THE SQUARE ROOT PROBLEM OF KATO IN ONE DIMENSION, AND FIRST ORDER ELLIPTIC SYSTEMS

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Abstract. Our aim is to prove that all non–degenerate second order elliptic operators L with Dirichlet, Neumann, or other two–point boundary conditions on an interval Ω satisfy the estimates $\| L^{1/2}u \|_p \approx \left\| \frac{du}{dx} \right\|_p + \| u \|_p$ when $1 < p < \infty$.

The study of the square root of L reduces to proving that the holomorphic functional calculus of a related first order elliptic system is bounded. The interpolation theory which is developed in [AM^cN] is a major tool in proving the L_2 theory. The L_p results follow once we have derived bounds on the Green's functions of the systems.

§1. Introduction.

Let Ω denote an interval of $\mathbb{R}$, and let $\mathcal{V}$ denote a subspace of the Sobolev space $H^1(\Omega)$ such that $\overset{o}{H}{}^1(\Omega) \subset \mathcal{V} \subset H^1(\Omega)$. Define the second order elliptic operator L in $L_2(\Omega)$ by

$$Lu(x) \;=\; a(x)\left\{ -\tfrac{d}{dx}\{ b(x)\tfrac{du}{dx}(x) + \alpha(x)u(x) \} \;+\; \beta(x)\tfrac{du}{dx}(x) + \gamma(x)u(x) \right\}$$

with domain arising from a sesquilinear form on $\mathcal{V} \times \mathcal{V}$. For example, $\mathcal{D}(L)$ could be $\{ u \in \overset{o}{H}{}^1(\Omega) : Lu \in L_2(\Omega) \}$ or $\{ u \in H^1(\Omega) : Lu \in L_2(\Omega)$ and $(b\frac{du}{dx} + \alpha u)|_{\partial\Omega} = 0 \}$.

Here $a, b, \alpha, \beta, \gamma \in L_\infty(\Omega)$, and there exists $\kappa > 0$ such that for almost all $x \in \Omega$,

$$\mathrm{Re}\; \begin{bmatrix} \overline{\zeta_1} & \overline{\zeta_2} \end{bmatrix} \begin{bmatrix} b(x) & \alpha(x) \\ \beta(x) & \gamma(x) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \;\geq \kappa\, |\zeta|^2 \;, \quad \zeta \in \mathbb{C}^2 ,$$

and $\mathrm{Re}\; a(x) \geq \kappa$.

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Under these conditions, L is a one–one operator of type 2ω in $L_2(\Omega)$ for some $\omega < \frac{\pi}{2}$. Such an operator has a unique square root $L^{\frac{1}{2}}$ which is an operator of type ω with the property that $(L^{\frac{1}{2}})^2 = L$.

One aim is to prove that $\mathcal{D}(L^{\frac{1}{2}}) = \mathcal{V}$ and $\| L^{\frac{1}{2}} u \|_2 \approx \| \frac{du}{dx} \|_2 + \| u \|_2$ for all $u \in \mathcal{V}$. In the case when $a = 1$, this was derived previously by Auscher and Tchamitchian by constructing suitable wavelets on Ω [AT1]. We achieve it by proving that a related first order system T has a bounded holomorphic functional calculus in $L_2(\Omega)$.

We then derive bounds on the Green's function of T and use them, together with a result of Duong and Robinson [DR], to prove the corresponding L_p estimates when $1 < p < \infty$. That is, we show that $\| L^{\frac{1}{2}} u \|_p \approx \| \frac{du}{dx} \|_p + \| u \|_p$ for all $u \in \mathcal{V} \cap W^1_p(\Omega)$.

We also show that L has a bounded H_∞ functional calculus in $L_2(\Omega)$, and indeed in $L_p(\Omega)$ when $1 < p < \infty$.

As a first step, we show how first order systems can be used to re-derive various results about the homogeneous operator $Lu = -a\frac{d}{dx}(b\frac{du}{dx})$ on $\mathbb{R}$. In particular we prove the homogeneous estimate $||L^{\frac{1}{2}}u||_2 \approx ||\frac{du}{dx}||_2$ for all $u \in H^1(\mathbb{R})$, which was first proved by Kenig and Meyer [KM], and also the corresponding L_p estimate which is due to Auscher and Tchamitchian [AT2]. They obtained it by proving that $(-a\frac{d}{dx}b\frac{d}{dx})^{\frac{1}{2}}(\frac{d}{dx})^{-1}$ is a Calderón–Zygmund operator [AT2], whereas we apply Calderón–Zygmund theory to functions of the system T.

