining (3.6) with the size condition of a_k gives

$$\begin{aligned}\int_{Q'_k} |\alpha_k(x)|^2\, dx &= \int_{Q'_k} |r^* a_k(x)|^2\, dx \\ &\le C\|a_k\|^2_{L^2(Q_k,\wedge^l)} \\ &\le C|Q_k|^{-1} \le C|Q'_k|^{-1}, \end{aligned} \tag{3.7}$$

where we used the fact that $l(Q_k) \sim l(Q'_k)$.

Case (c): $Q_k \cap \partial\Omega \ne \emptyset$. Let Q'_k be the smallest cube containing $(Q_k\cap\Omega)\cup r(Q_k\setminus\bar{\Omega}))$. Then supp $\alpha_k =$ supp $\beta_k \subset Q'_k$. Similar to the proof of (3.7) we have

$$\begin{aligned}\int_{Q'_k} |\alpha_k(x)|^2\, dx &\le 2\int_{Q_k\cap\Omega} |a_k(x)|^2\, dx + 2\int_{r(Q_k\setminus\Omega)} |r^* a_k(x)|^2\, dx \\ &\le C\int_{Q_k} |a_k(x)|^2\, dx \\ &\le C|Q_k|^{-1} \le C|Q'_k|^{-1}. \end{aligned}$$

Our next task is to decompose each atom α_k as a sum of $\mathcal{H}^1_{z,d}(\Omega,\wedge^l)$-atoms. Consider subdomains A of Ω of the type

$$\begin{aligned} A = \{(x', x_N) : x' = (x_1,\cdots,x_{N-1}),\ c_i < x_i < d_i, \\ i = 1,\cdots,N-1,\ \Phi(x') < x_N < \Phi(x') + s\}, \end{aligned}$$

where $s \in (0,+\infty)$ and $d_i - c_i = s$, $1 \le i \le N-1$. Let A_k be the smallest domain of this type such that $A_k \supset Q'_k \cap \Omega$. From properties of Lipschitz functions, there is an integer n depending only on the Lipschitz constant of the domain Ω such that $A_k \subset nQ'_k$ for all k. Also there exists a bilipschitz function Ψ such that Ψ maps A_k to the ball $B_k = B_k(x_k, s)$ and has bilipschitz constants which depend only on the Lipschitz constant of Ω. Applying the Whitney decomposition to the domain A_k with respect to its boundary ∂A_k [St, Chapter 6], A_k can be decomposed into a family of subcubes:

$$A_k = \bigcup_{i=0}^{\infty} Q_k^i$$

such that $8Q_k^i \subset \Omega$ and $|A_k| = \sum_{i=0}^{\infty} |Q_k^i|$. Denote $\Psi(Q_k^i) := B_k^i$, so that $B_k = \bigcup_{i=0}^{\infty} B_k^i$. Let

$$\sum_{i=0}^{\infty} \eta_k^i(x) \equiv 1 \quad \text{in} \quad A_k$$

be a smooth partition of unity satisfying $\eta_k^i(x) = 1$ if $x \in Q_k^i$, $\eta_k^i(x) = 0$ if $x \notin 2Q_k^i$, and

$$|D\eta_k^i(x)| \le C\ l(Q_k^i)^{-1}.$$

For $k = 0, 1, \cdots$, define

$$\tilde{\alpha}_k = (\Psi^{-1})^* \alpha_k \qquad \text{and} \qquad \tilde{\beta}_k = (\Psi^{-1})^* \beta_k.$$

Then properties (1) – (3) of α_k and β_k yield the following properties of $\tilde{\alpha}_k$ and $\tilde{\beta}_k$:

1) supp $\tilde{\beta}_k \subset B_k$;
2) $\tilde{\alpha}_k = d\tilde{\beta}_k$ in $\mathbb{R}^N$;
3) $\|\tilde{\alpha}_k\|_{L^2(B_k,\wedge^l)} \le C|B_k|^{-1/2}$, where C is independent of $\tilde{\alpha}_k$ and B_k.

Applying Lemma 2.11 to $\tilde{\alpha}_k$, one finds that there exists $\tilde{\psi}_k \in H_0^1(B_k, \wedge^{l-1})$ such that

$$\tilde{\alpha}_k = d\tilde{\psi}_k$$

and

$$\|D\tilde{\psi}_k\|_{L^2(B_k,\wedge^{l-1})} \le C\|\tilde{\alpha}_k\|_{L^2(B_k,\wedge^l)} \tag{3.8}$$

for constants C independent of B_k and $\tilde{\alpha}_k$. Let $\Psi^*\tilde{\psi}_k := \varphi_k$, so that supp $\varphi_k \subset A_k$ and

$$\alpha_k = \Psi^*\tilde{\alpha}_k = \Psi^*(d\tilde{\psi}_k) = d(\Psi^*\tilde{\psi}_k) = d\varphi_k.$$

From the smooth partition of unity, we have

$$\alpha_k = \sum_{i=0}^{\infty} d(\eta_k^i \varphi_k) := \sum_{i=0}^{\infty} \tau_k^i \alpha_k^i, \tag{3.9}$$

where

$$\alpha_k^i = \frac{d(\eta_k^i\varphi_k)}{|2Q_k^i|^{1/2}\|d(\eta_k^i\varphi_k)\|_{L^2(2Q_k^i,\wedge^l)}}$$

and

$$\tau_k^i = |2Q_k^i|^{1/2}\|d(\eta_k^i\varphi_k)\|_{L^2(2Q_k^i,\wedge^l)}.$$

It is clear that α_k^i is an $\mathcal{H}^1_{z,d}(\Omega,\wedge^l)$-atom, and f is a sum of α_k^i's by (3.3) and (3.9). Therefore Theorem 3.1 is proved if we can show that there exists a constant C depending only on the dimension N and the domain Ω such that

$$\sum_{i=0}^{\infty} \tau_k^i \le C < \infty. \tag{3.10}$$

To prove (3.10), we write

$$\begin{aligned}\sum_{i=0}^{\infty} \tau_k^i &\le 2^{1/2} \sum_{i=0}^{\infty} |2Q_k^i|^{1/2} \Big(\|\eta_k^i d\varphi_k\|_{L^2(2Q_k^i,\wedge^l)} + \|\varphi_k d\eta_k^i\|_{L^2(2Q_k^i,\wedge^{l-1})}\Big) \\ &:= I + II.\end{aligned}$$

