DIVERGENCE-FREE HARDY SPACE ON $\mathbb{R}^N_+$

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Abstract

We introduce a divergence-free Hardy space $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$ and prove its divergence-free atomic decomposition. We also characterize its dual space and establish a "div-curl" type theorem on $\mathbb{R}^3_+$ with an application to coercivity.

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CONTENTS

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1. Introduction

In [1], among other things, we studied the divergence-free Hardy space on $\mathbb{R}^N$ and obtained its divergence-free atomic decomposition. A natural question to ask is: can we define divergence-free Hardy spaces on domains and do these spaces have divergence-free atomic decompositions? In [2], we considered divergence-free Hardy spaces of vector-valued functions supported on bounded Lipschitz domains and proved their divergence-free atomic decompositions. In this paper we introduce a divergence-free Hardy space on $\mathbb{R}^N_+$, denoted by $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$, as the space of divergence-free vector-valued functions on $\mathbb{R}^N_+$ whose first $N-1$ components are in $\mathcal{H}^1_z(\mathbb{R}^N_+)$ and the last one is in $\mathcal{H}^1_r(\mathbb{R}^N_+)$. As the main result of this paper we prove the following divergence-free atomic decomposition of $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$ (Theorem 4.1):

Any element in $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$ can be decomposed into a sum of atoms, where the atoms are also divergence-free and have support in $\overline{\mathbb{R}^N_+}$.

From this result it is easy to see that $\mathcal{H}^1_{div}(\mathbb{R}^N_+, \mathbb{R}^N)$ coincides with the divergence-free Hardy space of vector-valued functions on $\mathbb{R}^N$ with support in $\overline{\mathbb{R}^N_+}$ (Corollary 4.2). A crucial element in the proof of the theorem is the use of even and odd functions on $\mathbb{R}^N$.

The contents of the paper are as follows. In Sections 2, we recall definitions of Hardy spaces on $\mathbb{R}^N_+$ and their links with the spaces and even or odd functions on $\mathbb{R}^N$. In Section 3, we provide definitions of $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$ and its atoms. Section 4 is devoted to the divergence-free atomic decomposition of $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$. In the final section, using the decomposition, we characterize its dual as a type of BMO space and establish a "div-curl" type theorem on $\mathbb{R}^3_+$ with an application to coercivity.

2. Preliminaries

Hardy spaces on $\mathbb{R}^N_+$ were introduced by Chang, Krantz and Stein in [3]. There are two kinds of Hardy spaces defined on $\mathbb{R}^N_+$, namely $\mathcal{H}^1_r(\mathbb{R}^N_+)$ and $\mathcal{H}^1_z(\mathbb{R}^N_+)$.

Definition 2.1. (i) A function f on $\mathbb{R}^N_+$ is said to be in $\mathcal{H}^1_r(\mathbb{R}^N_+)$ if it is the restriction to $\mathbb{R}^N_+$ of a function F in the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$. When $f \in \mathcal{H}^1_r(\mathbb{R}^N_+)$, define

$$\|f\|_{\mathcal{H}^1_r(\mathbb{R}^N_+)} = \inf \|F\|_{\mathcal{H}^1(\mathbb{R}^N)},$$

where the infimum is taken over all the functions $F \in \mathcal{H}^1(\mathbb{R}^N)$ such that $F|_{\mathbb{R}^N_+} = f$.

(ii) A function f on $\mathbb{R}^N_+$ is said to be in $\mathcal{H}^1_z(\mathbb{R}^N_+)$ if the function F defined by

$$F(x) = \begin{cases} f(x) & \text{in } \overline{\mathbb{R}^N_+} \\ 0 & \text{in } \mathbb{R}^N_- \end{cases}$$

belongs to $\mathcal{H}^1(\mathbb{R}^N)$. If $f \in \mathcal{H}^1_z(\mathbb{R}^N_+)$, its norm $\|f\|_{\mathcal{H}^1_z(\mathbb{R}^N_+)} = \|F\|_{\mathcal{H}^1(\mathbb{R}^N)}$.

Equipped with the norms $\|.\|_{\mathcal{H}^1_r(\mathbb{R}^N_+)}$ and $\|.\|_{\mathcal{H}^1_z(\mathbb{R}^N_+)}$, $\mathcal{H}^1_r(\mathbb{R}^N_+)$ and $\mathcal{H}^1_z(\mathbb{R}^N_+)$ are Banach spaces.