Note that for the inhomogeneous operator L on a bounded interval, we do not claim that there is such a representation in terms of a Calderón–Zygmund operator.

In the case when $a = 1$, the L_2 estimate was first obtained when $\Omega = \mathbb{R}$ by Coifman, M$^{\mathrm{c}}$Intosh and Meyer in conjunction with the proof of the L_2 boundedness of the Cauchy integral on a Lipschitz curve [CM$^{\mathrm{c}}$M], which is essentially the case $a = b$.

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The space of all bounded linear transformations from a Banach space $\mathcal{X}$ to a Banach space $\mathcal{Y}$ is denoted by $\mathcal{L}(\mathcal{X},\mathcal{Y})$, while the Banach algebra of all bounded linear operators on $\mathcal{X}$ is denoted by $\mathcal{L}(\mathcal{X})$. We often write a statement such as "$\mathcal{X} \subset \mathcal{Y}$ with $\| u \|_{\mathcal{Y}} \leq c \| u \|_{\mathcal{X}}$" concerning two normed spaces $\mathcal{X}$ and $\mathcal{Y}$ to mean that "$\mathcal{X} \subset \mathcal{Y}$ and there exists a constant c such that $\| u \|_{\mathcal{Y}} \leq c \| u \|_{\mathcal{X}}$ for all $u \in \mathcal{X}$".

Contents.

§2. Operators of type ω.

Here is what we need to know about operators of type ω and their functional calculi. See our previous paper [AM^cN] for details.

For $0 \leq \omega < \mu < \pi$, define the closed and open sectors

$$\begin{aligned} S_{\omega+} &= \{ \zeta \in \mathbb{C} : |\arg \zeta| \leq \omega \} \quad , \quad S_{\omega-} = S_{\omega+} , \\ S^0_{\mu+} &= \{ \zeta \in \mathbb{C} : \zeta \neq 0, |\arg \zeta| < \mu \} , \ S^0_{\mu-} = -S^0_{\mu+} , \end{aligned}$$

and, if $0 \leq \omega < \mu < \frac{\pi}{2}$, define the double sectors

$$S_\omega = S_{\omega+} \cup S_{\omega-} \quad \text{and} \quad S^0_\mu = S^0_{\mu+} \cup S^0_{\mu-} .$$

As usual, $H_\infty(S^0_{\mu+})$ denotes the Banach algebra consisting of all bounded holomorphic functions defined on $S^0_{\mu+}$, while $H_\infty(S^0_\mu)$ is defined similarly. In each case the norm is given by $\| f \|_\infty = \sup |f(z)|$.

A closed operator T in a Banach space $\mathcal{X}$ is said to be of *type* ω (or of *type* S_ω) if its spectrum $\sigma(T) \subset S_{\omega+}$ (or $\sigma(T) \subset S_\omega$) and for each $\mu > \omega$ there exists C_μ such that

$$\| (T - \zeta I)^{-1} \| \leq C_\mu |\zeta|^{-1} , \quad \zeta \notin S_{\mu+} \quad (\text{or } \zeta \notin S_\mu) .$$

In this paper $\mathcal{X}$ is always of the form $L_p(\Omega, \mathbb{C}^N)$ where Ω is a real interval and $1 < p < \infty$, in which case every one–one operator of type ω (or type S_ω) has dense domain $\mathcal{D}(T)$ and dense range $\mathcal{R}(T)$ in $\mathcal{X}$ [CDM^cY].

A one–one operator T of type ω (or type S_ω) in $\mathcal{X}$ has a *bounded* $H_\infty(S^0_{\mu+})$ (or $H_\infty(S^0_\mu)$) *functional calculus* in $\mathcal{X}$ provided there is a bounded algebra homomorphism $f \mapsto f(T)$ from $H_\infty(S^0_{\mu+})$ (or $H_\infty(S^0_\mu)$) to $\mathcal{L}(\mathcal{X})$ which satisfies $R_\lambda(T) = (T-\lambda I)^{-1}$ when $\lambda \notin S^0_{\mu+}$ (or S^0_μ), where $R_\lambda(\zeta) = (\zeta-\lambda)^{-1}$. If such a functional calculus exists, then it is unique, provided $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are dense in $\mathcal{X}$.