Note that $0 \leq \eta_k^i \leq 1$. By the size condition of α_k, we have

$$\begin{aligned} I &\leq 2^{1/2} \left(\sum_{i=0}^{\infty} |2Q_k^i| \right)^{1/2} \left(\sum_{i=0}^{\infty} \int_{2Q_k^i} |\alpha_k|^2 \, dx \right)^{1/2} \\ &\leq C|A_k|^{1/2} \|\alpha_k\|_{L^2(A_k, \wedge^l)} \\ &\leq C|Q_k'|^{1/2} \|\alpha_k\|_{L^2(Q_k', \wedge^l)} \leq C, \end{aligned}$$

where the constant C depends only on N and Ω. Since $d(x, \partial A_k) \sim d(y, \partial B_k)$ for $x \in A_k$, $y = \Psi(x)$, we obtain

$$\begin{aligned} \|\varphi_k d\eta_k^i\|_{L^2(2Q_k^i, \wedge^{l-1})} &\leq C \int_{2Q_k^i} \left| \frac{\varphi_k(x)}{d(x, \partial A_k)} \right|^2 dx \\ &\leq C \int_{\tilde{B}_k^i} \left| \frac{\varphi_k(\Psi^{-1}(y))}{d(y, \partial B_k)} \right|^2 dy \\ &\leq C \int_{\tilde{B}_k^i} \left| \frac{\tilde{\psi}_k(y)}{d(y, \partial B_k)} \right|^2 dy, \end{aligned}$$

where $\tilde{B}_k^i = \Psi(2Q_k^i)$ and C depends only on Ψ. Applying Hardy's inequality (see, for example, [D, Chapter 1, Section 5]), we get

$$\begin{aligned} II &\leq C \left(\sum_{i=0}^{\infty} |2Q_k^i| \right)^{1/2} \left(\sum_{i=0}^{\infty} \int_{\tilde{B}_k^i} \left| \frac{\tilde{\psi}_k(y)}{d(y, \partial B_k)} \right|^2 dy \right)^{1/2} \\ &\leq C|A_k|^{1/2} \left(\int_{B_k} \left| \frac{\tilde{\psi}_k(y)}{d(y, \partial B_k)} \right|^2 dy \right)^{1/2} \\ &\leq C|A_k|^{1/2} \|D\tilde{\psi}_k\|_{L^2(B_k, \wedge^{l-1})} \\ &\leq C|A_k|^{1/2} \|\tilde{\alpha}_k\|_{L^2(B_k, \wedge^l)} \quad \text{(by (3.8))} \\ &\leq C|A_k|^{1/2} |B_k|^{-1/2} \leq C, \end{aligned}$$

where C depends only on N and Ω. This completes the proof of (3.10). The proof of Theorem 3.1 is finished. □

4. Atomic Decompositions of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ for Bounded Lipschitz Domains

In this section we consider Hardy spaces $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ when Ω is a bounded strongly Lipschitz domain. As one of the main results of this paper we prove the following atomic decomposition theorem. Its proof uses atomic decompositions of tent spaces and a reflection map defined in a neighborhood of the boundary $\partial\Omega$ of the domain Ω (see Appendix A).

Theorem 4.1. *Suppose Ω is a bounded strongly Lipschitz domain in $\mathbb{R}^N$ and $1 \leq l \leq N$. An l−form f on $\mathbb{R}^N$ is in $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ if and only if f can be written as*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the a_k's are $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms and $\sum_{k=0}^{\infty} |\lambda_k| < \infty$.

Furthermore,

$$\|f\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^l)} \sim \inf \left(\sum_{k=0}^{\infty} |\lambda_k| \right),$$

where the infimum is taken over all such decompositions. The constants of the proportionality depend only on N and Ω.

To prove Theorem 4.1 we first prove a weaker result. Here and subsequently $\Omega_\gamma = \{x : \operatorname{dist}(x, \Omega) < \gamma\}$ for $\gamma > 0$.

Proposition 4.2. *Suppose Ω is a bounded strongly Lipschitz domain in $\mathbb{R}^N$, $1 \le l \le N$ and $\varepsilon > 0$. Then any $f \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ can be written as*

$$f = \sum_{j=0}^{\infty} \mu_j a_j$$

with

$$\sum_{j=0}^{\infty} |\mu_j| \le C \|f\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^l)}$$

for some constants C independent of f, where the functions a_j satisfy the following conditions:

(1) there exists $b_j \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in a cube $Q_j \subset \Omega_{6\sqrt{N}\varepsilon}$ such that $l(Q_j) < \varepsilon$ and $a_j = db_j$;

(2) $\|a_j\|_{L^2(Q_j,\wedge^l)} \le |Q_j|^{-1/2}$.

Proof. It follows from the proof of Theorem 2.3 in [LM2] that any $f \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ can be written as

$$f = -\int_0^\infty td\big(t\delta(f * \varphi_t) * \varphi_t\big)\, \frac{dt}{t},$$

where $\varphi \in C_0^\infty(\mathbb{R}^N)$ with support in the unit ball and $\int_0^\infty t|\xi|^2 \hat{\varphi}(t\xi)^2\, dt = 1$. For any $\varepsilon > 0$, write

$$\begin{aligned} f &= -\int_0^\varepsilon td\big(t\delta(f * \varphi_t) * \varphi_t\big)\, \frac{dt}{t} - \int_\varepsilon^\infty td\big(t\delta(f * \varphi_t) * \varphi_t\big)\, \frac{dt}{t} \\ &:= A + a. \end{aligned} \tag{4.1}$$

Define $F(y,t) = t\delta(f * \varphi_t)(y)$ for $0 < t \le \varepsilon$ and $F(y,t) = 0$ if $t > \varepsilon$. Similar to the proof of Theorem 6.4 in [LM2] we can show that $F \in \mathcal{N}^1(\mathbb{R}^{N+1}_+, \wedge^{l-1})$ and

$$\|F\|_{\mathcal{N}^1(\mathbb{R}^{N+1}_+,\wedge^{l-1})} \le C \|f\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^l)}.$$

Since $\operatorname{supp} f \subset \bar{\Omega}$ then $\operatorname{supp} F \subset \bar{\Omega}_\varepsilon \times (0, \varepsilon]$. Using Theorem 2.8, F has a decomposition

$$F = \sum_{k=1}^{\infty} \lambda_k \alpha_k$$

with

$$\sum_{k=1}^{\infty} |\lambda_k| \le C\|F\|_{\mathcal{N}^1(\mathbb{R}^{N+1}_+,\wedge^{l-1})} \le C\|f\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^l)},$$

where the α_k's are $\mathcal{N}^1(\mathbb{R}^{N+1}_+,\wedge^{l-1})$-atoms with supports in $T(B_k)\cap\big(\bar{\Omega}_\varepsilon\times(0,\varepsilon]\big)$ for some balls B_k satisfying $r(B_k)<\varepsilon/4$, and

$$\int_{T(B_k)} |\alpha_k(y,t)|^2\ \frac{dydt}{t} \le \frac{1}{|B_k|}. \tag{4.2}$$

Define

$$a_k = -\int_0^\varepsilon td\big(\alpha_k(\cdot,t)*\varphi_t\big)\frac{dt}{t}.$$

We have supp $a_k\subset 2B_k\cap\bar{\Omega}_{2\varepsilon}$ and

$$A=\sum_{k=1}^{\infty}\lambda_k a_k. \tag{4.3}$$

Let $\alpha_k:=\sum_J \alpha_k^J e_J\in\wedge^{l-1}$, where the α_k^J's are $\mathcal{N}^1(\mathbb{R}^{N+1}_+)$-atoms. Then $a_k=\sum_{i,J}a_k^{i,J}e_i\wedge e_J$, where

$$\begin{aligned} a_k^{i,J} &= -\int_0^\varepsilon \alpha_k^J(\cdot,t)*(\partial_i\varphi)_t\ \frac{dt}{t}\\ &= -\int_0^\infty \big(\alpha_k^J(\cdot,t)\chi(t/\varepsilon)\big)*(\partial_i\varphi)_t\ \frac{dt}{t}, \end{aligned}$$

where χ denotes the characteristic function in the unit ball. Given $\psi\in C_0^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\psi(x)\ dx=0$, it is proved in Theorem 6 of [CMS] that