Definition 2.2. For each $x = (x', x_N) \in \mathbb{R}^N$, where $x' = (x_1, ..., x_{N-1})$, define the corresponding point

$$\tilde{x} = (x', -x_N) \in \mathbb{R}^N.$$

A function f from $\mathbb{R}^N$ to $\mathbb{R}$ is called *even* when

$$f(\tilde{x}) = f(x),$$

while f is called *odd* when

$$f(\tilde{x}) = -f(x).$$

From Corollaries 1.6 and 1.8 in [3], we see that $\mathcal{H}^1_r(\mathbb{R}^N_+)$ and $\mathcal{H}^1_z(\mathbb{R}^N_+)$ are exactly the restrictions to $\mathbb{R}^N_+$of odd (respectively even) functions belong to $\mathcal{H}^1(\mathbb{R}^N)$. So the odd function

$$f_o(x) = \begin{cases} f(x) & \text{in } \overline{\mathbb{R}^N_+} \\ -f(\tilde{x}) & \text{in } \mathbb{R}^N_- \end{cases}$$

belongs to $\mathcal{H}^1(\mathbb{R}^N)$ if $f \in \mathcal{H}^1_r(\mathbb{R}^N_+)$ with

$$\|f\|_{\mathcal{H}^1_r(\mathbb{R}^N_+)} \sim \|f_o\|_{\mathcal{H}^1(\mathbb{R}^N)},$$

while the even function

$$f_e(x) = \begin{cases} f(x) & \text{in } \overline{\mathbb{R}^N_+} \\ f(\tilde{x}) & \text{in } \mathbb{R}^N_- \end{cases}$$

belongs to $\mathcal{H}^1(\mathbb{R}^N)$ with

$$\|f\|_{\mathcal{H}^1_z(\mathbb{R}^N_+)} \sim \|f_e\|_{\mathcal{H}^1(\mathbb{R}^N)}$$

when $f \in \mathcal{H}^1_z(\mathbb{R}^N_+)$.

3. Divergence-Free Hardy Space on $\mathbb{R}^N_+$

We now introduce a divergence-free Hardy space on $\mathbb{R}^N_+$. Let $f = (f_1, ..., f_N)$, div $f = 0$ in the sense of distributions, and let $f_1, ..., f_{N-1} \in \mathcal{H}^1_z(\mathbb{R}^N_+)$. Suppose $f_1, ..., f_{N-1}$ have even extensions $F_1, ..., F_{N-1}$ respectively and F_N is the extension of f_N. In order for $F = (F_1, ..., F_N)$ to preserve the divergence-free condition, F_N must be an odd extension. So we see that f_N should be in $\mathcal{H}^1_r(\mathbb{R}^N_+)$. Therefore it is natural to define the following divergence-free Hardy space.

Definition 3.1. The *divergence-free Hardy space on* $\mathbb{R}^N_+$ is defined as

$$\begin{aligned}\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N) =\{f :\ & f_1, ..., f_{N-1} \in \mathcal{H}^1_z(\mathbb{R}^N_+),\ f_N \in \mathcal{H}^1_r(\mathbb{R}^N_+), \\ & \text{div } f = 0 \text{ in } \mathbb{R}^N_+,\ \int_{\mathbb{R}^N_+} f \cdot D\varphi \, dx = 0 \text{ for all } \varphi \in C^\infty(\overline{\mathbb{R}^N_+})\}\end{aligned}$$

with norm

$$\|f\|_{\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)} = \sum_{i=1}^{N-1} \|f_i\|_{\mathcal{H}^1_z(\mathbb{R}^N_+)} + \|f_N\|_{\mathcal{H}^1_r(\mathbb{R}^N_+)},$$

where $D\varphi$ denotes the gradient of the function $\varphi : \mathbb{R}^N \to \mathbb{R}$.

We next define atoms for $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$.

Definition 3.2. An $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$-*atom* is a function a supported in a cube (or a ball) Q with $4Q \subset \mathbb{R}^N_+$ that satisfies

(i) $\int_Q a(x)\, dx = 0$;

(ii) $\|a\|_{L^2(Q,\mathbb{R}^N)} \le |Q|^{-1/2}$.

(iii) div $a = 0$ in the sense of distributions.

Here $|Q|$ denotes the Lebesgue volume of Q and $4Q$ the cube with same center as Q and four times side-length of Q.

Remark. It is easy to see that (iii) implies (i). Note that here we require that the supports of atoms are away from the boundary of $\mathbb{R}^N_+$, which is stronger than the usual definition of $\mathcal{H}^1_z(\mathbb{R}^N_+)$-atoms.