The important thing is the boundedness of the mapping $f \mapsto f(T)$, or in other words, the estimate $\| f(T) \| \le c_\mu \| f \|_\infty$.

In the case when $\mathcal{X}$ is a Hilbert space, then the existence of a bounded functional calculus is independent of μ. That is, if T is a one–one operator of type ω in a Hilbert space $\mathcal{H}$ which has a bounded $H_\infty(S^0_{\mu+})$ functional calculus in $\mathcal{H}$ for some $\mu > \omega$, then it has a bounded $H_\infty(S^0_{\nu+})$ functional calculus in $\mathcal{H}$ for all $\nu > \omega$ [M$^{\rm c}$]. In this case, we just say that T has a bounded H_∞ functional calculus. The same applies to operators of type S_ω.

There are close connections between the boundedness of the H_∞ functional calculus of an operator T in a Hilbert space, quadratic estimates associated with T, and the interpolation of the domains of its fractional powers. See [AM$^{\rm c}$N] for more details, including a proof of the following result which is basic to our purposes.

Theorem 2.1. *Let S and T be one–one operators of type ω (or type S_ω) in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and suppose there exist $\mathcal{E} \in \mathcal{L}(\mathcal{K},\mathcal{H})$ and $\mathcal{F} \in \mathcal{L}(\mathcal{H},\mathcal{K})$ such that $\mathcal{F}\mathcal{E} = I_{\mathcal{K}}$,*

$$\begin{aligned}
&\mathcal{E}\mathcal{D}(T) \subset \mathcal{D}(S) \text{ with } \| S\mathcal{E}u \| \le c\| Tu \| \text{, and} \\
&\qquad\qquad \mathcal{E}\mathcal{R}(T) \subset \mathcal{R}(S) \text{ with } \left\| S^{-1}\mathcal{E}u \right\| \le c\left\| T^{-1}u \right\| \text{, and} \\
&\mathcal{F}\mathcal{D}(S) \subset \mathcal{D}(T) \text{ with } \| T\mathcal{F}u \| \le c\| Su \| \text{, and} \\
&\qquad\qquad \mathcal{F}\mathcal{R}(S) \subset \mathcal{R}(T) \text{ with } \left\| T^{-1}\mathcal{F}u \right\| \le c\left\| S^{-1}u \right\| .
\end{aligned}$$

Suppose that S has a bounded H_∞ functional calculus in $\mathcal{H}$. Then T has a bounded H_∞ functional calculus in $\mathcal{K}$.

Every maximal accretive operator in a Hilbert space $\mathcal{H}$ is type $\frac{\pi}{2}$ and has a bounded H_∞ functional calculus. Another class of operators of type ω is given by the following result. To say that A is a *bounded invertible ω-accretive operator on $\mathcal{H}$* means that A, $A^{-1} \in \mathcal{L}(\mathcal{H})$ and $|\arg(Au,u)| \le \omega$ for all $u \in \mathcal{H}$.

Proposition 2.2. *(i) Let $T = AS$ where S is a positive self–adjoint operator in $\mathcal{H}$, and A is a bounded invertible ω-accretive operator on $\mathcal{H}$, and let $\underline{T} = SA$. Then T and $\underline{T}$ are one–one operators of type ω. Here $\mathcal{D}(T) = \mathcal{D}(S)$ and $\mathcal{D}(\underline{T}) = A^{-1}\mathcal{D}(S)$.*

(ii) If the condition on S is relaxed to the statement that S is a one–one maximal accretive operator with numerical range in $S_{\nu+}$ where $\nu < \pi - \omega$, then T and $\underline{T}$ are one–one operators of type $\omega + \nu$.

(iii) If the condition on S is changed to the statement that S is a one–one self–adjoint operator in $\mathcal{H}$, then T and $\underline{T}$ are one–one operators of type S_ω.

