THE MONOGENIC FUNCTIONAL CALCULUS

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Abstract. A study is made of a symmetric functional calculus for a system of bounded linear operators acting on a Banach space. Finite real linear combinations of the operators have real spectra, but the operators do not necessarily commute with each other. Analytic functions of the operators are formed by using functions taking their values in a Clifford algebra.

1. Introduction.

The notion of a monogenic functional calculus of commuting n-tuples of bounded operators was introduced by A. McIntosh and A. Pryde in order to give estimates on the solution of systems of operator equations [8, 9]. This led to the study of the monogenic functional calculus of noncommuting families by A. McIntosh and J. Picton–Warlow utilising plane-wave decompositions. V. Kisil and E. Ramírez de Arellano have also introduced a functional calculus for an n-tuple A of bounded selfadjoint elements of a C^*-algebra [5, 6], and for monogenic functions defined on a sufficiently large ball in $\mathbb{R}^{n+1}$. In this paper, we make precise the idea of the monogenic spectrum $\gamma(A)$ of an n-tuple A of noncommuting bounded operators on a Banach space. It is a compact subset of $\mathbb{R}^n$ characterised as being the smallest set about which a symmetric analytic functional calculus is defined. In further work we use the monogenic functional calculus to analyse the support of the Weyl functional calculus [3, 4].

The central idea is a natural extension of the Riesz-Dunford functional calculus for a single operator, but with functions of a single complex variable replaced by functions defined in $\mathbb{R}^{n+1}$ and taking values in a Clifford algebra. With the appropriate notion of the monogenic spectrum $\gamma(A) \subset \mathbb{R}^n$ of A, we find that the monogenic functional calculus coincides with the Weyl functional calculus $\mathcal{W}_A$ applied to functions of n real variables analytic in a neighbourhood of the support $\operatorname{supp}\mathcal{W}_A$ of $\mathcal{W}_A$. Furthermore, the equality $\gamma(A) = \operatorname{supp}\mathcal{W}_A$ holds [3, Theorem 6.2]. The Weyl functional calculus is a symmetric C^∞-functional calculus.

A $C^\infty(\mathbb{R}^n)$-functional calculus for an n-tuple A of bounded operators acting on a Banach space X exists whenever A satisfies an exponential bound. One would expect a *monogenic* functional calculus to exist even when such an exponential bound fails. In this case, it is not possible to identify the Cauchy kernel $G_\omega(A)$, $\omega \in \mathbb{R}^{n+1} \setminus (\{0\} \times \gamma(A))$ for A as the monogenic representation of a distribution $\mathcal{W}_A$ with compact support $\operatorname{supp}\mathcal{W}_A \subset \mathbb{R}^n$ [3].

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For a single bounded operator T, the resolvent $(\lambda I - T)^{-1}$ of T has a Neumann series expansion for all $\lambda \in \mathbb{C}$ with modulus $|\lambda| > \|T\|$. If the spectrum $\sigma(T)$ of T is real (so that $\mathbb{C} \setminus \sigma(T)$ is connected), then the resolvent function $\lambda \mapsto (\lambda I - T)^{-1}$, $\lambda \in \mathbb{C} \setminus \sigma(T)$, is the unique analytic function with maximal domain coinciding with the function defined by the Neumann series expansion for $|\lambda| > \|T\|$. The resolvent of a bounded linear operator T is the Cauchy kernel for the Riesz-Dunford functional calculus and its set of singularities is precisely the spectrum $\sigma(T)$ of T.

A similar strategy may be applied to an n-tuple A of bounded operators acting on a Banach space. The Cauchy kernel $G_\omega(A)$ may be defined by a multiple power series expansion for all $\omega \in \mathbb{R}^{n+1}$ with $|\omega|$ sufficiently large [5, Definition 3.11]. However, we need to know that $\omega \mapsto G_\omega(A)$ is the restriction of a monogenic function with a maximal connected domain in $\mathbb{R}^{n+1}$. In the case that A satisfies an exponential bound, so that a Weyl functional calculus $\mathcal{W}_A$ exists, the equality $\gamma(A) = \operatorname{supp} \mathcal{W}_A$ guarantees the existence of a unique maximal monogenic extension — the monogenic representation of the distribution $\mathcal{W}_A$. If the n-tuple A is a commutative system of operators, then the monogenic spectrum $\gamma(A)$ coincides with the Clifford spectrum considered in [5,8,9].

The purpose of the present work is to establish the existence of a monogenic functional calculus for an n-tuple A of bounded operators acting on a Banach space X, just under the condition that the spectrum $\sigma(\langle A, \xi\rangle)$ of the operator $\langle A, \xi\rangle = \sum_{j=1}^n A_j \xi_j$ is real for every $\xi \in \mathbb{R}^n$. This amounts to showing that there exists a compact nonempty set $\gamma(A) \subset \mathbb{R}^n$, the *monogenic spectrum* of A, and a monogenic function $\omega \mapsto G_\omega(A)$, $\omega \in \mathbb{R}^{n+1} \setminus (\{0\} \times \gamma(A))$, coinciding with the multiple power series expansion for $G_\omega(A)$ defined for all $\omega \in \mathbb{R}^{n+1}$ with $|\omega|$ sufficiently large.

The existence of the Cauchy kernel $G_\omega(A)$ for the n-tuple A and the set $\gamma(A)$ is proved in Theorem 2.2 and Theorem 2.6 by appealing to the plane wave decomposition for the Cauchy kernel [12, p.111]. In effect, we replace the Fourier transform in the definition of $G_\omega(A)$ via the Weyl functional calculus (if this makes sense) by a plane wave decomposition; we can do this provided that $\sigma(\langle A, \xi\rangle)$ is real for all real $\xi \in \mathbb{R}^n$. This is the key *algebraic* condition guaranteeing that the 'resolvent set' $\mathbb{R}^{n+1} \setminus (\{0\} \times \gamma(A)\}$ is connected — most of the analysis depends only on this topological property.

The verification that the function $f(A)$ of the n-tuple A by a real analytic function f defined in a neighbourhood in $\mathbb{R}^n$ of the monogenic spectrum $\gamma(A)$ is indeed a bounded linear operator on X is given in Theorem 3.5. The proof appeals to the fact that the unique monogenic extension $\tilde{f}$ of the real analytic function f may be approximated uniformly on compact subsets of the domain of $\tilde{f}$ in $\mathbb{R}^{n+1}$ by the monogenic extensions of scalar valued polynomials defined on $\mathbb{R}^n$. This function theory result is proved in Proposition 3.2.

In Theorem 3.6, we make precise the observation that the monogenic spectrum $\gamma(A)$ is the smallest set about which an analytic functional calculus is defined: if there exists a 'functional calculus' T_A defined for all functions of n real variables analytic in an open neighbourhood of a compact set $K \subset \mathbb{R}^n$ and with the property that $T_A(p(\langle \cdot, \xi\rangle)) = p(\langle A, \xi\rangle)$ for all polynomials $p : \mathbb{R} \to \mathbb{R}$ and $\xi \in \mathbb{R}^n$, then necessarily $\sigma(\langle A, \xi\rangle)$ is real for all $\xi \in \mathbb{R}^n$ and $\gamma(A) \subseteq K$. Moreover, T_A agrees with the monogenic functional calculus for those analytic functions defined in an open neighbourhood of K.

Even in the case of commuting systems of operators, the use of Clifford analysis leads to simplifications in the construction of functions of operators. In Theorem 3.9, we show that Taylor's functional calculus [13] for commuting systems $(A_1, \dots, A_n)$ of operators acting on a Banach space coincides with the monogenic functional calculus in the case that $\sigma(A_j)$ is real for every $j = 1, \dots, n$.

The notation of [3] concerning Clifford algebras is used. If $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$, then $\mathbb{F}_{(n)}$ denotes the Clifford algebra over $\mathbb{F}$ generated by $e_0, e_1, \dots, e_n$. Given a Banach space X, the family of sums $T = \sum_S T_S e_S$ for $T_S \in \mathcal{L}(X)$ and $S \subseteq \{1, \dots, n\}$ forms a Banach module $\mathcal{L}_{(n)}(X_{(n)})$ under left and right multiplication by elements of $\mathbb{F}_{(n)}$. The norm is given by $\|T\| = \left(\sum_S \|T_S\|^2_{\mathcal{L}(X)}\right)^{1/2}$. For each $x \in X$ and $\xi \in X'$, the element $\langle Tx, \xi\rangle$ of $\mathbb{F}_{(n)}$ is defined by $\langle Tx, \xi\rangle = \sum_S \langle T_S x, \xi\rangle e_S$.

Let D be the differential operator $D = \sum_{j=0}^n e_j \partial/\partial x_j$. A function $f : U \to \mathbb{F}_{(n)}$ is called *left monogenic* in an open set U if $Df = 0$ in U. It is *right monogenic* in U if $fD = 0$ in U. The expression *two-sided monogenic* is used for functions which are both left and right monogenic. For each $\omega \in \mathbb{R}^{n+1}$, the function G_ω defined by

$$G_\omega(x) = \frac{1}{\sigma_n} \frac{\overline{\omega - x}}{|\omega - x|^{n+1}}, \qquad \text{for each } x \neq \omega \tag{1}$$

is two-sided monogenic. Here $\sigma_n = 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ is the volume of the unit n-sphere in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1}$ is identified with a subspace of $\mathbb{R}_{(n)}$. The notation $E(\omega - x) = G_\omega(x)$ is used in [2].

Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set with smooth boundary $\partial\Omega$ and exterior unit normal $n(\omega)$ defined for all $\omega \in \partial\Omega$. For any left monogenic function f defined in a neighbourhood U of $\overline{\Omega}$, the Cauchy integral formula

$$\int_{\partial\Omega} G_\omega(x) n(\omega) f(\omega)\, d\mu(\omega) = \begin{cases} f(x), & \text{if} \quad x \in \Omega; \\ 0, & \text{if} \quad x \in U \setminus \overline{\Omega}. \end{cases} \tag{2}$$

is valid. Here μ is the surface measure of $\partial\Omega$. The result is proved in [2, Corollary 9.6]. If g is right monogenic in U then $\int_{\partial\Omega} g(\omega)n(\omega)f(\omega)\, d\mu(\omega) = 0$ [2, Corollary 9.3].

These results extend to the vector and operator valued setting in a routine fashion. In this case, 'monogenic' means that the partial derivatives are evaluated in the underlying topology of the space. We shall quote them without further discussion.

In the monogenic functional calculus for a suitable n-tuple A of bounded operators acting on a Banach space X, the operator $f(A)$ is defined for all $\mathbb{F}$-valued functions f of n-real variables analytic in an open neighbourhood U of the monogenic spectrum $\gamma(A)$, see Section 3. The operator $f(A)$ is defined analogously to the Cauchy integral formula (2), where G_ω is replaced by a suitable element $G_\omega(A)$ of $\mathcal{L}_{(n)}(X_{(n)})$ for each $\omega \in \mathbb{R}^{n+1} \setminus \gamma(A)$ and f is extended monogenically off $\{0\} \times U$ into $\mathbb{R}^{n+1}$.

In [3], the Cauchy kernel $\omega \mapsto G_\omega(A)$, $\omega \in \mathbb{R}^{n+1} \setminus \gamma(A)$, is identified by employing the Weyl functional calculus. In the present context, it is constructed in Lemma 2.5 by using the plane wave decomposition of the Cauchy kernel (1).

2. The Cauchy kernel for an n-tuple of operators

It is a simple matter to write down an example of a pair $A = (A_1, A_2)$ of bounded linear operators acting on $l^2(\mathbb{N})$ for which the bound

$$\|e^{i(\xi_1 A_1 + \xi_2 A_2)}\| \le C(1 + |\xi|)^s, \qquad \text{for all } \xi \in \mathbb{R}^2 \tag{3}$$

fails, but $\sigma(\xi_1 A_1 + \xi_2 A_2) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^2$.

2.1 Example. For each $n = 1, 2, \dots$, let U_n be the $n \times n$ matrix such that $(U_n)_{j,j+1} = 1$ for all $j = 1, \dots, n-1$, and $(U_n)_{k,j} = 0$ otherwise. Let I_n be the $n \times n$ identity matrix. Let $A_1 : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the direct sum of $(-1)^n I_n$ for $n = 2, 3, \dots$ and let $A_2 : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the direct sum of U_n for $n = 2, 3, \dots$. There exists no $C > 0$ and no $s > 0$ for which the commuting pair $A = (A_1, A_2)$ of operators on $l^2(\mathbb{N})$ satisfies the bound (3). Nevertheless, the spectrum $\sigma(\xi_1 A_1 + \xi_2 A_2)$ of the operator $\xi_1 A_1 + \xi_2 A_2$ is real for all $\xi \in \mathbb{R}^2$ because it is real on each common invariant subspace.

Let $x \mapsto G_\omega(x)$, $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ be the Cauchy kernel on $\mathbb{R}^3$ for $\omega \neq x$. The natural definition of $G_\omega(A)$ suggested by the matrix functional calculus is obtained by taking the direct sum of

$$\sum_{k=0}^{n} \frac{1}{k!} \partial_2^k G_\omega\big((0, (-1)^{n+1}, 0)\big)(U_{n+1})^k$$

for $n = 1, 2, \dots$ for each $\omega \in \mathbb{R}^3 \setminus (\{0\} \times \{-1, 1\} \times \{0\})$.

The example above suggests adopting a power series expansion as the definition of $G_\omega(A)$ for general n-tuples A.

For each $\omega \in \mathbb{R}^{n+1}$ such that $\omega \neq 0$, let

$$G_\omega(x) = \sum_{k=0}^{\infty} \left(\sum_{(l_1, \dots, l_k)} W_{l_1 \dots l_k}(\omega) V_{l_1 \dots l_k}(x) \right) \tag{4}$$

be the monogenic power series expansion of G_ω in the region $|x| < |\omega|$ [2, 11.4 pp. 77–81]. Here $W_{l_1 \dots l_k}(\omega)$ is given for each $\omega \in \mathbb{R}^{n+1}, \omega \neq 0$ by $(-1)^k \partial_{\omega_{l_1}} \cdots \partial_{\omega_{l_k}} G_\omega(0)$ and $V_{l_1 \dots l_k}(x)$ is the monogenic extension of $x_{l_1} \cdots x_{l_k}$ off $\mathbb{R}^n$ [2, Proposition 11.2.3].

Let A be an n-tuple of bounded operators acting on a Banach space X. Let

$$V_{l_1 \dots l_k}(A) = 1/k! \sum_{j_1, \dots, j_k} A_{j_1} \cdots A_{j_k},$$

the sum being over all distinguishable permutations of $(l_1, \dots, l_k)$. As in [5, Definition 3.11], the Cauchy kernel $G_\omega(A)$ is given by the expansion

$$G_\omega(A) = \sum_{k=0}^{\infty} \left(\sum_{(l_1, \dots, l_k)} W_{l_1 \dots l_k}(\omega) V_{l_1 \dots l_k}(A) \right) \tag{5}$$

in the case that $\omega \in \mathbb{R}^{n+1}$ and $|\omega| > (1 + \sqrt{2}) \| \sum_{j=1}^n A_j e_j \|$. The sum converges in $\mathcal{L}_{(n)}(X_{(n)})$ because $\sum_{k=0}^\infty \sum_{(l_1, \dots, l_k)} |W_{l_1 \dots l_k}(\omega)| \, \|V_{l_1 \dots l_k}(A)\|$ converges uniformly for $|\omega| \geq R, \omega \in \mathbb{R}^{n+1}$, for each $R > (1 + \sqrt{2}) \| \sum_{j=1}^n A_j e_j \|$ [3, Lemma 6.1]. Each function $W_{l_1 \dots l_k}$ is left and right monogenic, so (5) defines a left and right monogenic $\mathcal{L}_{(n)}(X_{(n)})$-valued function for all $\omega \in \mathbb{R}^{n+1}$ such that $|\omega| > (1 + \sqrt{2}) \| \sum_{j=1}^n A_j e_j \|$.

Although (5) makes sense for any n-tuple of bounded operators, the problem remains of enlarging the domain of definition of the monogenic function defined by (5) to be as large as possible in a unique way, such as in the case when the natural domain is connected. The following assertion allows us to define the monogenic functional calculus.

2.2 Theorem. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded operators acting on a Banach space X. Suppose that $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Then the $\mathcal{L}_{(n)}(X_{(n)})$-valued function $\omega \mapsto G_\omega(A)$ is the restriction to the region $|\omega| > (1 + \sqrt{2})\|\sum_{j=1}^n A_j e_j\|$ of a function which is two-sided monogenic on the set $\mathbb{R}^{n+1} \setminus \mathbb{R}^n$.*

To prove the theorem, we appeal to a series of key lemmas, in which we suppose that the n-tuple A satisfies the conditions of the theorem. Let μ be the surface measure of the unit $(n-1)$-sphere S_1 in $\mathbb{R}^n$. The following plane wave decomposition is given in [12, p.111]. Further proofs appear in [11] and [7]. The latter uses a general Fourier transform calculus for monogenic functions.

2.3 Proposition. *Let $\omega = x_0 e_0 + x$ be an element of $\mathbb{R}^{n+1}$ with $x \in \mathbb{R}^n$. If $x_0 > 0$, then*

$$\frac{\overline{\omega}}{\sigma_n |\omega|^{n+1}} = \frac{(n-1)!}{2} \left(\frac{i}{2\pi}\right)^n \int_{S_1} (e_0 + is)\left(\langle x, s\rangle - x_0 s\right)^{-n} d\mu(s).$$

If $x_0 < 0$, then

$$\frac{\overline{\omega}}{\sigma_n |\omega|^{n+1}} = -\frac{(n-1)!}{2} \left(\frac{-i}{2\pi}\right)^n \int_{S_1} (e_0 + is)\left(\langle x, s\rangle - x_0 s\right)^{-n} d\mu(s).$$

Remark. If n is odd, the integral of $\left(\langle x, s\rangle - x_0 e_0 s\right)^{-n}$ over S_1 is zero and as long as $(x_1, \ldots, x_n) \neq 0$, the integral of the other term $s\left(\langle x, s\rangle - x_0 e_0 s\right)^{-n}$ over S_1 is continuous at $x_0 = 0$. If n is even, the integral of $s\left(\langle x, s\rangle - x_0 e_0 s\right)^{-n}$ over S_1 is zero and the integral of $\left(\langle x, s\rangle - x_0 e_0 s\right)^{-n}$ suffers a jump as x_0 passes through 0.