4. Divergence-Free Atomic Decomposition

In this section we prove the divergence-free atomic decomposition for the space $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$ by using the decomposition of the divergence-free $\mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N)$ and properties of even and odd functions. Our main theorem of the paper is the following:

Theorem 4.1. *A function f on $\mathbb{R}^N_+$ belongs to $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$ if and only if it has a decomposition*

$$f = \sum_{k=0}^{\infty} \gamma_k \alpha_k, \tag{4.1}$$

where the α_k's are $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)-$atoms and $\sum_{k=0}^{\infty} |\gamma_k| < \infty$.

Furthermore

$$\|f\|_{\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)} \sim \inf \Big(\sum_{k=0}^{\infty} |\gamma_k| \Big),$$

where the infimum is taken over all such decompositions. The constants of the proportionality depend only on the dimension N.

By Theorem 4.1 we have the following interesting result which reveals that the space $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$ is in fact the same as the divergence-free Hardy space of vector-valued functions on $\mathbb{R}^N$ which support on $\overline{\mathbb{R}^N_+}$.

Corollary 4.2. *If $f \in \mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$, then $f_N \in \mathcal{H}^1_z(\mathbb{R}^N_+)$.*

The proof of Theorem 4.1 is based on the following lemmas. Proofs of Lemmas 4.3 and 4.4 are given at the end of this section.

Lemma 4.3. *Let $A = (a_1, ..., a_N)$ be an $\mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N)$-atom with* div $A = 0$ *in $\mathbb{R}^N$. Define*

$$b = \begin{cases} \Big(a_1(x) + a_1(\tilde{x}), ..., a_{N-1}(x) + a_{N-1}(\tilde{x}), a_N(x) - a_N(\tilde{x})\Big) & in\ \mathbb{R}^N_+ \\ 0 & in\ \ \overline{\mathbb{R}^N_-}. \end{cases}$$

Then

$$\text{div } b = 0 \quad in \quad \mathbb{R}^N.$$

Lemma 4.4. *Suppose $b \in L^2(\mathbb{R}^N, \mathbb{R}^N)$ is supported in a cube $Q \subset \mathbb{R}^N_+$ and* div $b = 0$ *in $\mathbb{R}^N$. Then b can be written as*

$$b = \sum_{i=0}^{\infty} \mu_i b_i,$$

where the b_i's are $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$-atoms and

$$\sum_{i=0}^{\infty} |\mu_i| \leq C|Q|^{1/2} \|b\|_{L^2(Q,\mathbb{R}^N)}$$

for some constants C independent of b and Q.

The following lemma is a special case of Theorem 3.3.3 in [4, Chapter 3], where $H^1_0(B, \mathbb{R}^N)$ denotes the closure of $C^\infty_0(B, \mathbb{R}^N)$ in the Sobolev space $H^1(B, \mathbb{R}^N)$ and $Du = \left(\frac{\partial u_i}{\partial x_j}\right)$ for $u = (u_1, ..., u_N)$, $N \geq 2$.

Lemma 4.5. *Let B be a ball in $\mathbb{R}^N$. If $u \in L^2(B, \mathbb{R}^N)$,* div $u = 0$ *in B and $n \cdot u|_{\partial B} = 0$, then there exists $\varphi \in H^1_0(B, \mathbb{R}^N)$ and a constant C independent of u and B such that*

$$u = \text{div } \varphi$$

and

$$\|D\varphi\|_{L^2(B,\mathbb{R}^N)} \leq C\|u\|_{L^2(B,\mathbb{R}^N)}.$$

We next recall the main result from [1, Theorem 6.4].

Theorem 4.6. *A function f with* div $f = 0$ *belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N)$ if and only if it has a decomposition*

$$f = \sum_{k=0}^{\infty} \lambda_k A_k \tag{4.1}$$

with

$$\sum_{k=0}^{\infty} |\lambda_k| \leq C\|f\|_{\mathcal{H}^1(\mathbb{R}^N,\mathbb{R}^N)},$$

where the A_k's are $\mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N)$-atoms and div $A_k = 0$ *on $\mathbb{R}^N$.*

Proof of Theorem 4.1. The easy part of the proof is the "if" part, where we assume f has a decomposition (4.1). For then, if the sum is finite,

$$\begin{aligned} \|f\|_{\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)} &= \|f\|_{\mathcal{H}^1_z(\mathbb{R}^N_+,\mathbb{R}^N)} \\ &\leq \sum_k |\gamma_k| \|\alpha_k\|_{\mathcal{H}^1_z(\mathbb{R}^N_+,\mathbb{R}^N)} \leq \sum_k |\gamma_k|, \end{aligned}$$

where we used the fact that $\|\alpha_k\|_{\mathcal{H}^1_z(\mathbb{R}^N_+,\mathbb{R}^N)} \leq 1$ when α_k is an $\mathcal{H}^1_z(\mathbb{R}^N_+\mathbb{R}^N)$-atom. This gives the convergence of the sum, and so $f \in \mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$.