$$\Big\|\int_0^\infty a(\cdot,t)*\psi_t\ \frac{dt}{t}\Big\|_{L^2(\mathbb{R}^N)} \le C\|a\|_{\mathcal{N}^2(\mathbb{R}^{N+1}_+)}$$

for a constant C independent of a. Thus

$$\begin{aligned} \|a_k^{i,J}\|^2_{L^2(2B_k)} &\le C\|\alpha_k^J\|^2_{\mathcal{N}^2(\mathbb{R}^{N+1}_+)}\\ &= C\int_{\mathbb{R}^N}\int_{\Gamma(x)}\Big|\alpha_k^J(y,t)\Big|^2\ \frac{dydt}{t^{N+1}}\ dx\\ &= C\int_{\mathbb{R}^{N+1}_+}\Big|\alpha_k^J(y,t)\Big|^2\ \frac{dydt}{t}\\ &\le C|2B_k|^{-1} \end{aligned}$$

by (4.2). This implies $\|a_k\|_{L^2(2B_k,\wedge^l)}\le C|2B_k|^{-1/2}$, where C is independent of k. From the definition of a_k, we see that a_k can be written as $a_k=d\tilde{b_k}$ for some $\tilde{b_k}\in L^1(\mathbb{R}^N,\wedge^{l-1})$ with support in $2B_k\cap\bar{\Omega}_{2\varepsilon}$ satisfying

$$\|\tilde{b}_k\|_{L^1(2B_k,\wedge^{l-1})}\le C\varepsilon \tag{4.4}$$

for a constant C independent of k. Applying Lemma 2.11 to a_k, there exists $b_k \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in $2B_k$ such that $a_k = db_k$. Let Q_k be the smallest cube containing $2B_k$, then $a_k = db_k$ for $b_k \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in $Q_k \subset \Omega_{6\sqrt{N}\varepsilon}$ with $l(Q_k) < \varepsilon$ and $\|a_k\|_{L^2(Q_k,\wedge^l)} \le C|Q_k|^{-1/2}$, where C does not depend on k.

We now deal with the second term in (4.1), i.e.

$$a = -\int_\varepsilon^\infty td\big(t\delta(f * \varphi_t) * \varphi_t\big)\ \frac{dt}{t}.$$

From this expression for a, there exists ψ with $\hat{\psi}(\xi) = \int_\varepsilon^\infty t|\xi|^2|\hat{\varphi}(t\xi)|^2\ dt$, $\xi \in \mathbb{R}^N$, such that

$$a = -\psi * f,$$

where we used the fact that $df = 0$. Next we prove $a \in L^2(\mathbb{R}^N, \wedge^l)$. Note that $f \in L^1(\mathbb{R}^N, \wedge^l)$. To prove $a \in L^2(\mathbb{R}^N, \wedge^l)$, we only need to show that $\psi \in L^2(\mathbb{R}^N)$ i.e. $\hat{\psi} \in L^2(\mathbb{R}^N)$. In fact

$$\hat{\psi}(\xi) \le C\int_\varepsilon^\infty t^{-1-2s}|\xi|^{-2s}\ dt = C|\xi|^{-2s}$$

for $s > N/4$, where C depends only on φ and ε. This gives

$$\begin{aligned}\int_{\mathbb{R}^N} |\hat{\psi}(\xi)|^2\ d\xi &= \int_{|\xi|<1} |\hat{\psi}(\xi)|^2\ d\xi + \int_{|\xi|\ge 1} |\hat{\psi}(\xi)|^2\ d\xi \\ &\le C + C\int_{|\xi|\ge 1} |\xi|^{-4s}\ d\xi = C < \infty.\end{aligned}$$

That is, $\hat{\psi} \in L^2(\mathbb{R}^N)$. Hence $a \in L^2(\mathbb{R}^N, \wedge^l)$.

Let Q_0 be the smallest cube containing $\Omega_{3\varepsilon}$. Then

$$\begin{aligned}\|a\|_{L^2(Q_0,\wedge^l)} &\le \|\psi\|_{L^2(\mathbb{R}^N)}\|f\|_{L^1(\mathbb{R}^N,\wedge^l)} \\ &:= \lambda_0|Q_0|^{-1/2},\end{aligned}$$

where $\lambda_0 = C\|f\|_{L^1(\mathbb{R}^N,\wedge^l)} > 0$, the constant C depends only on $\Omega_{3\varepsilon}$ and N. Letting

$$a_0 = \frac{a}{\lambda_0},$$

then $\|a_0\|_{L^2(Q_0,\wedge^l)} \le |Q_0|^{-1/2}$. Since $f \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)$, there exists $g \in \mathcal{D}'(\mathbb{R}^N, \wedge^{l-1})$ supported in $\bar{\Omega}$ such that $f = dg$. Now we define

$$\tilde{b}_0 = \frac{g - \sum_{k=1}^\infty \lambda_k \tilde{b}_k}{\lambda_0}.$$

Then $\tilde{b}_0 \in \mathcal{D}'(\mathbb{R}^N, \wedge^{l-1})$ by (4.4). Since supp $g \subset \bar{\Omega}$ and supp $\tilde{b}_k \subset \bar{\Omega}_{2\varepsilon}$, we have supp $\tilde{b}_0 \subset \bar{\Omega}_{2\varepsilon}$. Thus $\tilde{b}_0$ is compactly supported in $\Omega_{3\varepsilon}$. Applying Lemma 2.11 to $a_0 = d\tilde{b}_0$, there exists $b_0 \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in $\Omega_{3\varepsilon}$ such that

$$a_0 = db_0$$

and

$$\|b_0\|_{L^2(Q_0,\wedge^{l-1})} \le C\|a_0\|_{L^2(Q_0,\wedge^l)},$$

where the constant C depends only on $\Omega_{3\varepsilon}$ and N.

Let n be a positive integer. Splitting Q_0 to subcubes Q_0^i: $Q_0 = \bigcup_{i=1}^{2^{nN}} Q_0^i$ such that $l(Q_0^i) = \frac{l(Q_0)}{2^n} < \frac{\varepsilon}{2}$ (when n is sufficient large) and Q_0^i and Q_0^j are disjoint for any $i \ne j$. Suppose $Q_0^1, \cdots, Q_0^M$ are all cubes which intersect the domain $\Omega_{3\varepsilon}$, then

$$\Omega_{3\varepsilon} \subset \bigcup_{i=1}^{M} Q_0^i.$$

Let $\sum_{i=1}^M \eta_i = 1$ in $\Omega_{3\varepsilon}$ be a smooth partition of unity such that $\eta_i = 1$ in Q_0^i, 0 outside $2Q_0^i$ and $|D\eta_i(x)| \le C\ l(Q_0^i)^{-1}$. Then a_0 has a decomposition

$$a_0 = db_0 = \sum_{i=1}^{M} d(\eta_i b_0) := \sum_{i=1}^{M} \gamma_i a_0^i, \tag{4.5}$$

where $a_0^i = \frac{d(\eta_i b_0)}{|2Q_0^i|^{1/2}\|d(\eta_i b_0)\|_{L^2(2Q_0^i,\wedge^l)}} := db_0^i$ for $b_0^i \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in $2Q_0^i \cap \Omega_{3\varepsilon}$ (note that $2Q_0^i \subset \Omega_{6\sqrt{N}\varepsilon}$), and $\gamma_i = |2Q_0^i|^{1/2}\|d(\eta_i b_0)\|_{L^2(2Q_0^i,\wedge^l)}$. We next prove that