We view the n-tuple $A = (A_1, \ldots, A_n)$ of bounded linear operators acting on a Banach space X as an element $A = \sum_{j=1}^n A_j e_j$ of the Banach module $\mathcal{L}_{(n)}(X_{(n)})$. In the following statement, $\mathbb{R}^n$ is identified, as usual, with the set of all $x \in \mathbb{R}^{n+1}$ for which $x = (0, x_1, \ldots, x_n)$, and in turn, $\mathbb{R}^{n+1}$ is identified with a subspace of the Clifford algebra $\mathbb{R}_{(n)}$.

2.4 Lemma. *Let $y = \sum_{j=1}^n y_j e_j$ and $y_0 \neq 0$. Then for each $s \in S_1$, $\langle yI - A, s\rangle - y_0 sI$ is invertible in $\mathcal{L}_{(n)}(X_{(n)})$.*

Proof. The inverse of $\langle yI - A, s\rangle - y_0 sI$ is given by

$$(\langle yI - A, s\rangle - y_0 sI)^{-1} = (\langle yI - A, s\rangle + y_0 sI)(\langle yI - A, s\rangle^2 + y_0^2 I)^{-1}.$$

We see that this makes sense as follows.

Let $s \in S_1$, $y_0 \in \mathbb{R}, y_0 \neq 0$ and $y \in \mathbb{R}^n$. Let $f : \mathbb{R} \to (0, \infty)$ be defined by $f(x) = (\langle y, s\rangle - x)^2 + y_0^2$ for all $x \in \mathbb{R}$. Then applying the Spectral Mapping Theorem to the bounded operator $\langle A, s\rangle$,

$$\sigma(\langle yI - A, s\rangle^2 + y_0^2 I) = f[\sigma(\langle A, s\rangle)] \subset f(\mathbb{R}) \subset (0, \infty).$$

Hence, the operator $\langle yI - A, s\rangle^2 + y_0^2 I$ is invertible for $y_0 \neq 0$. Moreover, it commutes with $\langle yI - A, s\rangle \pm y_0 sI$, since all three operators involve only multiples of the identity I and the single operator $\langle yI - A, s\rangle$. By direct calculation,

$$(\langle yI - A, s\rangle + y_0 sI)(\langle yI - A, s\rangle - y_0 sI) = (\langle yI - A, s\rangle^2 + y_0^2 I),$$

because under Clifford multiplication, $s^2 = -1$ for all $s \in S_1$. $\square$

Thus, for each $s \in S_1$, $\big(\langle yI - A, s\rangle - y_0 s\big)^{-1}$ is an element of $\mathcal{L}_{(n)}(X_{(n)})$ and so

$$\big(\langle yI - A, s\rangle - y_0 s\big)^{-n} = \Big(\big(\langle yI - A, s\rangle - y_0 s\big)^{-1}\Big)^n$$

is an element of $\mathcal{L}_{(n)}(X_{(n)})$ too.

The following lemma completes the proof of Theorem 2.2.

2.5 Lemma. *For each real number $y_0 \neq 0$, and each $y \in \mathbb{R}^n$, the $\mathcal{L}_{(n)}(X_{(n)})$-valued function $s \mapsto (e_0 + is)\big(\langle yI - A, s\rangle - y_0 s\big)^{-n}$ defined for $s \in S_1$ is Bochner μ-integrable on S_1, and the function $y + y_0 e_0 \mapsto \int_{S_1} (e_0 + is)\big(\langle yI - A, s\rangle - y_0 s\big)^{-n}\, d\mu(s)$ is left and right monogenic on $\mathbb{R}^{n+1} \setminus \mathbb{R}^n$.*

Furthermore, if $y_0 > 0$ and $|y| > (1 + \sqrt{2})\|A\|$, then

$$G_{y+y_0 e_0}(A) = \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \int_{S_1} (e_0 + is)\left(\langle yI - A, s\rangle - y_0 s\right)^{-n} d\mu(s). \tag{6}$$

If $y_0 < 0$ and $|y| > (1 + \sqrt{2})\|A\|$, then

$$G_{y+y_0 e_0}(A) = -\frac{(n-1)!}{2}\left(\frac{-i}{2\pi}\right)^n \int_{S_1} (e_0 + is)\left(\langle yI - A, s\rangle - y_0 s\right)^{-n} d\mu(s).$$

Here the left-hand sides of the equations are defined by formula (5).

Proof. The function $s \mapsto (e_0 + is)\big(\langle yI - A, s\rangle - y_0 s\big)^{-n}$ is continuous on S_1, and so Bochner μ-integrable. The monogenicity of the function follows by differentiation under the integral sign.

We shall establish the equality

$$G_{y+y_0 e_0}(tA) = \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \int_{S_1} (e_0 + is)\left(\langle yI - tA, s\rangle - y_0 s\right)^{-n} d\mu(s) \tag{7}$$

for all $0 \leq t \leq 1$, $y_0 > 0$ and $|y| > (1 + \sqrt{2})\|A\|$. The case $y_0 < 0$ is similar.

For $t = 0$, the left hand side of equation (7) is equal to $G_{y+y_0 e_0}(0)$. An appeal to Proposition 2.3 ensures that the right hand side equals $G_{y+y_0 e_0}(0)$ at $t = 0$. By differentiation under the integral sign, for $y_0 > 0$, the right hand side of (7) is a solution of the equation

$$\partial_t u(y,t) = -\langle A, \nabla_y\rangle u(y,t) \tag{8}$$

in the Banach module $\mathcal{L}_{(n)}(X_{(n)})$ with the initial condition $u(y,0) = G_{y+y_0 e_0}(0)I$. Then

$$u(y,t) = G_{y+y_0 e_0}(0)I - \int_0^t \langle A, \nabla_y\rangle u(y,s)\, ds. \tag{9}$$

In the case that $|y_0| > |y| + \|A\|$, a power series expansion shows that the right hand side of (7) is analytic in t for all $|t| \leq 1$.

Let $y \in \mathbb{R}^n$ satisfy $|y| > (1+\sqrt{2})\|A\|$ and set $\omega = y_0 e_0 + y$. In the notation used in formulae (4) and (5), the series

$$\sum_{k=0}^{\infty} t^k \left(\sum_{(l_1,\dots,l_k)} W_{l_1\dots l_k}(\omega) V_{l_1\dots l_k}(A) \right) \tag{10}$$

represents $e^{-\langle A,\nabla_y\rangle t} G_{y+y_0e_0}(0)$ and iterating equation (9), we find that

$$u(y,t) = e^{-\langle A,\nabla_y\rangle t} G_{y+y_0e_0}(0),$$

that is, the solution of equation (8) with the initial condition $u(y,0) = G_{y+y_0e_0}(0)I$ has the series representation (10).

In the region $\Gamma \subset \mathbb{R}^{n+1}$ where $|y| > (1+\sqrt{2})\|A\|$ and $|y_0| > |y| + \|A\|$, the right-hand side of equation (7) and the expression (10) are analytic in t for $0 \le |t| \le 1$, so equality follows for all $0 \le |t| \le 1$ in the region Γ by the uniqueness of the Taylor series expansion. Both sides of equation (7) are monogenic in their domains, so by unique continuation, the equality (7) must be true for all $0 \le |t| \le 1$ and all $y_0 > 0$ and $|y| > (1+\sqrt{2})\|A\|$. $\square$

The maximal monogenic extension of the function $\omega \mapsto G_\omega(A)$ is denoted by the same symbol, that is, let Ω be the union of all open sets containing the open set $\Gamma = \{|\omega| > (1+\sqrt{2})\|A\|\}$ on which is defined a two-sided monogenic function equal to $\omega \mapsto G_\omega(A)$ on Γ. Then a two-sided monogenic function equal to $\omega \mapsto G_\omega(A)$ on Γ is defined on all of Ω. It is unique because Ω is connected and contains Γ – a compact subset of $\mathbb{R}^n$ cannot disconnect $\mathbb{R}^{n+1}$.

The complement $\gamma(A)$ of the domain Ω of $\omega \mapsto G_\omega(A)$ is called the *monogenic spectrum* of A. According to Lemma 2.4, $\gamma(A)$ is contained in the closed ball of radius $(1+\sqrt{2})\big(\sum_{j=1}^n \|A_j\|^2\big)^{1/2}$ about zero in $\mathbb{R}^n$, so it is compact by the Heine-Borel theorem. The following result was mentioned in [5, Lemma 3.13], but with a different definition of the spectrum.

2.6 Theorem. *Let A be an n-tuple of bounded operators acting on a nonzero Banach space X such that $\sigma\big(\langle A,\xi\rangle\big) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Then $\gamma(A)$ is a nonempty compact subset of $\mathbb{R}^n$.*

Proof. It only remains to show that $\gamma(A)$ is nonempty. The norms of the coefficients $W_{l_1\dots l_k}(\omega)$ of the expansion (5) decrease monotonically with $|\omega|$, so the function $\omega \mapsto G_\omega(A)$ is bounded and monogenic outside a ball. If $\gamma(A) = \emptyset$, then for each $x \in X$ and $\xi \in X'$, the function $\omega \mapsto \langle G_\omega(A)x, \xi\rangle$ is two-sided monogenic inside any ball, and so it is bounded and two-sided monogenic everywhere. By Liouville's Theorem [2, 12.3.11], it is a constant function. However, by the Hahn-Banach theorem we can obtain $x \in X$ and $\xi \in X'$ and $\omega_1, \omega_2 \in \mathbb{R}^{n+1}$ such that $\langle G_{\omega_1}(A)x, \xi\rangle \ne \langle G_{\omega_2}(A)x, \xi\rangle$, a contradiction. $\square$

2.7 Proposition. *Let A be an n-tuple of bounded operators acting on a Banach space X such that $\sigma\big(\langle A,\xi\rangle\big) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Then $\gamma(A) \subset \mathbb{R}^n$ is the complement in $\mathbb{R}^{n+1}$ of the set of all points $\omega \in \mathbb{R}^{n+1}$ at which the function*

$$(y + y_0e_0) \mapsto \operatorname{sgn}(y_0)^{n-1} \int_{S_1} (e_0 + is)\left(\langle yI - A, s\rangle - y_0 s\right)^{-n} d\mu(s)$$

is continuous in a neighbourhood of ω.