We next prove the "only if" part. Let $f \in \mathcal{H}^1_{z,div}(\mathbb{R}^N_+, \mathbb{R}^N)$. We consider even extensions in the last coordinate for $f_1, ..., f_{N-1}$ and the odd extension in the last coordinate for f_N. Let

$$F_i(x) = \begin{cases} f_i(x) & \text{in } \overline{\mathbb{R}^N_+} \\ f_i(\tilde{x}) & \text{in } \mathbb{R}^N_- \end{cases}$$

for $i = 1, ..., N-1$ and

$$F_N(x) = \begin{cases} f_N(x) & \text{in } \overline{\mathbb{R}^N_+} \\ -f_N(\tilde{x}) & \text{in } \mathbb{R}^N_-. \end{cases}$$

Then $F = (F_1, ..., F_N) \in \mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N)$. We know that

$$\int_{\mathbb{R}^N_+} f \cdot D\varphi \; dx = 0$$

for all $\varphi \in C^\infty(\overline{\mathbb{R}^N_+})$. Combining this with Green's formula gives

$$\begin{aligned} \int_{\mathbb{R}^N} F \cdot D\varphi \; dx &= \int_{\mathbb{R}^N_+} f \cdot D\varphi \; dx + \int_{\mathbb{R}^N_-} \Big(f_1(\tilde{x}), ..., f_{N-1}(\tilde{x}), -f_N(\tilde{x}) \Big) \cdot D\varphi(x) \; dx \\ &= \int_{\mathbb{R}^N_+} f(x) \cdot D\varphi(\tilde{x}) \; dx = 0 \end{aligned}$$

for all $\varphi \in C^\infty_0(\mathbb{R}^N)$. That is, div $F = 0$ in $\mathbb{R}^N$. Applying Lemma 4.6 to F, one finds that any $F \in \mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N)$ can be written as

$$F = \sum_{k=0}^{\infty} \lambda_k A_k \tag{4.2}$$

with

$$\sum_{k=0}^{\infty} |\lambda_k| \leq C \|F\|_{\mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N)}, \tag{4.3}$$

where the A_k's are $\mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N)$-atoms and div $A_k = 0$ in $\mathbb{R}^N$. Let $A_k := (a^1_k, ..., a^N_k)$, then (4.2) implies $F_i = \sum_{k=0}^\infty \lambda_k a^i_k$, $i = 1, ..., N$. Since $F_1, ..., F_{N-1}$ are even and F_N is odd in the last coordinate, we may write, for $i = 1, ..., N-1$,

$$\begin{aligned} F_i(x) &= \frac{F_i(x) + F_i(\tilde{x})}{2} \\ &= \sum_{k=0}^{\infty} \lambda_k \frac{a^i_k(x) + a^i_k(\tilde{x})}{2} := \sum_{k=0}^{\infty} \lambda_k \tilde{a}^i_k \end{aligned}$$

and

$$\begin{aligned} F_N(x) &= \frac{F_N(x) - F_N(\tilde{x})}{2} \\ &= \sum_{k=0}^{\infty} \lambda_k \frac{a^N_k(x) - a^N_k(\tilde{x})}{2} := \sum_{k=0}^{\infty} \lambda_k \tilde{a}^N_k. \end{aligned}$$

Note that $F_i|_{\mathbb{R}^N_+}=f_i$. Combining this together with expressions of F_i $(i=1,...,N)$ above, we obtain $f_i=\sum_k \lambda_k \tilde{a}^i_k|_{\mathbb{R}^N_+}$, $i=1,...,N$. Hence

$$f=\sum_{k=0}^{\infty}\lambda_k a_k, \tag{4.4}$$

where $a_k=\left(\tilde{a}^1_k|_{\mathbb{R}^N_+},...,\tilde{a}^N_k|_{\mathbb{R}^N_+}\right)$.