$$\sum_{i=1}^{M} \gamma_i \le C$$

for a constant C depends only on $\Omega_{3\varepsilon}$, n and N. In fact

$$\begin{aligned}
\sum_{i=1}^{M} \gamma_i &\le 2^{1/2} \sum_{i=1}^{M} |2Q_0^i|^{1/2} \left(\|\eta_i db_0\|_{L^2(2Q_0^i,\wedge^l)} + \|b_0 d\eta_i\|_{L^2(2Q_0^i,\wedge^{l-1})} \right) \\
&\le C|Q_0|^{1/2}\|a_0\|_{L^2(Q_0,\wedge^l)} + C\ 2^n \mathrm{diam}(\Omega_{3\varepsilon})^{-1}|Q_0|^{1/2}\|b_0\|_{L^2(Q_0,\wedge^{l-1})} \\
&\le C + C\ 2^n \mathrm{diam}(\Omega_{3\varepsilon})^{-1},
\end{aligned}$$

where we used the size estimates of a_0 and b_0. Combining (4.1) with (4.3) and (4.5), we have

$$f = \sum_{k=1}^{\infty} \lambda_k a_k + \sum_{i=1}^{M} \lambda_0 \gamma_i a_0^i := \sum_{j=0}^{\infty} \lambda_j a_j.$$

It is easy to check that a_j and λ_j satisfy the conditions of Proposition 4.2. The proof is finished. $\square$

In Proposition 4.2 we have shown that any $f \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ can be decomposed into a sum of a_k with support in $\Omega_{6\sqrt{N}\varepsilon}$. In order to prove Theorem 4.1, we define a reflection map in a neighborhood of $\partial\Omega$ to reflect the part of the support of a_k outside Ω into its inside. Finally using the Whitney decomposition we get the atomic decomposition of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$.

Proof of Theorem 4.1. The proof of the "if" part is rather similar to that of Theorem 3.1 and so is skipped. We now prove the "only if" part. Suppose f has the decomposition of Proposition 4.2. Let $f := g + h$, where

$$g = \sum_{j \in J'} \mu_j a_j$$

is the sum of those a_j supported in cubes Q_j satisfying $4Q_j \subset \Omega$, J' is a subset of $\{0, 1, \cdots\}$, and

$$h = \sum_{j \in J''} \mu_j a_j$$

is the sum of remaining a_j, where $J'' = \{0, 1, \cdots\} \setminus J'$. It is obvious that a_j is an $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atom when $j \in J'$ and supp $a_j \subset N_{6\sqrt{N}\varepsilon}(\partial\Omega)$ when $j \in J''$, where $N_{6\sqrt{N}\varepsilon}(\partial\Omega) := \{x : \text{dist}(x, \partial\Omega) < 6\sqrt{N}\varepsilon\}$. So to prove Theorem 4.1 we need only to show that h is also a sum of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms.

Choose ε sufficiently small so that there exists a bilipschitz reflection map

$$r : \ N_{6\sqrt{N}\varepsilon}(\partial\Omega) \ \to \ N_{6\sqrt{N}\varepsilon}(\partial\Omega),$$

which depends only on the domain Ω. See Appendix A for the construction and properties of the map. Since supp $h \subset \bar{\Omega}$, then

$$\begin{aligned} h|_\Omega &= (h - r^*h)|_\Omega \\ &= \sum_{j \in J''} \mu_j (a_j - r^*a_j)|_\Omega = \sum_{j \in J''} \mu_j \Big(d(b_j - r^*b_j) \Big)\Big|_\Omega. \end{aligned} \tag{4.7}$$

For $j \in J''$, define

$$\alpha_j = \begin{cases} a_j - r^*a_j & \text{in } \Omega; \\ 0 & \text{outside } \Omega \end{cases}$$

and

$$\beta_j = \begin{cases} b_j - r^*b_j & \text{in } \Omega; \\ 0 & \text{outside } \Omega. \end{cases}$$

Then (4.7) gives

$$h = \sum_{j \in J''} \mu_j \alpha_j. \tag{4.8}$$

Let $P_j = r(Q_j)$ when $Q_j \cap \Omega = \emptyset$; $P_j = (Q_j \cap \Omega) \cup r(Q_j \setminus \bar{\Omega})$ when $Q_j \cap \partial\Omega \neq \emptyset$ and $P_j = Q_j$ when $Q_j \subset \Omega$. Then supp $\beta_j \subset P_j$. Suppose Q'_j is the smallest cube containing P_j. Similar to the proof of Theorem 3.1, we can show, for $j \in J''$, that α_j and β_j satisfy

(1) α_j and β_j are supported in Q'_j;

(2) $\alpha_j = d\beta_j$ in $\mathbb{R}^N$;

(3) $\|\alpha_j\|_{L^2(Q'_j, \wedge^l)} \leq C|Q'_j|^{-1/2}$, where constants C do not depend on j.

From the definition of a strongly Lipschitz domain, there are a finite number of open sets $O_1, \cdots, O_M$ so that $\Omega \subset \bigcup_{m=1}^M O_m$. For each m there exists a special Lipschitz domain Ω_m and a rotation rot_m such that

$$O_m \cap \Omega = O_m \cap rot_m(\Omega_m).$$

Let ε be sufficiently small, so that

$$N_{6\sqrt{N}\varepsilon}(\partial\Omega) \subset \bigcup_{m=1}^{M} O_m.$$

For $j \in J''$, supp $a_j \subset Q_j \subset N_{6\sqrt{N}\varepsilon}(\partial\Omega)$ and $l(Q_j) < \varepsilon$. Since ε is sufficiently small, for any a_j, there exists an open set O_m such that both the support of a_j and that of α_j are contained in O_m. Therefore we can write (4.8) as

$$h := \sum_{j \in J_1} \mu_j \alpha_j + \cdots + \sum_{j \in J_M} \mu_j \alpha_j, \tag{4.9}$$

where $\sum_{j \in J_m} \mu_j \alpha_j$ is a sum of α_j's whose support P_j and the support of a_j are both entirely contained in O_m.

Let $h_m := \sum_{j \in J_m} \mu_j \alpha_j$. We next prove that h_m can be written as a sum of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms. Without loss of generality we suppose that $O_m \cap \partial\Omega$ lies on the graph of a Lipschitz function. Repeat the arguments in the proof of Theorem 3.1, we can show that for any $j \in J_m$, α_j has a decomposition

$$\alpha_j = \sum_{i=0}^{\infty} \tau_j^i \alpha_j^i,$$

where the α_j^i's are $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms and $\sum_{i=0}^{\infty} \tau_j^i \le C$ for a constant C depending only on the domain Ω and the dimension N. The proof of Theorem 4.1 is finished. $\square$

5. Dual Spaces

We first introduce BMO-spaces $BMO_{r,d}(\Omega, \wedge^k)$. Then use them to characterize the dual spaces of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ when Ω is a special Lipschitz domain or a bounded strongly Lipschitz domain by using the duality between $\mathcal{H}^1(\mathbb{R}^N, \wedge^l)$ and $BMO(\mathbb{R}^N, \wedge^{N-l})$.