Proof. Suppose that the function is continuous in a neighbourhood $U \subset \mathbb{R}^{n+1}$ of $\omega \in \mathbb{R}^{n+1}$. By Lemma 2.5 and Painlevé's Theorem [2, Theorem 10.6, p. 64], the function

$$y + y_0 e_0 \mapsto \operatorname{sgn}(y_0)^{n-1} \int_{S_1} (e_0 + is)\left(\left(\langle yI - A, s\rangle - y_0 s\right)^{-n} x, \xi\right\rangle d\mu(s)$$

is two-sided monogenic for each $x \in X$ and $\xi \in X'$. The statement now follows from the equality

$$\begin{aligned}\int_{S_1} &(e_0 + is)\left(\left(\langle yI - A, s\rangle - y_0 s\right)^{-n} x, \xi\right\rangle d\mu(s) \\ &= \left\langle \left(\int_{S_1} (e_0 + is)\left(\langle yI - A, s\rangle - y_0 s\right)^{-n} d\mu(s)\right) x, \xi \right\rangle\end{aligned}$$

and the observation that an $\mathcal{L}_{(n)}(X_{(n)})$-valued function is left or right monogenic for the norm topology if and only if it is left or right monogenic for the weak operator topology. $\square$

As a consequence of Proposition 2.7, the set $\gamma(A)$ remains the same, if, in the definition of $\gamma(A)$, the term "two-sided monogenic" is replaced by either "left monogenic" or "right monogenic".

We have established the following representation for the Cauchy kernel $G_\omega(A)$, $\omega \in \mathbb{R}^{n+1} \setminus \gamma(A)$, of an n-tuple A of bounded linear operators on X with the property that $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. In the case $\omega \in \mathbb{R}^{n+1}$ and $\omega = y + y_0 e_0$ with $y \in \mathbb{R}^n$ and y_0 a nonzero real number, we have

(11)
$$G_\omega(A) = \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \operatorname{sgn}(y_0)^{n-1} \int_{S_1} (e_0 + is)\left(\langle yI - A, s\rangle - y_0 s\right)^{-n} d\mu(s).$$

If $\omega \in \mathbb{R}^n \setminus \gamma(A)$, then

$$\begin{aligned} G_\omega(A) &= \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \lim_{y_0 \to 0+} \int_{S_1} (e_0 + is)\left(\langle \omega I - A, s\rangle - y_0 s\right)^{-n} d\mu(s). \\ &= -\frac{(n-1)!}{2}\left(\frac{-i}{2\pi}\right)^n \lim_{y_0 \to 0-} \int_{S_1} (e_0 + is)\left(\langle \omega I - A, s\rangle - y_0 s\right)^{-n} d\mu(s). \end{aligned} \tag{12}$$

3. The Cauchy integral formula for an n-tuple of operators

Let A be an n-tuple of bounded operators acting on a Banach space X such that $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Let Ω be a bounded open neighbourhood of $\gamma(A)$ in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$ and exterior unit normal $n(\omega)$ defined for all $\omega \in \partial\Omega$. Let μ be the surface measure of Ω. Suppose that f is left monogenic in a neighbourhood of the closure $\overline{\Omega} = \Omega \cup \partial\Omega$ of Ω. Then we *define*

$$f(A) = \int_{\partial\Omega} G_\omega(A) n(\omega) f(\omega)\, d\mu(\omega) \tag{13}$$

Because $\omega \mapsto G_\omega(A)$ is right monogenic, the element $f(A)$ of $\mathcal{L}_{(n)}(X_{(n)})$ is defined independently of the set Ω with the properties mentioned above. This may be seen by taking $x \in X$ and $\xi \in X'$. Then by the properties of Bochner integrals

$$\langle f(A)x, \xi\rangle = \int_{\partial\Omega} \langle G_\omega(A)x, \xi\rangle n(\omega) f(\omega)\, d\mu(\omega)$$

and the $\mathbb{F}_{(n)}$-valued function $\omega \mapsto \langle G_\omega(A)x, \xi\rangle$ is two-sided monogenic off $\gamma(A)$. The analogue for monogenic functions of Cauchy's Theorem [2, Corollary 9.3] ensures that the open set Ω can be changed as long as the boundary of the set Ω does not cross $\gamma(A)$. Because this is true for all $x \in X$ and $\xi \in X'$, the Hahn-Banach theorem ensures that the values of the integrals (13) do not change when Ω is so modified.

Moreover, a similar argument shows that if $f : V \to \mathbb{C}$ is a function analytic in a neighbourhood V of $\gamma(A)$ in $\mathbb{R}^n$ and $\tilde{f}_1 : U_1 \to \mathbb{C}_{(n)}$ and $\tilde{f}_2 : U_2 \to \mathbb{C}_{(n)}$ are left monogenic functions defined in neigbourhoods U_1, U_2 of $\gamma(A)$ in $\mathbb{R}^{n+1}$ such that $\tilde{f}_1(x) = f(x)$ for all $x \in U_1 \cap V$ and $\tilde{f}_2(x) = f(x)$ for all $x \in U_2 \cap V$, then $\tilde{f}_1(A) = \tilde{f}_2(A)$. It therefore makes sense to define $f(A) = \tilde{f}_1(A)$. In Theorem 3.5 (iv), we show that $f(A)$ actually belongs to the closed linear subspace $\mathcal{L}(X)$ of the Banach module $\mathcal{L}_{(n)}(X_{(n)})$.

For any open subset U of $\mathbb{R}^{n+1}$, let $M(U, \mathbb{F}_{(n)})$ be the collection of all $\mathbb{F}_{(n)}$-valued functions which are left monogenic in U. It is a right $\mathbb{F}_{(n)}$-module. The space $M(U, \mathbb{F}_{(n)})$ is given the compact-open topology (uniform convergence on every compact subset of U). If K is a closed subset of $\mathbb{R}^n$, then $M(K, \mathbb{F}_{(n)})$ is the union of all spaces $M(U, \mathbb{F}_{(n)})$, as U ranges over the open sets in $\mathbb{R}^{n+1}$ containing K. The space $M(K, \mathbb{F}_{(n)})$ is equipped with the inductive limit topology.

Equipped with the C-K product [2, p. 113], $M(K, \mathbb{F}_{(n)})$ becomes a topological algebra and the closed linear subspace $M(K, \mathbb{F})$ of $M(K, \mathbb{F}_{(n)})$ consisting of left monogenic extensions of $\mathbb{F}$-valued functions on K is a commutative topological algebra. Then the topological algebra $M(K, \mathbb{F})$ is isomorphic, via monogenic extension, to the topological algebra $H(K, \mathbb{F})$ of $\mathbb{F}$-valued functions analytic in an open neighbourhood of K in $\mathbb{R}^n$ with pointwise multiplication. We write just $H(K)$ for $H(K, \mathbb{C})$. The induced topology on $H(K)$ is convergence of the left (or right) monogenic extensions on compact subsets of a neighbourhood of K in $\mathbb{R}^{n+1}$, rather than the usual topology of convergence on compact subsets of a neighbourhood of K in $\mathbb{R}^n$—formula (13) forces us into this somewhat unusual terminology.

We shall need a result on the approximation of a special class of $\mathbb{F}^{n+1}$-valued monogenic functions by monogenic polynomials in the same class. Let $f = \sum_{j=0}^n f_j e_j$ be an $\mathbb{F}^{n+1}$-valued function defined in an open subset U of $\mathbb{R}^{n+1}$. The equation $Df = 0$ implies that the one-form $\alpha = f_0 dx_0 - f_1 dx_1 - \cdots - f_n dx_n$ is closed in U. The left monogenic function f is called *conservative* if $\int_\gamma \alpha = 0$ for every closed contour γ in U, that is, α is exact in U.

Let L be a compact subset of $\mathbb{R}^{n+1}$. The closed linear subspace of $M(L, \mathbb{F}_{(n)})$ consisting of all conservative left monogenic functions defined in a neighbourhood of L in $\mathbb{R}^{n+1}$ and with values in the linear span $\mathbb{F}^{n+1}$ over $\mathbb{F}$ of the basis vectors $e_0, \ldots, e_n$ is denoted by $\mathcal{M}(L, \mathbb{F}^{n+1})$. Note that if L is the closure of a disjoint union of finitely many simply connected domains, then $\mathcal{M}(L, \mathbb{F}^{n+1}) = M(L, \mathbb{F}^{n+1})$.