Next we need to decompose a_k into $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)$-atoms. For any k, define

$$b_k=\begin{cases}\left(\frac{a^1_k(x)+a^1_k(\tilde{x})}{2},\ ...,\ \frac{a^{N-1}_k(x)+a^{N-1}_k(\tilde{x})}{2},\ \frac{a^N_k(x)-a^N_k(\tilde{x})}{2}\right) & \text{in } \overline{\mathbb{R}^N_+}\\ 0 & \text{in } \mathbb{R}^N_-.\end{cases}$$

Since a^1_k, ..., a^N_k are $\mathcal{H}^1(\mathbb{R}^N)$-atoms, it is easy to show that there exists a cube $Q_k\subset\mathbb{R}^N_+$ such that supp $b_k\subset Q_k$ and

$$\|b_k\|_{L^2(Q_k,\mathbb{R}^N)}\le C|Q_k|^{-1/2}$$

for a constant C independent of b_k and Q_k. Applying Lemma 4.3 to b_k, we have

$$\text{div } b_k=0 \quad \text{on} \quad \mathbb{R}^N.$$

It follows from Lemma 4.4 that b_k can be written as

$$b_k=\sum_{i=0}^{\infty}\mu^i_k b^i_k, \tag{4.5}$$

where the b^i_k's are $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)$-atoms and for any k

$$\sum_{i=0}^{\infty}|\mu^i_k|\le C|Q_k|^{1/2}\|b_k\|_{L^2(Q_k,\mathbb{R}^N)}\le C. \tag{4.6}$$

Note that $a_k=b_k|_{\mathbb{R}^N_+}$. We obtain

$$f=\sum_{k=0}^{\infty}\lambda_k\sum_{i=0}^{\infty}\mu^i_k b^i_k \tag{4.7}$$

by (4.4) and (4.5). Combining (4.3) with (4.6) yields

$$\begin{aligned}\sum_{k,i=0}^{\infty}|\lambda_k\mu^i_k| &\le C\|F\|_{\mathcal{H}^1(\mathbb{R}^N,\mathbb{R}^N)}\\ &\le C\Big(\sum_{i=1}^{N-1}\|f_i\|_{\mathcal{H}^1_z(\mathbb{R}^N_+)}+\|f_N\|_{\mathcal{H}^1_r(\mathbb{R}^N_+)}\Big)\\ &= C\|f\|_{\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)}.\end{aligned}$$

The proof of Theorem 4.1 is finished. $\square$

Proof of Lemma 4.3. For $\varphi \in C_0^\infty(\mathbb{R}^N)$, define

$$\psi(x) = \begin{cases} \varphi(x) & \text{in } \overline{\mathbb{R}^N_+}; \\ \varphi(\tilde{x}) & \text{in } \mathbb{R}^N_-. \end{cases}$$

It is easy to check that $\psi \in H^1(\mathbb{R}^N)$. Note that the following Green's formula

$$\int_{\mathbb{R}^N} A \cdot D\varphi \; dx = -\int_{\mathbb{R}^N} \text{div } A \cdot \varphi \; dx$$

for all $A \in L^2(\mathbb{R}^N, \mathbb{R}^N)$ with div $A \in L^2(\mathbb{R}^N)$ and $\varphi \in H^1(\mathbb{R}^N)$ (ref. [5, pages 27-28]). We have

$$\begin{aligned} \int_{\mathbb{R}^N} b \cdot D\varphi \; dx &= \int_{\mathbb{R}^N_+} A(x) \cdot D\varphi(x) \; dx + \int_{\mathbb{R}^N_-} A(x) \cdot D\varphi(\tilde{x}) \; dx \\ &= \int_{\mathbb{R}^N} A(x) \cdot D\psi(x) \; dx = 0 \end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. That is, div $b = 0$ in $\mathbb{R}^N$. $\square$

Proof of Lemma 4.4. Suppose Ψ is a bilipschitz function from $\mathbb{R}^N$ to $\mathbb{R}^N$ which maps the cube Q to a ball B with the same center. Let $\beta = (\Psi^{-1})^* b$ be the pull-back of b. Then $\beta \in L^2(\mathbb{R}^N, \mathbb{R}^N)$ with support B and div $\beta = 0$ in $\mathbb{R}^N$. Applying Lemma 4.5 to β, there exists $\psi \in H_0^1(B, \mathbb{R}^N)$ and a constant C independent of β and B such that

$$\beta = \text{curl } \psi$$

and

$$\|D\psi\|_{L^2(B,\mathbb{R}^N)} \leq C\|\beta\|_{L^2(B,\mathbb{R}^N)}. \tag{4.8}$$