Definition 5.1. Suppose $0 \le k \le N$. Let $BMO_{r,d}(\Omega, \wedge^k)$ be the space of measurable functions G for which

$$\|G\|_{BMO_{r,d}(\Omega,\wedge^k)} = \sup_B \inf_{g_B} \left(\frac{1}{|B|} \int_B |G - g_B|^2 \, dx \right)^{1/2} < \infty,$$

where the supremum is taken over all balls B with $2B \subset \Omega$ and the infimum is taken over all functions $g_B \in L^2(B, \wedge^k)$ with $dg_B = 0$ in B.

Consider $BMO_{r,d}(\Omega, \wedge^k)/X_0$ with norm

$$\|G + X_0\|_{BMO_{r,d}(\Omega,\wedge^k)/X_0} = \|G\|_{BMO_{r,d}(\Omega,\wedge^k)},$$

where $X_0 = \{G \in BMO_{r,d}(\Omega, \wedge^k) : \|G\|_{BMO_{r,d}(\Omega,\wedge^k)} = 0\}$. When $k = 0$, $BMO_{r,d}(\Omega, \wedge^k)/X_0$ reduces to a BMO-space on domains Ω which is isomorphic to $BMO_r(\Omega)$ [L, Theorem 3.1]. The following lemma is used to prove the main theorem of this section, its proof is based on Theorems 3.1 and 4.1 and is similar to Lemma 2.11 in [LM2], and so is skipped. Let $D_{z,d}(\Omega, \wedge^l)$ denote the vector space finitely generated by $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms, which is dense in $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ by Theorems 3.1 and 4.1.

Lemma 5.2. *Let $1 \le l \le N$. For $g \in BMO(\mathbb{R}^N, \wedge^{N-l})$,*

$$\int_\Omega g \wedge h = 0 \quad \text{for all} \quad h \in D_{z,d}(\Omega, \wedge^l)$$

if and only if

$$dg = 0 \quad \text{in} \quad \Omega.$$

Theorem 5.3. *Let Ω be a special Lipschitz domain or a bounded strongly Lipschitz domain in $\mathbb{R}^N$ and $1 \le l \le N$. If $G + X_0 \in BMO_{r,d}(\Omega, \wedge^{N-l})/X_0$, then the linear functional L defined by*

$$L(h) = \int_\Omega G \wedge h, \tag{5.1}$$

initially defined on $D_{z,d}(\Omega, \wedge^l)$, has a unique bounded extension to $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$.

Conversely, if $L \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)^*$, then there exists a unique $G+X_0 \in BMO_{r,d}(\Omega, \wedge^{N-l})/X_0$ such that (5.1) holds. The map $G + X_0 \mapsto L$ given by (5.1) is a Banach isomorphism between $BMO_{r,d}(\Omega, \wedge^{N-l})/X_0$ and $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)^*$.

Proof. Let $G \in BMO_{r,d}(\Omega, \wedge^{N-l})$. If $\|G\|_{BMO_{r,d}(\Omega,\wedge^{N-l})} = 0$, then for any ball B with $2B \subset \Omega$

$$\inf_{g \in L^2(B,\wedge^{N-l}), dg=0} \int_B |G - g|^2 \, dx = 0.$$

This implies that $G \in L^2(B, \wedge^{N-l})$ and $dG = 0$ in B. Hence $dG = 0$ in Ω. So

$$L(h) = 0 \quad \text{for all} \quad h \in D_{z,d}(\Omega, \wedge^l)$$

by Lemma 5.2. Thus one can define $\rho_1 : \ BMO_{r,d}(\Omega, \wedge^{N-l})/X_0 \to D_{z,d}(\Omega, \wedge^l)^*$ by

$$\rho_1(G + X_0)(h) = \int_\Omega G \wedge h, \qquad h \in D_{z,d}(\Omega, \wedge^l).$$

To prove that ρ_1 is bounded from $BMO_{r,d}(\Omega, \wedge^{N-l})/X_0$ to $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)^*$, it is sufficient to show that

$$\left| \int_\Omega G \wedge h \right| \le C \|G\|_{BMO_{r,d}(\Omega,\wedge^{N-l})} \|h\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^l)} \tag{5.2}$$

for all $G \in BMO_{r,d}(\Omega, \wedge^{N-l})$ and h in the dense subspace $D_{z,d}(\Omega, \wedge^l) \subset \mathcal{H}^1_{z,d}(\Omega, \wedge^l)$.

Suppose $h = \sum_k \lambda_k a_k \in D_{z,d}(\Omega, \wedge^l)$, where the a_k's are $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$-atoms: $a_k = db_k$ for some $b_k \in L^2(\mathbb{R}^N, \wedge^{l-1})$ with support in a ball B_k with $2B_k \subset \Omega$ and $\|a_k\|_{L^2(B_k,\wedge^l)} \le |B_k|^{-1/2}$. So for all $g_k \in L^2(B_k, \wedge^{N-l})$ with $dg_k = 0$ in B_k,

$$\begin{aligned}
\left| \int_\Omega G \wedge h \right| &\le \sum_k |\lambda_k| \left| \int_{B_k} (G - g_k) \wedge a_k \right| \\
&\le \sum_k |\lambda_k| \left(\int_{B_k} |G - g_k|^2 \, dx \right)^{1/2} \|a_k\|_{L^2(B_k,\wedge^l)} \\
&\le \sum_k |\lambda_k| \left(\frac{1}{|B_k|} \int_{B_k} |G - g_k|^2 \, dx \right)^{1/2}.
\end{aligned}$$

This gives (5.2).

Define

$$Y := BMO(\mathbb{R}^N, \wedge^{N-l})/Y_0,$$

where

$$Y_0 := \{g \in BMO(\mathbb{R}^N, \wedge^{N-l}) : dg = 0 \text{ in } \Omega\}.$$

It follows from the Hahn-Banach Theorem that the map

$$\rho_2 : \quad L \ \mapsto \quad G + Y_0,$$

is a Banach isomorphism between $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)^*$ and Y and $\|L\|_{op} \sim \|G + Y_0\|_Y$, where $L \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)^*$ defined as in (5.1).

Define

$$\rho_3 : \ G + Y_0 \ \mapsto \ G|_\Omega + X_0$$

from Y to $BMO_{r,d}(\Omega, \wedge^{N-l})/X_0$. We now prove that ρ_3 is well-defined and bounded. Let $G \in BMO(\mathbb{R}^N, \wedge^{N-l})$, $g \in Y_0$ and B be a ball with $2B \subset \Omega$. Then

$$\inf_{g_B} \left(\frac{1}{|B|} \int_B |G|_\Omega - g_B|^2 \, dx \right)^{1/2} \leq \inf_{c \in \wedge^{N-l}} \left(\frac{1}{|B|} \int_B |G - g - c|^2 \, dx \right)^{1/2},$$

where the infimum in the left-hand side is taken over all $g_B \in L^2(B, \wedge^{N-l})$ with $dg_B = 0$ in B and that in the right-hand side is taken over all constant forms c in $\wedge^{N-l}$. Thus

$$\begin{aligned}
\|G|_\Omega\|_{BMO_{r,d}(\Omega, \wedge^{N-l})} &\leq \sup_{2B \subset \Omega} \inf_{c \in \wedge^{N-l}} \left(\frac{1}{|B|} \int_B |G - g - c|^2 \, dx \right)^{1/2} \\
&\leq \sup_{B \subset \mathbb{R}^N} \inf_{c \in \wedge^{N-l}} \left(\frac{1}{|B|} \int_B |G - g - c|^2 \, dx \right)^{1/2} \\
&= \|G - g\|_{BMO(\mathbb{R}^N, \wedge^{N-l})}.
\end{aligned}$$