3.1 Lemma. *Let L be a compact subset of $\mathbb{R}^{n+1}$ with connected complement. Then* *the linear space of all* $\mathbb{F}^{n+1}$*-valued left monogenic polynomials is dense in the space* $\mathcal{M}(L, \mathbb{F}^{n+1})$ *for the topology of uniform convergence on* L.

Proof. The result is a version of the Runge approximation theorem for left monogenic functions [2, Corollary 18.5]. We shall describe where the proof of [2, Theorem 18.4] needs to be adapted to the present context.

The topology on the space $\mathcal{M}(L, \mathbb{F}^{n+1})$ of uniform convergence on L is induced by the uniform norm of the space $C(L, \mathbb{F}^{n+1})$ of $\mathbb{F}^{n+1}$-valued continuous functions defined on the compact set L. According to the Riesz representation theorem, the dual space of $C(L, \mathbb{F}^{n+1})$ is identifiable with the space of all $\mathbb{F}^{n+1}$-valued Borel measures on L equipped with the total variation norm.

Let B be an open ball in $\mathbb{R}^{n+1}$ such that $L \subset B$. An argument analogous to the proof of [2, Theorem 18.4] works once we establish that every element of $\mathcal{M}(L, \mathbb{F}^{n+1})$ may be approximated uniformly on L by elements of $M(\overline{B}, \mathbb{F}^{n+1}) = \mathcal{M}(\overline{B}, \mathbb{F}^{n+1})$. By means of the usual Hahn-Banach theorem (rather than the left module version [2, Theorem 2.10]), it suffices to establish that every $\mathbb{F}^{n+1}$-valued measure which annihilates $M(\overline{B}, \mathbb{F}^{n+1})$ is also zero on $\mathcal{M}(L, \mathbb{F}^{n+1})$. The remainder of this proof is devoted to establishing this fact.

If μ is an $\mathbb{F}^{n+1}$-valued Borel measure on L, we set

$$\langle f, \mu \rangle = \int_L \langle f, d\mu \rangle = \sum_{j=0}^{n} \int_L f_j \, d\mu_j$$

for all functions $f = \sum_{j=0}^{n} f_j e_j$ belonging to $C(L, \mathbb{F}^{n+1})$. Suppose that μ annihilates $M(\overline{B}, \mathbb{F}^{n+1})$, that is, $\langle f, \mu \rangle = 0$ for all $f \in M(\overline{B}, \mathbb{F}^{n+1})$. Then for all $\omega \in \mathbb{R}^{n+1} \setminus \overline{B}$, the function G_ω belongs to $M(\overline{B}, \mathbb{F}^{n+1})$, so we have $\langle G_\omega, \mu \rangle = 0$. The function $\omega \mapsto \langle G_\omega, \mu \rangle$ is an $\mathbb{F}$-valued harmonic function defined in $\mathbb{R}^{n+1}$ off the support L of μ. Since $\mathbb{R}^{n+1} \setminus L$ is connected, unique continuation for harmonic functions implies that $\langle G_\omega, \mu \rangle = 0$ for all $\omega \in \mathbb{R}^{n+1} \setminus L$.

If we can represent any function f belonging to the space $\mathcal{M}(L, \mathbb{F}^{n+1})$ as

$$f(x) = \int_{\mathbb{R}^{n+1}} G_\omega(x) \phi(\omega) \, d\omega, \quad x \in L, \tag{14}$$

for a smooth *scalar valued* function ϕ with compact support in $\mathbb{R}^{n+1} \setminus L$, then by Fubini's theorem, we have

$$\begin{aligned} \langle f, \mu \rangle &= \int_L \left\langle \int_{\mathbb{R}^{n+1}} G_\omega(x) \phi(\omega) \, d\omega, d\mu(x) \right\rangle \\ &= \int_{\mathbb{R}^{n+1} \setminus L} \left(\int_L \langle G_\omega, d\mu \rangle \right) \phi(\omega) \, d\omega = 0. \end{aligned}$$

It remains to show that the representation (14) is valid for all $f \in \mathcal{M}(L, \mathbb{F}^{n+1})$.

A closed one-form α such that $\int_\gamma \alpha = 0$ for all closed contours γ in U is exact, so there exists a scalar valued function $F : U \mapsto \mathbb{C}$ such that $\alpha = dF$, that is, the equality $f = \overline{D}F$ holds. The function F is harmonic in U because $\Delta F = D\overline{D}F = Df = 0$ in U.

Let u be a smooth function with compact support in U and equal to F on the open neighbourhood Ω of L in $\mathbb{R}^{n+1}$. Let $w = \Delta u$. Because $u = F$ in Ω and F is harmonic, w vanishes in Ω and is supported in U.

If g denotes the fundamental solution of the Laplacian in $\mathbb{R}^{n+1}$, then $\Delta g = \delta$ in the sense of distributions and we have $u = g * w$. But $u = F$ in Ω, so $F(x) = g * w(x)$ for all $x \in \Omega$. From the identity $G_\omega(x) = (\overline{D}g)(\omega - x)$ for all $\omega, x \in \mathbb{R}^{n+1}$ with $\omega \neq x$, we have

$$f(x) = \overline{D}F(x) = -\int_{\mathbb{R}^{n+1}} G_\omega(x) w(\omega)\, d\omega, \quad x \in \Omega.$$

Hence, the representation (14) is valid with $\phi = -w$.

The remainder of the proof [2, Theorem 18.4] works in the present context, so that f may be approximated uniformly on L by elements g of $M(\mathbb{R}^{n+1}, \mathbb{F}^{n+1})$. The Taylor series of g converges uniformly on compact subsets of $\mathbb{R}^{n+1}$ [2, Section 11.5.2]. Comparison with the Taylor series of $t \mapsto g(tx)$ shows that g may be approximated in $M(\mathbb{R}^{n+1}, \mathbb{F}^{n+1})$ by $\mathbb{F}^{n+1}$-valued left monogenic polynomials. Alternatively, we can see this directly from the representation (14) by expanding G_ω in its Taylor series (4). $\square$

Remark. It is easily checked that a left monogenic function with values in $\mathbb{F}^{n+1}$ is automatically right monogenic.

The next statement would follow from the Stone-Weierstrass approximation theorem if $H(K)$ had the topology of uniform convergence on K. The point is that $H(K)$ has the topology, inherited from $M(K, \mathbb{F}_{(n)})$, of uniform convergence of monogenic extensions on compact subsets of $\mathbb{R}^{n+1}$.

3.2 Proposition. *Let K be a compact subset of $\mathbb{R}^n$. The linear space of all scalar valued polynomials is dense in $H(K)$.*

Proof. It suffices to prove the result for real valued functions $f \in H(K)$ defined in a neighbourhood of K, otherwise f can be decomposed into real and imaginary parts. Let U be a bounded open neighbourhood of K in $\mathbb{R}^{n+1}$ for which f has a left monogenic extension $\tilde{f}$ to U. According to [2, Theorem 11.3.4, Remark 11.2.7 (ii)], the left monogenic extension $\tilde{f}$ of f into $\mathbb{R}^{n+1}$ takes its values in the real linear subspace $\mathbb{R}^{n+1}$ of $\mathbb{R}_{(n)}$ spanned by $e_0, \dots, e_n$. The function $\omega \mapsto \overline{\tilde{f}(-\overline{\omega})}$ is left monogenic and the equality $\overline{\tilde{f}(-\overline{\omega})} = \tilde{f}(\omega)$ holds for all $\omega \in U$ by unique continuation from points of K.

Let L be a compact subset of U such that $\mathbb{R}^{n+1} \setminus L$ is connected and L is invariant under the mapping $J : \omega \mapsto -\overline{\omega}$. According to [2, Theorem 14.8], the open set U in which $\tilde{f}$ is monogenic may be chosen to be a J-invariant set in which every closed contour γ in U is homotopic to a closed contour in $U \cap \mathbb{R}^n$. Here we are allowing the possibility that U may not be a simply connected domain. Then every compact subset of U is contained in such a set L.

To check that $\tilde{f}$ is conservative in U, let γ be a closed contour in U and let γ' be a closed contour in $U \cap \mathbb{R}^n$ homotopic to γ. The one form α associated with $\tilde{f}$ is closed in U, so $\int_\gamma \alpha = \int_{\gamma'} \alpha = 0$, because $\alpha = f\,dx_0$ on $\gamma' \subset U \cap \mathbb{R}^n$. By Lemma 3.1, $\tilde{f}$ can be approximated uniformly on L by polynomials $p \in M(L, \mathbb{R}^{n+1})$ and so by polynomials $\omega \mapsto (p(\omega) + \overline{p(-\overline{\omega})})/2$. The coefficients of the expansion of p in left inner spherical monogenics lie in $\mathbb{R}^{n+1}$, so $p(\omega) + \overline{p(-\overline{\omega})} \in \mathbb{R}$ for all $\omega \in \mathbb{R}^n$. Hence the polynomial $\omega \mapsto (p(\omega) + \overline{p(-\overline{\omega})})/2$ is scalar valued on $\mathbb{R}^n$ and approximates $\tilde{f}$ uniformly on L. $\square$

The operation $f \mapsto f(A)$ defined on $H(\gamma(A))$ extends to analytic functions with values in a finite dimensional vector space V over $\mathbb{C}$ by application to the component functions of f. In particular, if $f : U \to \mathbb{C}_{(n)}$ is an analytic function defined on a neighbourhood U of $\gamma(A)$ in $\mathbb{R}^n$ and $f = \sum_S f_S e_S$ for the scalar component functions f_S defined for $S \subset \{1, \dots, n\}$, then $f(A) = \sum_S f_S(A) e_S$. If the term 'analytic' is replaced by 'C^∞', then this property is shared with the Weyl functional calculus, see [3].