Let $\Psi^*\psi := \varphi$. Note that $b = \Psi^*\beta$. We have

$$b = \Psi^*(\text{curl } \psi) = \text{curl } (\Psi^*\psi) = \text{curl } \varphi.$$

Applying a Whitney decomposition to the cube Q with respect to its boundary [6, Chapter 6], Q can be decomposed into a family of subcubes $\{Q_i\}$ from a dyadic grid of $\mathbb{R}^N$:

$$Q = \bigcup_{i=0}^{\infty} Q_i$$

such that $8Q_i \subset Q$ and $|Q| = \sum_{i=0}^\infty |Q_i|$. Denote $\Psi(Q_i) := B_i$, then B has a similar decomposition

$$B = \bigcup_{i=0}^{\infty} B_i.$$

Let

$$\sum_{i=0}^{\infty} \eta_i(x) \equiv 1$$

be the smooth partition of unity such that $\eta_i(x) = 1$ if $x \in Q_i$, $\eta_i(x) = 0$ if $x \notin 2Q_i$ and

$$|D\eta_i(x)| \le C\ l(Q_i)^{-1},$$

where $l(Q)$ denotes the side-length of Q. Applying the smooth partition of unity to φ, we have

$$b = \sum_{i=0}^{\infty} \text{curl } (\eta_i \varphi) := \sum_{i=0}^{\infty} \mu_i b_i,$$

where

$$b_i = \frac{\text{curl}(\eta_i\varphi)}{|2Q_i|^{1/2} \|\text{curl}(\eta_i\varphi)\|_{L^2(2Q_i,\mathbb{R}^N)}}$$

and

$$\mu_i = |2Q_i|^{1/2} \|\text{curl}(\eta_i\varphi)\|_{L^2(2Q_i,\mathbb{R}^N)}.$$

It is straightforward to check that the function b_i is an $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)$-atom. We next prove that there exists a constant C independent of b and Q such that

$$\sum_{i=0}^{\infty} \mu_i \le C|Q|^{1/2} \|b\|_{L^2(Q,\mathbb{R}^N)}. \tag{4.9}$$

We write the left-hand side of (4.9) into two parts

$$\begin{aligned}\sum_{i=0}^{\infty} \mu_i &\le 2^{1/2} \sum_{i=0}^{\infty} |2Q_i|^{1/2} \Big(\big\| \eta_i \text{ curl } \varphi \|_{L^2(2Q_i,\mathbb{R}^N)} + \sum_{l,m=1}^{N} \Big\| \varphi_l \frac{\partial \eta_i}{\partial x_m} \Big\|_{L^2(2Q_i)} \Big) \\ &:= I + II,\end{aligned}$$

where φ_l and x_m denote respectively the l^{th} and m^{th} component of φ and x.

Note that $0 \le \eta_i \le 1$. The Cauchy-Schwartz inequality gives

$$\begin{aligned} I &\le 2^{1/2} \sum_{i=0}^{\infty} |2Q_i|^{1/2} \| \text{ curl } \varphi\|_{L^2(2Q_i,\mathbb{R}^N)} \\ &\le 2^{1/2} \left(\sum_{i=0}^{\infty} |2Q_i| \right)^{1/2} \left(\sum_{i=0}^{\infty} \int_{2Q_i} |b|^2\, dx \right)^{1/2} \\ &\le C|Q|^{1/2} \|b\|_{L^2(Q,\mathbb{R}^N)} \end{aligned}$$

for constants C independent of Q and b.

Let $y = \Psi(x)$ for $x \in Q$, then $d(x, \partial Q) \sim d(y, \partial B)$ and the implicit constants do not depend on Q and B. Note that

$$|D\eta_i(x)| \le C\ l(Q_i)^{-1} \le C\ d(x, \partial Q)^{-1}, \quad x \in 2Q_i.$$