Here are some results about T^2 and $|T|_* = (T^2)^{\frac{1}{2}}$. See [AM^cN] for details.

Theorem 2.3. *Let T be a one–one operator of type S_ω in $\mathcal{H}$. Then T^2 is type 2ω and $|T|_*$ is type ω.*

If in addition T has a bounded H_∞ functional calculus in $\mathcal{H}$, then so does T^2. Moreover $\mathcal{D}(|T|_*) = \mathcal{D}(T)$ with $\| \, |T|_* u \| \approx \| Tu \|$.

Let us see why the final result is valid. If T has a bounded H_∞ functional calculus in $\mathcal{H}$, then in particular, $\operatorname{sgn}(T) \in \mathcal{L}(\mathcal{H})$ with $\| \operatorname{sgn}(T)u \| \le c \| u \|$ where sgn is the bounded holomorphic function on S^0_μ defined by $\operatorname{sgn}(\zeta) = 1$ when $\zeta \in S^0_{\mu+}$ and $\operatorname{sgn}(\zeta) = -1$ when $\zeta \in S^0_{\mu-}$. Now $|T|_* = \operatorname{sgn}(T)T$ and $T = \operatorname{sgn}(T)\,|T|_*$. Therefore $\| \, |T|_* u \| = \| \operatorname{sgn}(T)Tu \| \le c \| Tu \|$ and $\| Tu \| \le c \| \, |T|_* u \|$ as required.

§3. Homogeneous second order differential operators in $L_2(\mathbb{R})$.

In this section $0 \le \omega < \pi/2$, and a and b denote bounded ω-accretive functions on $\mathbb{R}$ with bounded reciprocals, meaning that $a, b, \frac{1}{a}, \frac{1}{b} \in L_\infty(\mathbb{R}, \mathbb{C})$ and $|\arg a|, |\arg b| \le \omega$. The operator of multiplication by b is a bounded invertible ω-accretive operator on $L_2(\mathbb{R})$, as is multiplication by a.

The aim of this section is to show how Theorem 2.1 can be applied to derive the following result of Kenig and Meyer [KM] from known results about $-ib\frac{d}{dx}$. This material appears in [AM^cN], but is repeated here as the whole paper depends on it.

Theorem 3.1. *Suppose that a and b are functions with the properties given above, and let L denote the operator in $L_2(\mathbb{R})$ defined by $Lw = -a\frac{d}{dx}(b\frac{dw}{dx})$ with domain $\mathcal{D}(L) = \{w \in H^1(\mathbb{R}) : b\frac{dw}{dx} \in H^1(\mathbb{R})\}$. Then L is one–one of type 2ω in $L_2(\mathbb{R})$, its square root $L^{\frac{1}{2}}$ has domain $\mathcal{D}(L^{\frac{1}{2}}) = H^1(\mathbb{R})$, and $\left\| L^{\frac{1}{2}} w \right\|_2 \approx \left\| \frac{dw}{dx} \right\|_2$ for all $w \in H^1(\mathbb{R})$.*

The fact that this result can be deduced from the same quadratic estimates as those already known for $-ib\frac{d}{dx}$, was first proved by Pipher using direct arguments involving integration by parts. Her work led to Theorem 3.1 of [AM^cN] (which is essentially Theorem 2.1 with $\mathcal{E} = \mathcal{F} = I$).

Our initial approach had been to show that bounded holomorphic functions of T are Calderón–Zygmund operators. As this is of independent interest, details are presented in Section 5.

Let us record a related result of independent interest.

Theorem 3.2. *The operator* $L = -a\frac{d}{dx}b\frac{d}{dx}$ *has a bounded* H_∞ *functional calculus in* $L_2(\mathbb{R})$.

First consider the operator $-ib\frac{d}{dx}$ in $L_2(\mathbb{R})$ with domain $\mathcal{D}(-ib\frac{d}{dx}) = H^1(\mathbb{R}) = \{ w \in L_2(\mathbb{R}) : \frac{dw}{dx} \in L_2(\mathbb{R}) \}$ where the derivative is in the weak or distributional sense. The operator $-i\frac{d}{dx}$ with domain $H^1(\mathbb{R})$ is a one–one self–adjoint operator in $L_2(\mathbb{R})$. Thus, by Proposition 2.2(iii), $-ib\frac{d}{dx}$ is a one–one operator of type S_ω in $L_2(\mathbb{R})$.