The following statement follows from formula (13) and the estimate

$$\|f(A)\| \leq 2^{n/2} \mu(\partial\Omega) \sup_{\omega \in \partial\Omega} \|G_\omega(A)\| \sup_{\omega \in \partial\Omega} |f(\omega)|. \tag{15}$$

3.3 Proposition. *Let A be an n-tuple of bounded operators acting on a Banach space X. Suppose that $\sigma(\langle A, \xi \rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Then the mapping $f \mapsto f(A)$ is continuous from $M(\gamma(A), \mathbb{F}_{(n)})$ to $\mathcal{L}_{(n)}(X_{(n)})$.*

3.4 Proposition. *Let A be an n-tuple of bounded operators acting on a Banach space X such that $\sigma(\langle A, \xi \rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Suppose that $f : U \to \mathbb{C}_{(n)}$ is left monogenic in an open neighbourhood U in $\mathbb{R}^{n+1}$ of the closed unit ball of radius $(1 + \sqrt{2})\big(\sum_{j=1}^n \|A_j\|^2\big)^{1/2}$ about zero.*

If the Taylor series of f restricted to $U \cap \mathbb{R}^n$ is given by

$$f(x) = \sum_{k=0}^\infty \frac{1}{k!} \sum_{l_1=1}^n \cdots \sum_{l_k=1}^n a_{l_1 \dots l_k} x_{l_1} \cdots x_{l_k}, \tag{16}$$

with $a_{l_1 \dots l_k} \in \mathbb{C}_{(n)}$, then

$$f(A) = \sum_{k=0}^\infty \left(\sum_{(l_1, \dots, l_k)} V_{l_1 \dots l_k}(A) \right) a_{l_1 \dots l_k}. \tag{17}$$

Proof. Let Ω be an open set in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$ such that $\Omega \subset B_r(0) \subset U$ and Ω contains the closed unit ball of radius $(1 + \sqrt{2})\big(\sum_{j=1}^n \|A_j\|^2\big)^{1/2}$ in $\mathbb{R}^{n+1}$. The series

$$f(x) = \sum_{k=0}^\infty \frac{1}{k!} \sum_{l_1=1}^n \cdots \sum_{l_k=1}^n V_{l_1 \dots l_k}(x) a_{l_1 \dots l_k},$$

representing the left monogenic extension of (16), converges normally in Ω [2, 11.5.2], so

$$f(A) = \sum_{k=0}^\infty \sum_{(l_1, \dots, l_k)} \left(\int_{\partial\Omega} G_\omega(A) n(\omega) V_{l_1 \dots l_k}(\omega) \, d\mu(\omega) \right) a_{l_1 \dots l_k}.$$

It follows from the expansion (5) and formula (12.2) of [2, p. 86] that

$$\int_{\partial\Omega} G_\omega(A) n(\omega) V_{l_1 \dots l_k}(\omega) \, d\mu(\omega) = V_{l_1 \dots l_k}(A)$$

for all $l_1, \dots, l_k = 1, \dots, n$ and $k = 1, 2, \dots$. The equality (17) follows. $\square$

3.5 Theorem. *Let A be an n-tuple of bounded operators acting on a Banach space X such that $\sigma(\langle A,\xi\rangle)\subset\mathbb{R}$ for all $\xi\in\mathbb{R}^n$.*

(i) *Suppose that $k_1,\dots,k_n=0,1,2,\dots$, $k=k_1+\dots+k_n$ and $f(x)=x_1^{k_1}\cdots x_n^{k_n}$ for all $x=(x_1,\dots,x_n)\in\mathbb{R}^n$. Then*

$$f(A)=\frac{k_1!\cdots k_n!}{k!}\sum_{\pi}A_{\pi(1)}\cdots A_{\pi(k)},$$

where the sum is taken over every map π of the set $\{1,\dots,k\}$ into $\{1,\dots,n\}$ which assumes the value j exactly k_j times, for each $j=1,\dots,n$.

(ii) *Let $p:\mathbb{C}\to\mathbb{C}$ be a polynomial and $\zeta\in\mathbb{C}^n$. Set $f(z)=p(\langle z,\zeta\rangle)$, for all $z\in\mathbb{C}^n$. Then $f(A)=p(\langle A,\zeta\rangle)$.*

(iii) *Let Ω be an open set in $\mathbb{R}^{n+1}$ containing $\gamma(A)$ with a smooth boundary $\partial\Omega$. Then for all $\omega\notin\overline{\Omega}$,*

$$G_\omega(A)=\int_{\partial\Omega}G_\zeta(A)n(\zeta)G_\omega(\zeta)\,d\mu(\zeta).$$

(iv) *Suppose that U is an open neighbourhood of $\gamma(A)$ in $\mathbb{R}^n$ and $f:U\to\mathbb{C}$ is an analytic function. Then $f(A)\in\mathcal{L}(X)$.*

Proof. (i) Let Γ be the set of all k-tuples $(l_1,\dots,l_k)$ in $\{1,\dots,n\}^k$ for which j appears exactly k_j times, for each $j=1,\dots,n$. Let $a_\gamma=k_1!\cdots k_n!$ for all $\gamma\in\Gamma$ and $a_\gamma=0$ for all $\gamma\in\{1,\dots,n\}^k\backslash\Gamma$. Then $x_1^{k_1}\cdots x_n^{k_n}=\frac{1}{k!}\sum_{l_1=1}^n\cdots\sum_{l_k=1}^n a_{(l_1,\dots,l_k)}x_{l_1}\cdots x_{l_k}$, so by the proposition above,

$$f(A)=\sum_{(l_1,\dots,l_k)}a_{(l_1\dots l_k)}V_{l_1\dots l_k}(A)=\frac{k_1!\cdots k_n!}{k!}\sum_{\pi}A_{\pi(1)}\cdots A_{\pi(k)}.$$

Statement (ii) follows from (i) because only symmetric products of the $\langle A_j\rangle$ appear in both $f(A)$ and $p(\langle A,\zeta\rangle)$.

(iii) On appealing to equations (4), (5), the equality follows directly from Proposition 3.3 for all $\omega\notin\overline{\Omega}$ such that $|\omega|>(1+\sqrt{2})\|A\|$. Both sides of the equation are right monogenic in ω in the complement of the set $\overline{\Omega}$, so equality follows there by unique continuation.

(iv) According to (i), $p(A)\in\mathcal{L}(X)$ for any scalar valued polynomial p on $\mathbb{R}^n$. By Proposition 3.2, there exists an open neighbourhood V of U in $\mathbb{R}^{n+1}$ such that the left monogenic extension $\tilde{f}$ of f can be approximated on compact subsets of V by monogenic extensions of scalar polynomials on $\mathbb{R}^n$. An appeal to Proposition 3.3 shows that $f(A)$ belongs to the closed linear subspace $\mathcal{L}(X)$ of $\mathcal{L}_{(n)}(X_{(n)})$. $\square$

As follows from [1], the Weyl functional calculus $\mathcal{W}_A$ for an n-tuple A of bounded operators acting on a Banach space X is determined by the following two properties:

a) $\mathcal{W}_A:C^\infty(\mathbb{R}^n)\to\mathcal{L}(X)$ is a continuous linear map for the operator norm;
b) $\mathcal{W}_A(p(\langle\cdot,\xi\rangle))=p(\langle A,\xi\rangle)$ for every polynomial $p:\mathbb{R}\to\mathbb{R}$ and $\xi\in\mathbb{R}^n$.

The Paley-Wiener Theorem ensures that the inverse Fourier transform $(\mathcal{W}_A)\check{}$ of $\mathcal{W}_A$ extends to an entire analytic function on $\mathbb{C}^n$ satisfying an exponential bound and b) guarantees that that $(\mathcal{W}_A)\check{}(\xi)=(2\pi)^{-n/2}e^{i\langle A,\xi\rangle}$ for all $\xi\in\mathbb{R}^n$. Hence $\mathcal{W}_A=(2\pi)^{-n/2}(e^{i\langle A,\xi\rangle})\hat{}$. In particular, $\sigma(\langle A,\xi\rangle)\subset\mathbb{R}$ for all $\xi\in\mathbb{R}^n$ (see, for example, [9, Corollary 7.5]).

The analogous statement for the monogenic functional calculus follows.

3.6 Theorem. *Let A be an n-tuple of bounded linear operators acting on a Banach space X. Suppose that there exists a compact subset K of $\mathbb{R}^n$ and a map T such that*

a) $T : H(K) \to \mathcal{L}(X)$ *is a continuous linear map;*
b) $T(p(\langle \cdot , \xi\rangle)) = p(\langle A, \xi\rangle)$ *for every polynomial* $p : \mathbb{R} \to \mathbb{R}$ *and* $\xi \in \mathbb{R}^n$.

Then $\sigma(\langle A, \xi\rangle)$ is real for each $\xi \in \mathbb{R}^n$, $\gamma(A) \subseteq K$ and $T(f) = f(A)$ for every $f \in H(K)$.