We obtain

$$\begin{aligned}\int_{2Q_i}\left|\varphi_l(x)\frac{\partial\eta_i(x)}{\partial x_m}\right|^2 dx &\le C\int_{2Q_i}\left|\frac{\varphi_l(x)}{d(x,\partial Q)}\right|^2 dx\\ &= C\int_{\Psi^{-1}(\tilde{B}_i)}\left|\frac{\varphi_l(x)}{d(x,\partial Q)}\right|^2 dx\\ &\le C\int_{\tilde{B}_i}\left|\frac{\varphi_l(\Psi^{-1}(y))}{d(y,\partial B)}\right|^2 |\det(\Psi^{-1})'(y)|\, dy\\ &\le C\int_{\tilde{B}_i}\left|\frac{\psi_l(y)}{d(y,\partial B)}\right|^2 dy,\end{aligned}$$

where $\tilde{B}_i = \Psi(2Q_i)$ and C depends only on Ψ. Applying Hardy's inequality (see, for example, [7, Chapter 1, Section 5]), we get

$$\begin{aligned}II &\le C\left(\sum_{i=0}^{\infty}|2Q_i|\right)^{1/2}\sum_{l=1}^{N}\left(\sum_{i=0}^{\infty}\int_{\tilde{B}_i}\left|\frac{\psi_l(y)}{d(y,\partial B)}\right|^2 dy\right)^{1/2}\\ &\le C|Q|^{1/2}\sum_{l=1}^{N}\left(\int_B\left|\frac{\psi_l(y)}{d(y,\partial B)}\right|^2 dy\right)^{1/2}\\ &\le C|Q|^{1/2}\sum_{l=1}^{N}\|D\psi_l\|_{L^2(B,\mathbb{R}^N)} \quad \text{(by}\\ &\le C|Q|^{1/2}\|\beta\|_{L^2(B,\mathbb{R}^N)} \quad (4.8))\\ &\le C|Q|^{1/2}\|b\|_{L^2(Q,\mathbb{R}^N)},\end{aligned}$$

where C does not depend on Q and b, in the last step we used the fact that

$$\|\beta\|_{L^2(B,\mathbb{R}^N)} \sim \|b\|_{L^2(Q,\mathbb{R}^N)}.$$

This proves (4.9). The proof of Lemma 4.4 is finished. □

5. Dual Space and an Application to Coercivity

The aim of this section is to characterize the dual space of $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)$ and establish a "div-curl" type theorem on $\mathbb{R}^3_+$ with an application to coercivity. The proofs of these results are similar to the corresponding results in [1] and so are skipped.

Definition 5.1. Let $BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)$ to be the space of measurable functions G on $\mathbb{R}^N_+$ for which

$$\|G\|_{BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)} = \sup_B \inf_{g_B}\left(\frac{1}{|B|}\int_B |G-g_B|^2\, dx\right)^{1/2} < \infty,$$

where the supremum is taken over all balls B with $2B \subset \mathbb{R}^N_+$, the infimum is taken over all functions $g_B \in L^2(B,\mathbb{R}^N)$ with curl $g_B = 0$ in B.

Consider $BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)/X_0$ with norm

$$\|G+X_0\|_{BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)/X_0}=\|G\|_{BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)},$$

where $X_0=\{G\in BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N):\|G\|_{BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)}=0\}$. Let $D_z(\mathbb{R}^N_+,\mathbb{R}^N)$ denote the vector space finitely generated by $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)$-atoms, which is dense in $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)$ by Theorem 4.1. We have the following duality theorem.

Theorem 5.2. *If $G+X_0\in BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)/X_0$, then the linear functional L defined by*

$$L(h)=\int_{\mathbb{R}^N_+}G\cdot h, \tag{5.1}$$

initially defined on $D_z(\mathbb{R}^N_+,\mathbb{R}^N)$, has a unique bounded extension to $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)$. Conversely, if $L\in\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)^*$, then there exists a unique $G+X_0\in BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)/X_0$ such that (5.1) holds. The map $G+X_0\to L$ given by (5.1) is a Banach isomorphism between $BMO_{r,curl}(\mathbb{R}^N_+,\mathbb{R}^N)/X_0$ and $\mathcal{H}^1_{z,div}(\mathbb{R}^N_+,\mathbb{R}^N)^*$.

Using Theorem 5.2 we obtain the following "div-curl" type theorem on $\mathbb{R}^3_+$.

Theorem 5.3. *Suppose $b\in L^2_{loc}(\mathbb{R}^3_+,\mathbb{R}^3)$. Then*

$$\sup_{u,v}\int_{\mathbb{R}^3_+}b\cdot(\nabla u\times\nabla v)\sim\|b\|_{BMO_{r,curl}(\mathbb{R}^3_+,\mathbb{R}^3)}, \tag{5.2}$$

where the supremum is taken over all u and v in $H^1_0(\mathbb{R}^3_+)$ with $\|\nabla u\|_{L^2(\mathbb{R}^3_+,\mathbb{R}^3)}$, $\|\nabla v\|_{L^2(\mathbb{R}^3_+,\mathbb{R}^3)}$ ≤ 1. The implicit constants in (5.2) are independent of b.