This yields

$$\|G|_\Omega\|_{BMO_{r,d}(\Omega, \wedge^{N-l})} \leq \|G + Y_0\|_Y.$$

Hence ρ_3 is bounded. It is clear that $\rho_3 \circ \rho_2 \circ \rho_1 = I$ and $\rho_1 \circ \rho_3 \circ \rho_2 = I$. The proof of Theorem 5.3 is completed. $\square$

From the proof above, we see that Y is isomorphic to $BMO_{r,d}(\Omega, \wedge^{N-l})/X_0$ with equivalent norms. For $l = N$, the result was proved by Jones [J] for a more general class of domains:

Corollary 5.4. *If $G \in BMO(\mathbb{R}^N)$, then $G|_\Omega \in BMO_r(\Omega)$. Conversely, if $g \in BMO_r(\Omega)$ then there exists $G \in BMO(\mathbb{R}^N)$ such that $g = G|_\Omega$ and*

$$\|G\|_{BMO(\mathbb{R}^N)} \leq C \|g\|_{BMO_r(\Omega)},$$

where C depends only on the domain Ω.

6. The "Div-Curl" Type Theorem on Domains

Applying the duality properties of $\mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ discussed in the previous section we establish a "div-curl" type theorem for special Lipschitz domains or bounded strongly Lipschitz domains, which is an extension of the "div-curl" theorem by Coifman, Lions, Meyer and Semmes to the case of domains.

Theorem 6.1. *Suppose Ω is a special Lipschitz domain or a bounded strongly Lipschitz domain in $\mathbb{R}^N$, $k,\ l,\ m \geq 0$, $k+m+l+2=N$ and $b \in L^2_{loc}(\Omega, \wedge^k)$. Then*

$$\sup_{u,v} \int_\Omega b \wedge du \wedge dv \sim \|b\|_{BMO_{r,d}(\Omega,\wedge^k)}, \tag{6.1}$$

where the supremum is taken over all u and v such that

$$\left.\begin{aligned} &u \in H^1_0(\Omega, \wedge^m), \quad \|du\|_{L^2(\Omega,\wedge^{m+1})} \leq 1; \\ &v \in H^1_0(\Omega, \wedge^l), \quad \|dv\|_{L^2(\Omega,\wedge^{l+1})} \leq 1. \end{aligned}\right\} \tag{6.2}$$

The implicit constants in (6.1) depend only on N and Ω.

To prove Theorem 6.1 we need the following:

Lemma 6.2 ([LM2, Lemma 6.9]). *If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < m+l \leq N$, $u \in L^p(\mathbb{R}^N, \wedge^m)$, $v \in L^q(\mathbb{R}^N, \wedge^l)$, $du = 0$, $dv = 0$ in $\mathbb{R}^N$. Then $u \wedge v \in \mathcal{H}^1(\mathbb{R}^N, \wedge^{m+l})$ and there exists a constant C independent of u and v such that*

$$\|u \wedge v\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^{m+l})} \leq C\|u\|_{L^p(\mathbb{R}^N,\wedge^m)}\|v\|_{L^q(\mathbb{R}^N,\wedge^l)}.$$

Proof of Theorem 6.1. Suppose that u and v satisfy (6.2). Let U and V be the zero extensions to $\mathbb{R}^N$ of u and v. Then $U \in H^1(\mathbb{R}^N, \wedge^m)$ and $V \in H^1(\mathbb{R}^N, \wedge^l)$. From Lemma 6.2, $dU \wedge dV \in \mathcal{H}^1(\mathbb{R}^N, \wedge^{m+l+2})$ and

$$\|dU \wedge dV\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^{m+l+2})} \leq C\|du\|_{L^2(\Omega,\wedge^{m+1})}\|dv\|_{L^2(\Omega,\wedge^{l+1})} \leq C. \tag{6.3}$$

Note that $dU\wedge dV = d(U\wedge dV)$ and $U\wedge dV$ is supported in Ω, so $dU\wedge dV \in \mathcal{H}^1_{z,d}(\Omega, \wedge^{m+l+2})$ with $\|dU \wedge dV\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^{m+l+2})} \leq C$ for a constant C depending only on of N and Ω by (6.3). Suppose $b \in BMO_{r,d}(\Omega, \wedge^k)$. Then (5.2) yields

$$\begin{aligned} \left|\int_\Omega b \wedge du \wedge dv\right| &= \left|\int_\Omega b \wedge dU \wedge dV\right| \\ &\leq \|b\|_{BMO_{r,d}(\Omega,\wedge^k)}\|dU \wedge dV\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^{N-k})} \\ &\leq C\|b\|_{BMO_{r,d}(\Omega,\wedge^k)}. \end{aligned}$$

Conversely, the proof is similar to the case of $\mathbb{R}^N$. We need to show, for all balls B with $2B \subset \Omega$, that there exist $u \in H^1_0(B, \wedge^m)$ and $v \in H^1_0(2B, \wedge^l)$ with $\|du\|_{L^2(\mathbb{R}^N,\wedge^{m+1})}$, $\|dv\|_{L^2(\mathbb{R}^N,\wedge^{l+1})} \leq 1$ such that

$$\inf_{g_B}\left(\frac{1}{|B|}\int_B |b - g_B|^2\, dx\right)^{1/2} \leq C\left|\int_B b \wedge du \wedge dv\right|, \tag{6.4}$$

the infimum being taken over all $g_B \in L^2(B, \wedge^k)$ with $dg_B = 0$ in B and C is a constant independent of b and B. By scaling we need only to show that (6.4) holds for the unit ball, which follows the lines of the $\mathbb{R}^N$ case in [LM2]. Note that we used $2B \subset \Omega$ in the construction of u and v. We skip the details of the proof. □

Remark. The equivalence in (6.1) holds if the supremum is taken over all $u \in H^1_0(\Omega, \wedge^m)$ and $v \in H^1_0(\Omega, \wedge^l)$ with $\|Du\|_{L^2(\Omega,\wedge^{m+1})}$, $\|Dv\|_{L^2(\Omega,\wedge^{l+1})} \leq 1$.

Let $k = m = 0$ and $N = 3$ in Theorem 6.1, we get

Corollary 6.3. *Let Ω be a special Lipschitz domain or a bounded strongly Lipschitz domain in $\mathbb{R}^3$ and $b \in L^2_{loc}(\Omega)$, then*

$$\|b\|_{BMO_r(\Omega)} \sim \sup_{E,F} \int_\Omega b\ E \cdot F\ dx, \tag{6.5}$$

where the supremum is taken over all $E,\ F \in L^2(\Omega, \mathbb{R}^3)$, $E = Du$ for some $u \in H^1_0(\Omega)$, $F = \text{curl } v$ for some $v \in H^1_0(\Omega, \mathbb{R}^3)$, and $\|E\|_{L^2(\Omega,\mathbb{R}^3)},\ \|F\|_{L^2(\Omega,\mathbb{R}^3)} \le 1$.

When Ω is replaced by $\mathbb{R}^3$, Corollary 6.3 is a special case of a result by Coifman, Lions, Meyer and Semmes in [CLMS, page 262].