Proof. Denote the tensor product $T \otimes I_{(n)}$ of T with the identity $I_{(n)}$ on $\mathbb{F}_{(n)}$ by T again and define $T : M(K, \mathbb{F}_{(n)}) \to \mathcal{L}_{(n)}(X_{(n)})$ by $T(f) = T(f \restriction U)$, $f \in M(K, \mathbb{F}_{(n)})$, for an open neighbourhood U of K in $\mathbb{R}^n$ in which f is defined.

Let $\xi \in \mathbb{R}^n$ and $\langle K, \xi\rangle := \{\langle x, \xi\rangle : x \in K\} \subset \mathbb{R}$. For all $\lambda \in \mathbb{C} \setminus \langle K, \xi\rangle$, the function $x \mapsto (\lambda - \langle x, \xi\rangle)^{-1}$ belongs to $H(K)$ and the function $\lambda \mapsto (\lambda - \langle \cdot , \xi\rangle)^{-1}$ is an $H(K)$-valued analytic function on $\mathbb{C} \setminus \langle K, \xi\rangle$, so $\int_\Gamma (\lambda - \langle \cdot , \xi\rangle)^{-1}\, d\lambda = 0$ in $H(K)$ for all closed contours Γ contained in $\mathbb{C} \setminus \langle K, \xi\rangle$. The integral converges as a Bochner integral, so that

$$\int_\Gamma T\big((\lambda - \langle \cdot , \xi\rangle)^{-1}\big)\, d\lambda = T \int_\Gamma (\lambda - \langle \cdot , \xi\rangle)^{-1}\, d\lambda = 0.$$

By Morera's Theorem, $\lambda \mapsto T\big((\lambda - \langle \cdot , \xi\rangle)^{-1}\big)$ is an $\mathcal{L}(X)$-valued analytic function defined in $\mathbb{C} \setminus \langle K, \xi\rangle$. By b) and the continuity of T, the equality

$$(\lambda - \langle A, \xi\rangle)^{-1} = T\big((\lambda - \langle \cdot , \xi\rangle)^{-1}\big)$$

holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| > \sup |\langle K, \xi\rangle|$. It follows that the resolvent set of the operator $\langle A, \xi\rangle$ contains the set $\mathbb{C} \setminus \langle K, \xi\rangle$, that is, $\sigma(\langle A, \xi\rangle) \subseteq \langle K, \xi\rangle \subset \mathbb{R}$.

As in the proof of [1, Theorem 2.4], property b) and the continuity of T on $H(K)$ guarantee that $T(f)$ is equal to (17) for all complex valued analytic functions f with a power series given by (16) in an open neighbourhood of K with $a_{l_1 \dots l_k} \in \mathbb{C}$.

Let $R > (1 + \sqrt{2})\|A\|$ be so large that K is contained in the open ball $B_R(0)$ of radius R in $\mathbb{R}^{n+1}$. According to equations (4) and (5), it follows that $G_\omega(A) = T(G_\omega)$ for all $\omega \in \mathbb{R}^{n+1}$ with $|\omega| \geq R$.

Now the function $\omega \mapsto G_\omega$ is monogenic from $\mathbb{R}^{n+1} \setminus K$ into $M(K, \mathbb{F}_{(n)})$, because for each $\alpha \in \mathbb{R}^{n+1} \setminus K$ there exist disjoint open sets U and V in $\mathbb{R}^{n+1}$ such that $\alpha \in U$, $K \subset V$ and $\nabla_\omega G_\omega(x)$ is uniformly bounded and uniformly continuous for all $\omega \in U$ and $x \in V$. Consequently, $\omega \mapsto T(G_\omega)$ is monogenic from $\mathbb{R}^{n+1} \setminus K$ into $\mathcal{L}_{(n)}(X_{(n)})$ and the function defined by equation (5) has a monogenic extension off K, that is, $\gamma(A) \subseteq K$ and $G_\omega(A) = T(G_\omega)$ for all $\omega \in \mathbb{R}^{n+1} \setminus K$.

Let $f \in H(K)$ and suppose that $\tilde{f}$ is a left monogenic extension of f to an open neighbourhood of K in $\mathbb{R}^{n+1}$. We may suppose further that $\tilde{f}$ is defined in a neighbourhood of the closure $\overline{\Omega}$ of a bounded open set $\Omega \supset K$ in $\mathbb{R}^{n+1}$, for which the Cauchy integral formula (2) holds for $\tilde{f}$. Then by formula (2), we have

$$\begin{aligned} T(f) &= T\left(\int_{\partial\Omega} G_\omega(\,\cdot\,) n(\omega) \tilde{f}(\omega)\, d\mu(\omega)\right) \\ &= \int_{\partial\Omega} T(G_\omega) n(\omega) \tilde{f}(\omega)\, d\mu(\omega) \\ &= \int_{\partial\Omega} G_\omega(A) n(\omega) \tilde{f}(\omega)\, d\mu(\omega) \\ &= f(A). \end{aligned}$$

□

The monogenic functional calculus, when it exists, is therefore the *richest* analytic functional calculus satisfying b) that can be defined over a compact subset of $\mathbb{R}^n$. Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is an affine transformation given by $(Lx)_k = \sum_{j=1}^n c_{kj}x_j + d_k$ for all $x \in \mathbb{R}^n$ and $k = 1, \dots, m$. The m-tuple LA is given by $(LA)_k = \sum_{j=1}^n c_{kj}A_j + d_k I$ and $Lf = f \circ L$ for a function defined on a subset of $\mathbb{R}^m$.

The following properties of the Weyl functional calculus [1, Theorem 2.9], suitably interpreted, are also enjoyed by the monogenic functional calculus.

Let $\pi_j : \mathbb{R}^n \to \mathbb{R}$ be the j'th projection $\pi_j(x) = x_j$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

3.7 Theorem. *Let A be an n-tuple of bounded operators acting on a Banach space X such that $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$.*

(a) **Affine covariance:** *if $L : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map, then $\gamma(LA) \subseteq L\gamma(A)$ and for any function f analytic in a neighbourhood in $\mathbb{R}^m$ of $L\gamma(A)$, the equality $f(LA) = (f \circ L)(A)$ holds.*

(b) **Consistency with the one-dimensional calculus:** *if $g : \mathbb{R} \to \mathbb{C}$ is analytic in a neighbourhood of the projection $\pi_1\gamma(A)$ of $\gamma(A)$ onto the first ordinate, and $f = g \circ \pi_1$, then $f(A) = g(A_1)$. We also have consistency with the k-dimensional calculus, $1 < k < n$.*

(c) **Continuity:** *The mapping $(T, f) \mapsto f(T)$ is continuous for $T = \sum_{j=1}^n T_j e_j$ from $\mathcal{L}_{(n)}(X_{(n)}) \times M(\mathbb{R}^{n+1}, \mathbb{C}_{(n+1)})$ to $\mathcal{L}_{(n)}(X_{(n)})$ and from $\mathcal{L}(X) \times H(\mathbb{R}^n)$ to $\mathcal{L}(X)$.*

(d) **Covariance of the Range:** *If T is an invertible continuous linear map on X and TAT^{-1} denotes the n-tuple with entries TA_jT^{-1} for $j = 1, \dots n$, then $\gamma(TAT^{-1}) = \gamma(A)$ and $f(TAT^{-1}) = Tf(A)T^{-1}$ for all functions f analytic in a neighbourhood of $\gamma(A)$ in $\mathbb{R}^n$.*

Proof. (a) The mapping $f \mapsto f \circ L(A)$ defined for all $f \in H(L\gamma(A))$ satisfies the conditions of Theorem 3.5 for the m-tuple LA, so $\gamma(LA) \subseteq L\gamma(A)$ and $f \circ L(A) = f(LA)$ for all $f \in H(L\gamma(A))$.

(b) Set $L = \pi_1$ and apply (a).

(c) Let $A = \sum_{j=1}^n A_j e_j$ and choose $R > (\sqrt{2}+1)\|A\|$. Let U_R be the intersection of the open unit ball of radius R in $\mathcal{L}_{(n)}(X_{(n)})$ with the subspace $\{\sum_{j=1}^n S_j e_j : S_j \in \mathcal{L}(X)\}$. According to equation (5), the mapping $(\omega, T) \mapsto G_\omega(T)$ is continuous from $\mathbb{R}^{n+1} \times U_R$ into $\mathcal{L}_{(n)}(X_{(n)})$ for all $|\omega| > R$.

Let $B_r(0)$ be the open ball of radius $r > R$ in $\mathbb{R}^{n+1}$. Then from (16) we have

$$
\begin{aligned}
\|f_1(T_1) - f_2(T_2)\| &\le \int_{\partial B_r(0)} \|G_\omega(T_1)n(\omega)f_1(\omega) - G_\omega(T_2)n(\omega)f_2(\omega)\|\, d\mu(\omega) \\
&\le 2^{n/2}\mu(\partial B_r(0))\Big(\sup_{\omega\in\partial B_r(0)} \|G_\omega(T_1) - G_\omega(T_2)\| \max\{ \sup_{\omega\in\partial B_r(0)} |f_1(\omega)|, \sup_{\omega\in\partial B_r(0)} |f_2(\omega)|\} \\
&\qquad + \sup_{\omega\in\partial B_r(0)} \|f_1(\omega) - f_2(\omega)\| \max\{ \sup_{\omega\in\partial B_r(0)} |G_\omega(T_1)|, \sup_{\omega\in\partial B_r(0)} |G_\omega(T_2)|\}\Big)
\end{aligned}
$$

for all $T_1, T_2 \in U_R$. The spaces $M(\mathbb{R}^n, \mathbb{C}_{(n)})$ and $M(\mathbb{R}^{n+1}, \mathbb{C}_{(n)})$ are isomorphic [2, Corollary 14.6]. Combined with Corollary 3.4 (iii), this completes the proof of (c).