Next we give an application of the "div-curl" type theorem to coercivity. In the study of homogenization of linearized elasticity, Geymonat, Müller and Triantafyllidis [8] introduced the following quantity Λ which gives a criterion of whether an elliptic system satisfying the Legendre-Hadamard condition can be homogenized, namely

$$\Lambda=\inf\left\{\frac{\int_{\mathbb{R}^N}A^{i,j}_{\alpha,\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta}dx}{\int_{\mathbb{R}^N}|Du|^2dx}:u\in C_0^\infty(\mathbb{R}^N,\mathbb{R}^N)\right\}$$

and proved that if $\Lambda>0$ some homogenization results can be obtained for the corresponding system. If $\Lambda<0$, the system cannot be homogenized. Zhang asked the following question: what conditions on the coefficient $A^{i,j}_{\alpha,\beta}$ of the system imply that $\Lambda\ge 0$?

Let $N=3$ and suppose that $A^{i,j}_{\alpha,\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta}$ can be written as

$$A^{i,j}_{\alpha,\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta}=B^{i,j}_{\alpha,\beta}(x)\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta}+b_{ij}(x)(\text{adj }Du)_{i,j}, \tag{5.3}$$

where $A^{i,j}_{\alpha,\beta}$, $B^{i,j}_{\alpha,\beta}\in L^\infty(\mathbb{R}^3)$, $B^{i,j}_{\alpha,\beta}\frac{\partial u_i}{\partial x_\alpha}\frac{\partial u_j}{\partial x_\beta}\ge C|Du|^2$, adj Du denotes the adjoint matrix of Du for $u\in H^1(\mathbb{R}^3,\mathbb{R}^3)$ (the summation convention is understood). We are interested in forms of this type in three dimensions, because they arrive naturally from the linearization of polyconvex variational integrals studied in nonlinear elasticity by Ball in [9].

By a simple calculation (see [1] for details), the last term in (5.3) can be written as a sum of terms like $b \cdot (\nabla u \times \nabla v)$. Let $a(u)$ denote the following polyconvex quadratic form

$$a(u) = |Du|^2 + b_1 \cdot (\nabla u_2 \times \nabla u_3) + b_2 \cdot (\nabla u_1 \times \nabla u_3) + b_3 \cdot (\nabla u_1 \times \nabla u_2). \tag{5.4}$$

So the question when $\Lambda \geq 0$ becomes: find necessary conditions of b_i such that

$$\int_{\mathbb{R}^3} a(u)\ dx \geq 0 \quad \text{ for all } \quad u \in H^1(\mathbb{R}^3, \mathbb{R}^3).$$

We answered this question in [1] (see [10] for the two dimensional case). It is natural to consider the same question on $\mathbb{R}^3_+$. That is, find necessary conditions of b_i such that

$$\int_{\mathbb{R}^3_+} a(u)\ dx \geq 0 \quad \text{ for all } \quad u \in H^1_0(\mathbb{R}^3_+, \mathbb{R}^3). \tag{5.5}$$

Applying Theorem 5.3 and an idea of Zhang in [10], we give an “almost” necessary and sufficient condition on b_i such that (5.5) holds.

Theorem 5.4. *Let $a(u)$ be the expression shown in (5.4). Then*

(1) There exists a constant C_1 depending only on N such that $\max_{1\leq i\leq 3} \|b_i\|_{BMO_{r,curl}(\mathbb{R}^3_+, \mathbb{R}^3)} \leq C_1$ *implies that*

$$\int_{\mathbb{R}^3_+} a(u)\ dx \geq \frac{1}{2}\|Du\|^2_{L^2(\mathbb{R}^3_+, \mathbb{R}^3)} \quad \text{ for all } \quad u \in H^1_0(\mathbb{R}^3_+, \mathbb{R}^3).$$

(2) If $\int_{\mathbb{R}^3_+} a(u)\ dx \geq 0$ for all $u \in H^1_0(\mathbb{R}^3_+, \mathbb{R}^3)$, then there exists an constant C_2 depending only on N such that

$$\max_{1\leq i\leq 3} \|b_i\|_{BMO_{r,curl}(\mathbb{R}^3_+, \mathbb{R}^3)} \leq C_2.$$

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