Corollary 6.4. *Let Ω be a special Lipschitz domain or a bounded strongly Lipschitz domain in $\mathbb{R}^2$ and $b \in L^2_{loc}(\Omega)$, then we have the following Jacobian determinant estimate*

$$\|b\|_{BMO_r(\Omega)} \sim \sup_u \int_\Omega b \det\, Du\ dx,$$

the supremum being taken over all $u = (u_1, u_2) \in H^1_0(\Omega, \mathbb{R}^2)$ with $\|du_i\|_{L^2(\mathbb{R}^2,\mathbb{R}^2)} \le 1,\ i = 1, 2$.

When $\Omega = \mathbb{R}^2$, Corollary 6.4 can be derived from Theorems II.1 and III.2 in [CLMS].

The following theorem gives a decomposition of $f \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ into "$du \wedge dv$" quantities. This is an extension of Theorem III.2 by Coifman, Lions, Meyer and Semmes in [CLMS] to the case of domains. The proof of the theorem uses Theorems 3.1 and 4.1 and is similar to that of Theorem 6.11 in [LM2], and so is omitted.

Theorem 6.5. *Let Ω be a special Lipschitz domain or a bounded strongly Lipschitz domain in $\mathbb{R}^N$, $1 \le l \le N$ and $0 \le m \le l-2$. Then any $f \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ can be written as*

$$f = \sum_{k=0}^{\infty} \lambda_k\ du_k \wedge dv_k,$$

where $u_k \in H^1_0(\Omega, \wedge^m)$, $v_k \in H^1_0(\Omega, \wedge^{l-m-2})$ with $\|Du_k\|_{L^2(\Omega,\wedge^m)}$, $\|Dv_k\|_{L^2(\Omega,\wedge^{l-m-2})} \le 1$, and

$$\sum_{k=0}^{\infty} |\lambda_k| \le C \|f\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^l)}$$

for some constants C independent of f.

Remark. If u_k and v_k satisfy the conditions of Theorem 6.5, then $du_k \wedge dv_k \in \mathcal{H}^1_{z,d}(\Omega, \wedge^l)$ and

$$\|du_k \wedge dv_k\|_{\mathcal{H}^1_{z,d}(\Omega,\wedge^l)} \le C$$

for a constant C independent of u_k and v_k by Proposition 4.8 of [HLMZ].

Corollary 6.6. *Let Ω be a special Lipschitz domain or a bounded strongly Lipschitz domain in $\mathbb{R}^N$, then any $f \in \mathcal{H}^1_z(\Omega)$ can be written as*

$$f = \sum_{k=0}^{\infty} \lambda_k\ E_k \cdot F_k$$

with

$$\sum_{k=0}^{\infty} |\lambda_k| \le C\|f\|_{\mathcal{H}^1_z(\Omega)}$$

where C is independent of f, where $E_k = Du_k$ for some $u_k \in H^1_0(\Omega)$, $F_k = \text{curl } v_k$ for some $v_k \in H^1_0(\Omega, \mathbb{R}^3)$ and $\|E_k\|_{L^2(\Omega,\mathbb{R}^3)}$, $\|F_k\|_{L^2(\Omega,\mathbb{R}^3)} \le 1$.

When we replace Ω by $\mathbb{R}^3$, Corollary 6.6 becomes the three-dimensional case of Theorem III.2 in [CLMS].

7. An Application to Coercivity

In the study of homogenization of linearized elasticity, Geymonat, Müller and Triantafyllidis [GMT] considered the following system

$$\left.\begin{aligned} &\operatorname{div}_\alpha A^{i,j}_{\alpha,\beta}(\frac{x}{\varepsilon})\frac{\partial u_j}{\partial x_\beta} = f \text{ in } \Omega \\ &u|_{\partial\Omega} = 0, \end{aligned}\right\} \tag{7.1}$$

where $A^{i,j}_{\alpha,\beta}(x)$ is a periodic measurable function, $1 \le i,\ j,\ \alpha,\ \beta \le N$. A quantity Λ is introduced which gives a criterion of whether an elliptic system satisfying the Legendre-Hadamard condition can be homogenized, namely

$$\Lambda = \inf\left\{ \frac{\int_{\mathbb{R}^N} A^{i,j}_{\alpha,\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta}dx}{\int_{\mathbb{R}^N}|Du|^2dx} : u \in C^\infty_0(\mathbb{R}^N, \mathbb{R}^N) \right\}.$$

It was proved in [GMT] that if $\Lambda > 0$ some homogenization results can be obtained for the system (7.1). If $\Lambda < 0$, the system cannot be homogenized. Zhang asked the following question: what conditions on the coefficient $A^{i,j}_{\alpha,\beta}$ of the system imply that $\Lambda \ge 0$?

Let $N = 3$ and suppose that $A^{i,j}_{\alpha,\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta}$ can be written in the form

$$A^{i,j}_{\alpha,\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta} = B^{i,j}_{\alpha,\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta} + b_{ij}(x)(\text{adj } Du)_{i,j}, \tag{7.2}$$

where $A^{i,j}_{\alpha,\beta}$, $B^{i,j}_{\alpha,\beta} \in L^\infty(\mathbb{R}^3)$, $B^{i,j}_{\alpha,\beta}\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta} \ge C|Du|^2$ and adj Du denotes the adjoint matrix of Du for $u \in H^1(\mathbb{R}^3, \mathbb{R}^N)$ (the summation convention is understood). We are interested in forms of this type in three dimensions, because they arrive naturally from the linearization of polyconvex variational integrals studied in nonlinear elasticity by Ball in [B].

By a simple calculation (see [LM2] for details), the last term in (7.2) can be written as $b_1 \wedge du_2 \wedge du_3 + b_2 \wedge du_1 \wedge du_3 + b_3 \wedge du_1 \wedge du_2$, where $b_i = (b_{i1}, b_{i2}, b_{i3})$, $i = 1, 2, 3$. Let $a(u)$ denote the following polyconvex quadratic form

$$a(u) = |Du|^2 + b_1 \wedge du_2 \wedge du_3 + b_2 \wedge du_1 \wedge du_3 + b_3 \wedge du_1 \wedge du_2. \tag{7.3}$$

So the question when $\Lambda \ge 0$ becomes: find necessary conditions of b_i such that

$$\int_{\mathbb{R}^3} a(u)\ dx \ge 0 \quad \text{for all} \quad u \in H^1(\mathbb{R}^3, \mathbb{R}^3).$$

We answered this question in [LM2] (See [Z] for the two dimensional case). Now we consider the same problem on domains Ω. That is, find necessary conditions of b_i such that

$$\int_\Omega a(u)\; dx \geq 0 \quad \text{for all} \quad u \in H_0^1(\Omega, \mathbb{R}^N). \tag{7.4}$$

Using Theorem 6.1 and an idea of [Z], we give an "almost" necessary and sufficient condition on b_i such that (7.4) holds. The proof of the result is similar to Proposition 5.1 in [LM2] and so is skipped.