(d) follows from the identity $G_\omega(TAT^{-1}) = TG_\omega(A)T^{-1}$ valid from (5) for $|\omega|$ large enough. Then $\gamma(TAT^{-1}) \subseteq \gamma(A)$. The reverse inclusion comes from writing $G_\omega(A) = T^{-1}G_\omega(TAT^{-1})T$ for $|\omega|$ large enough. $\square$

The inclusion in (a) may be proper, as may be seen from the equality $\gamma(\pi_1 A) = \sigma(A_1)$. The next assertion shows that property b) of Theorem 3.5 can be extended from polynomials to analytic functions.

3.8 Proposition. *Let A be an n-tuple of bounded operators acting on a Banach space X such that $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$. Let $\zeta \in \mathbb{C}^n$ and set $\langle \gamma(A), \zeta\rangle := \{\langle x, \zeta\rangle : x \in \gamma(A)\}$. Then $\sigma(\langle A, \zeta\rangle) \subseteq \langle \gamma(A), \zeta\rangle$.*

Suppose that $U \subset \mathbb{C}$ is a bounded open set with connected complement containing the set $\langle \gamma(A), \zeta\rangle$. Suppose that $g : U \to \mathbb{C}$ is analytic. Set $f(z) = g(\langle z, \zeta\rangle)$, for all $z \in \mathbb{C}^n$ such that $\langle z, \zeta\rangle \in U$. Then $f(A) = g(\langle A, \zeta\rangle)$.

Proof. The proof of the inclusion $\sigma(\langle A, \zeta\rangle) \subseteq \langle \gamma(A), \zeta\rangle$ follows the argument of Theorem 3.6. By Runge's Theorem for functions of a single complex variable, g can be approximated uniformly on compact subsets of U by polynomials $\langle p_n\rangle_n$ on $\mathbb{C}$. Hence f can be approximated by $\{p_n \circ \zeta\}_n$ uniformly on sets $\langle \cdot, \zeta\rangle^{-1} K$ for $K \subset U$ compact.

Now take K to be a compact subset of U whose interior K° contains $\langle \gamma(A), \zeta\rangle$. Let V be an open subset of $\mathbb{R}^{n+1}$ such that $\gamma(A) \subset V$ and $\overline{V}$ is contained in $\langle \cdot, \zeta\rangle^{-1} K^\circ$. Then f can be approximated uniformly on $\overline{V}$ by functions $\{p_n \circ \zeta\}_n$ with $\{p_n\}_n$ a sequence of polynomials on $\mathbb{C}$. The equality $f(A) = g(\langle A, \zeta\rangle)$ is a consequence of Corollary 3.4 (ii) and Proposition 3.3. $\square$

In the case that A is a commuting n-tuple of bounded operators acting on a Banach space X, it is shown in [9, Corollary 3.4] that for $\lambda \in \mathbb{R}^n$, the operator $\sum_{j=1}^n (\lambda_j I - A_j)^2$ is invertible in $\mathcal{L}(X)$ if and only if $\sum_{j=1}^n (\lambda_j I - A_j) e_j$ is an invertible element of $\mathcal{L}_{(n)}(X_{(n)})$.

The following result was announced in [5, Lemma 3.2, Corollary 3.17] for commuting selfadjoint operators.

3.9 Theorem. *Let A be a commuting n-tuple of bounded operators acting on a Banach space X such that $\sigma(A_j) \subset \mathbb{R}$ for all $j = 1, \ldots, n$.*

Then $\gamma(A)$ is the complement in $\mathbb{R}^n$ of the set of all $\lambda \in \mathbb{R}^n$ for which the operator $\sum_{j=1}^n (\lambda_j I - A_j)^2$ is invertible in $\mathcal{L}(X)$.

Moreover, $\gamma(A)$ is the Taylor spectrum of A. If the complex valued function f is analytic in a neighbourhood of $\gamma(A)$ in $\mathbb{R}^n$, then the operator $f(A) \in \mathcal{L}(X)$ coincides with the operator obtained from Taylor's functional calculus [13].

Proof. Let $\rho_{(n)}(A)$ be the set of all $\lambda \in \mathbb{R}^{n+1}$ such that either $\lambda_0 \neq 0$ or if $\lambda_0 = 0$, then the operator $\sum_{j=1}^n (\lambda_j I - A_j)^2$ is invertible in $\mathcal{L}(X)$. Set $\sigma_{(n)}(A) = \mathbb{R}^n \setminus \rho_{(n)}(A)$.

Each of the operators A_j has real spectrum, so $\sigma(\langle A, \xi\rangle) \subset \mathbb{R}$ [9, Proposition 10.1]. Suppose first that n is odd. In this case, the Cauchy kernel $G_\omega(A)$ for A can be written down directly. The element

$$1/\sigma_n |\omega I - A|^{-n-1} (\overline{\omega I - A}) \tag{18}$$

of $\mathcal{L}_{(n)}(X_{(n)})$ has the power series expansion (5) for $|\omega|$ large enough. Here

$$|\omega I - A|^{-m} = \left(\left(\omega_0^2 I + \sum_{j=1}^n (\omega_j I - A_j)^2\right)^{-1} \right)^{m/2}$$

for an even integer m and $\overline{\omega I - A} = \omega_0 I - \sum_{j=1}^n (\omega_j I - A_j) e_j$.

The operator $\omega_0^2 I + \sum_{j=1}^n (\omega_j I - A_j)^2$ is invertible for each $\omega \in \rho_{(n)}(A)$ because A_j has real spectrum for each $j = 1, \dots, n$ [9, Proposition 10.1]. As stated in [9, Example 5.4], it is easily verified that the function $\omega \mapsto 1/\sigma_n |\omega I - A|^{-n-1} \overline{\omega I - A}$, $\omega \in \rho_{(n)}(A)$, is monogenic in $\mathcal{L}_{(n)}(X_{(n)})$. Hence $\gamma(A) \subseteq \sigma_{(n)}(A)$ and $G_\omega(A)$ is given by the expression (18) for all $\omega \in \rho_{(n)}(A)$.

Now suppose that $x \in \mathbb{R}^n \setminus \gamma(A)$. Then $\omega \mapsto G_\omega(A)$ is norm-continuous in a neighbourhood U of x in $\mathbb{R}^{n+1}$ and it is given by (18) for $\omega_0 \neq 0$. The function

$$\omega \mapsto \sigma_n |\omega I - A|^{n-1} G_\omega(A)$$

is also continuous in U. For $\omega_0 \neq 0$, $\sigma_n |\omega I - A|^{n-1} G_\omega(A) = |\omega I - A|^{-2} \overline{\omega I - A}$ and the equality $(\omega I - A)^{-1} = |\omega I - A|^{-2} \overline{\omega I - A}$ holds in $\mathcal{L}_{(n)}(X_{(n)})$, so the $\mathcal{L}_{(n)}(X_{(n)})$-valued function $\omega \mapsto (\omega I - A)^{-1}$ has a continuous extension J from $U \setminus \mathbb{R}^n$ to U. Continuity ensures that the equalities $J(\omega)(\omega I - A) = (\omega I - A) J(\omega) = I e_0$ hold for all $\omega \in U$, so $xI - A$ is invertible in $\mathcal{L}_{(n)}(X_{(n)})$, that is, $x \in \rho_{(n)}(A)$. This completes the proof that $\gamma(A) = \sigma_{(n)}(A)$ for the case in which n is odd.

For n even, we have to define $(\omega_0^2 I + \sum_{j=1}^n (\omega_j I - A_j)^2)^{-(n+1)/2}$ in some fashion. A convenient way is to use the plane wave decomposition formula (6) to define $G_\omega(A)$. To identify the set $\gamma(A)$, we use Taylor's functional calculus [13].

That $\sigma_{(n)}(A)$ is the Taylor spectrum of A is proved in [10, Theorem 1]. A continuous linear map $T : H(\sigma_{(n)}(A)) \to \mathcal{L}(X)$ such that $T(p) = p(A)$ for all polynomials $p : \mathbb{R}^n \to \mathbb{C}$ is constructed in [13].

The function $\omega \mapsto |\omega - \cdot|^{-n-1}$ is analytic from $\rho_{(n)}(A)$ into $H(\sigma_{(n)}(A))$, so on application of the mapping T, it follows that $\omega \mapsto T\big(|\omega - \cdot|^{-n-1}\big)$ is analytic from $\rho_{(n)}(A)$ into $\mathcal{L}(X)$. The analytic functional calculus ensures that the function

$$\omega \mapsto 1/\sigma_n T\big(|\omega - \cdot|^{-n-1}\big) \overline{\omega I - A} \tag{19}$$

has the power series expansion (5) for $|\omega|$ large enough and is monogenic in $\rho_{(n)}(A)$. Hence $\gamma(A) \subseteq \sigma_{(n)}(A)$ and $G_\omega(A)$ is given by formula (19) for all $\omega \in \rho_{(n)}(A)$. The proof that $\sigma_{(n)}(A) \subseteq \gamma(A)$ follows the case for n odd.

Equality of the monogenic functional calculus and Taylor's functional calculus T [13] is a consequence of Theorem 3.6. □

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