It is known that $-ib\frac{d}{dx}$ has a bounded H_∞ functional calculus in $L_2(\mathbb{R})$. This is equivalent to the fact that the operator $-i\frac{d}{dz}|_\gamma$ satisfies quadratic estimates on the Lipschitz curve γ in $\mathbb{C}$ parametrised by $z = g(x)$, where $g' = \frac{1}{b}$. These results are intimately connected with the L_2 boundedness of the Cauchy integral C_γ on $L_2(\gamma)$ which was first proved by Calderón [C] when $\operatorname{Re} g(x) = x$ and $\| \operatorname{Im} b \|_\infty$ is small, and by Coifman, M^cIntosh and Meyer [CM^cM] in the general case. There are now many proofs of this fact. Connections between such estimates and the holomorphic functional calculus of D_γ are treated in [CM], [M^cQ] and [ADM^c].

Second, consider $S = -ia\frac{d}{dx} \oplus -ib\frac{d}{dx}$ in $\mathcal{H} = L_2(\mathbb{R}, \mathbb{C}^2) \approx L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$. Clearly S has a bounded H_∞ functional calculus in $\mathcal{H}$.

Third, consider

$$T = \begin{bmatrix} 0 & -a\frac{d}{dx} \\ b\frac{d}{dx} & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix} = BD$$

with $\mathcal{D}(T) = H^1(\mathbb{R}, \mathbb{C}^2)$. Now $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a bounded invertible ω-accretive operator on $\mathcal{H}$, while D is a one–one self–adjoint operator. So, by Proposition 2.2(iii), T is a one–one operator of type S_ω. Clearly $\mathcal{D}(T) = \mathcal{D}(S)$ with $\| Tu \| \approx \| Su \|$ and $\mathcal{R}(T) = \mathcal{R}(S)$ with $\| T^{-1}u \| \approx \| S^{-1}u \|$, where S is defined in the previous paragraph. On applying Theorem 2.1 with $\mathcal{E} = \mathcal{F} = I$, we conclude that T has a bounded H_∞ functional calculus in $\mathcal{H}$.

Proof of Theorem 3.1. By Theorem 2.3, $T^2 = \begin{bmatrix} -a\frac{d}{dx}b\frac{d}{dx} & 0 \\ 0 & -b\frac{d}{dx}a\frac{d}{dx} \end{bmatrix}$ is type 2ω in $\mathcal{H}$ and

$$|T|_* = (T^2)^{\frac{1}{2}} = \begin{bmatrix} (-a\frac{d}{dx}b\frac{d}{dx})^{\frac{1}{2}} & 0 \\ 0 & (-b\frac{d}{dx}a\frac{d}{dx})^{\frac{1}{2}} \end{bmatrix}.$$

Since T has a bounded H_∞ functional calculus in $\mathcal{H}$, it follows that $\mathcal{D}(|T|_*) = \mathcal{D}(T)$ and $\| \, |T|_* u \, \| \approx \| Tu \|$ for all $u \in \mathcal{D}(T)$.

Therefore $-a\frac{d}{dx}b\frac{d}{dx}$ is type 2ω in $L_2(\mathbb{R})$, and $\left\| (-a\frac{d}{dx}b\frac{d}{dx})^{\frac{1}{2}} w \right\| \approx \| \frac{dw}{dx} \|$ (and of course $\left\| (-b\frac{d}{dx}a\frac{d}{dx})^{\frac{1}{2}} w \right\| \approx \| \frac{dw}{dx} \|$ as well). $\square$

Proof of Theorem 3.2. This is a consequence of the diagonal nature of T^2. $\square$

§4. Hardy spaces and generalised Cauchy–Riemann systems.