Theorem 7.1. *Let $a(u)$ be the expression shown in (7.3).*

(1) There exists a constant C_1 depending only on N and Ω such that

$$\max_{1\leq i\leq 3} \|b_i\|_{BMO_{r,d}(\Omega,\mathbb{R}^3)} \leq C_1$$

implies that

$$\int_\Omega a(u)\; dx \geq \frac{1}{2}\|Du\|_{L^2(\Omega,\mathbb{R}^3)} \quad \text{for all} \quad u \in H_0^1(\Omega, \mathbb{R}^3).$$

(2) If $\int_\Omega a(u)\; dx \geq 0$ for all $u \in H_0^1(\Omega, \mathbb{R}^3)$, then there exists a constant C_2 depending only on N and Ω such that

$$\max_{1\leq i\leq 3} \|b_i\|_{BMO_{r,d}(\Omega,\mathbb{R}^3)} \leq C_2.$$

Appendix A. Reflection Maps in Neighborhoods of Lipschitz Boundaries

In this appendix we construct a reflection map in a neighborhood of the boundary $\partial\Omega$ of a bounded strongly Lipschitz domain Ω in $\mathbb{R}^N$.

By the definition of a strongly Lipschitz domain, for any $p \in \partial\Omega$, there exists a neighborhood U_p of p, an isometry $\psi_p : \mathbb{R}^N \to \mathbb{R}^N$ and a Lipschitz function $\varphi_p : \mathbb{R}^{N-1} \to \mathbb{R}$ such that $\psi_p(0) = p$ and $\psi_p(V_p \cap \{(x', \varphi_p(x')) : x' \in \mathbb{R}^{N-1}\}) = U_p \cap \partial\Omega$, where $V_p = \psi_p^{-1}(U_p)$. From compactness, there are finite many $U_{p_1}, \cdots, U_{p_m}$ such that $\partial\Omega \subset \bigcup_{i=1}^m U_{p_i}$. Let

$$\sum_{i=1}^m \eta_i(x) = 1$$

be a smooth partition of unity in a neighborhood of $\partial\Omega$ such that supp $\eta_i \subset U_{p_i}$. Let $\theta_i := (\psi_{p_i})_*(-e_N)$, the push-forward of $-e_N$. Define

$$\theta(x) = \sum_{i=1}^m \eta_i(x)\theta_i(x).$$

Then $\theta \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and

$$n(x)\cdot\theta(x) = \sum_{i=1}^m \eta_i(x) n(x)\cdot\theta_i(x) \geq c\sum_{i=1}^m \eta_i(x) = c > 0$$

for a.e. $x \in \partial\Omega$. Consider the corresponding ODE:

$$\frac{dy}{dt} = \theta(y), \qquad y(0) = x \tag{A.1}$$

with $x \in \partial\Omega$. By the theory of differential equations (see, for example, [T, Chapter 1]), (A.1) has a unique solution $y = y(t, x)$ on some $t-$interval. For fixed t, write $y(t,x) := \mathcal{F}_\theta^t(x)$.

From the construction of the vector field θ, for any $p \in \partial\Omega$ there exists a hypersurface M_p and a neighborhood of p, we still denote it by U_p, such that for all $q \in U_p \cap \partial\Omega$, $\theta(q)$ is not tangent to M_p. We can choose coordinates near p so that p is the origin and M_p is given by $\{x_N = 0\}$. Thus, we can identify a point $x' \in \mathbb{R}^{N-1}$ near the origin with $x' \in M_p$. Define a map

$$\mathcal{F}_\theta : \ M_p \times (-t_p, t_p) \quad \rightarrow \quad U_p$$

by $\mathcal{F}_\theta(x', t) = \mathcal{F}_\theta^t(x')$. This is a C^∞ and surjective derivative and so by the inverse function theorem is a local diffeormorphism.

Let $V := \mathcal{F}_\theta^{-1}(U_p)$. Without loss of generality, suppose that $\mathcal{F}_\theta^{-1}\big(U_p \cap \partial\Omega\big)$ lies on a Lipschitz graph and can be expressed as

$$\mathcal{F}_\theta^{-1}\big(U_p \cap \partial\Omega\big) = \{(x', x_N) : x_N = \varphi(x'),\ x' \in M_p \cap V\}, \tag{A.2}$$

where φ is a Lipschitz function on $\mathbb{R}^{N-1}$. Choosing a suitable $l > 0$ such that

$$\mathcal{V} := \{(x', x_N) : \varphi(x') - l < x_N < \varphi(x') + l,\ x' \in M_p \cap V\} \subset V.$$

Let $\mathcal{U}_p := \mathcal{F}_\theta(\mathcal{V})$ and let ε and l be sufficiently small so that $N_\varepsilon(\partial\Omega) \subset \bigcup_{i=1}^m \mathcal{U}_{p_i}$.

Set $\mathcal{V}^+ := \{(x', x_N) \in \mathcal{V} : x_N > \varphi(x')\}$ and $\mathcal{V}^- := \{(x', x_N) \in \mathcal{V} : x_N < \varphi(x')\}$. Define

$$R : \ \mathcal{V} \ \rightarrow \ \mathcal{V}$$

by $R(x', x_N) = (x', 2\varphi(x') - x_N)$. It is obvious that R is a bilipschitz map and satisfies (1) $R \circ R = I$ in $\mathcal{V}$; (2) $R = I$ on $\{(x', x_N) \in \mathcal{V} : x_N = \varphi(x')\}$, where I denotes the identity map; (3) $R : \mathcal{V}^\pm \rightarrow \mathcal{V}^\mp$. We call a bilipschitz map with these conditions a reflection map in $\mathcal{V}$.

Now define

$$r_p = \mathcal{F}_\theta \circ R \circ \mathcal{F}_\theta^{-1} \quad \text{in} \quad \mathcal{U}_p. \tag{A.3}$$

It is easy to check that r_p is a reflection map in $\mathcal{U}_p$. From (A.2), we have

$$\mathcal{U}_p \cap \partial\Omega = \{\mathcal{F}_\theta^{\varphi(x')}(x') : x' \in M_p \cap \mathcal{V}\}.$$

This implies that any point $x \in \mathcal{U}_p$ can be written as

$$x = \mathcal{F}_\theta^{t+\varphi(x')}(x'), \qquad x' \in M_p,\ t \in (-\infty, +\infty).$$

Then (A.3) yields

$$\begin{aligned} r_p(x) &= \mathcal{F}_\theta \circ R \circ \mathcal{F}_\theta^{-1}\big(\mathcal{F}_\theta^{\varphi(x')+t}(x')\big) \\ &= \mathcal{F}_\theta \circ R\big(x', t + \varphi(x')\big) \\ &= \mathcal{F}_\theta\big(x', 2\varphi(x') - t - \varphi(x')\big) \\ &= \mathcal{F}_\theta^{\varphi(x')-t}(x'). \end{aligned} \tag{A.4}$$

Define

$$r: \ N_\varepsilon(\partial\Omega) \ \to \ N_\varepsilon(\partial\Omega)$$

by $r(x) = r_{p_i}(x)$ if $x \in \mathcal{U}_{p_i}$. Then it follows from (A.4) that r is well defined. Moreover, r is a bilipschitz map satisfying 1) $r \circ r = I$ in $N_\varepsilon(\partial\Omega)$; 2) $r = I$ on $\partial\Omega$; 3) $r : N_\varepsilon^{\pm}(\partial\Omega) \to N_\varepsilon^{\mp}(\partial\Omega)$, where $N_\varepsilon^{+}(\partial\Omega) = N_\varepsilon(\partial\Omega) \cap \Omega$ and $N_\varepsilon^{-}(\partial\Omega) = \Omega_\varepsilon \setminus \bar{\Omega}$. We see that r is a reflection map in $N_\varepsilon(\partial\Omega)$.

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