There are intimate connections between functional calculi and quadratic estimates. Let us illustrate this by proving the following consequence of the fact that the operator T of the previous section has a bounded H_∞ functional calculus in $\mathcal{H} = L_2(\mathbb{R}, \mathbb{C}^2)$. It concerns quadratic estimates for solutions of the generalised Cauchy–Riemann equations

$$(\mathrm{GCR}_\pm) \qquad \left\{ \begin{array}{rcll} \frac{\partial w}{\partial t}(x,t) - a(x)\frac{\partial v}{\partial x}(x,t) & = & 0 & \pm t > 0 \\ \frac{\partial v}{\partial t}(x,t) + b(x)\frac{\partial w}{\partial x}(x,t) & = & 0 & \pm t > 0 \\ \lim\limits_{t\to 0} \begin{bmatrix} w(x,t) \\ v(x,t) \end{bmatrix} & = & u(x)\,, & \lim\limits_{t\to\pm\infty} \begin{bmatrix} w(x,t) \\ v(x,t) \end{bmatrix} = 0 \end{array} \right.$$

where a and b have the properties specified in Section 3.

Theorem 4.1. *There is a Hardy decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ of $\mathcal{H}$ into closed subspaces $\mathcal{H}^\pm$ with the following property. For every $u \in \mathcal{H}^+$ there exists a solution w, v of (GCR$_+$) which satisfies the quadratic estimates*

$$\left\{ \int_0^\infty \int_{\mathbb{R}} \left\{ \left|\tfrac{\partial w}{\partial t}(x,t)\right|^2 + \left|\tfrac{\partial v}{\partial t}(x,t)\right|^2 \right\} t\, dx dt \right\}^{\frac12} \approx \| u \|_2 \ .$$

Similarly, for every $u \in \mathcal{H}^-$, there exists a solution (w, v) of (GCR$_-$) which satisfies the corresponding estimates with the integral in t taken from $-\infty$ to 0.

Note that the decomposition is typically not orthogonal. Note also that we are thinking of the solution $w(\,.\,,t)$, $v(\,.\,,t) \in L_2(\mathbb{R})$ for each t and are taking derivatives and limits with respect to t in the sense of L_2 convergence. We leave the consideration of other kinds of limits, and of the uniqueness of the solution, to the interested reader.

Let us construct the Hardy spaces $\mathcal{H}^\pm$ by considering the spectral projections $E_\pm$ associated with the parts of $\sigma(T)$ in each sector $S_{\omega\pm}$.

For some $\mu > \omega$, define the functions $\chi_+, \chi_- \in H_\infty(S^0_\mu)$ by $\chi_+(\zeta) = 1$ if $\mathrm{Re}\,\zeta > 0$ and $\chi_+(\zeta) = 0$ if $\mathrm{Re}\,\zeta < 0$, $\chi_-(\zeta) = 1 - \chi_+(\zeta)$, so that $\mathrm{sgn}(\zeta) = \chi_+(\zeta) - \chi_-(\zeta)$. On using the fact that T has a bounded H_∞ functional calculus, $E_+ = \chi_+(T) \in \mathcal{L}(\mathcal{H})$ and $E_- = \chi_-(T) \in \mathcal{L}(\mathcal{H})$. Moreover, by the identities of the functional calculus, ${E_+}^2 = E_+$, ${E_-}^2 = E_-$, $E_+E_- = 0 = E_-E_+$, $E_+ + E_- = I$ and $E_+ - E_- = \mathrm{sgn}(T)$.

The operators E_+ and E_- form a pair of bounded spectral projections corresponding to the parts of the spectra in $S_{\omega+}$ and $S_{\omega-}$ respectively. Therefore $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^+ = E_+(\mathcal{H})$ and $\mathcal{H}^- = E_-(\mathcal{H})$ are the corresponding spectral subspaces.

We now consider quadratic estimates. It is fundamental to our purposes that an operator T of type S_ω in a Hilbert space $\mathcal{H}$ has a bounded H_∞ functional calculus if and only if T satisfies quadratic estimates in the sense that the norm $\| u \|$ is equivalent to the quadratic norm

$$||u||_T \;=\; \left\{ \int_0^\infty ||\psi(tT)u||^2 \;\tfrac{dt}{t} \right\}^{\frac{1}{2}}$$

where $\psi \in H_\infty(S_\mu^0)$ satisfies $|\psi(\zeta)| \le C|\zeta|^s(1+|\zeta|^{2s})^{-1}$ for some C, $s>0$, and is not identically zero on either sector [M^c]. See [ADM^c] for further details.

Proof of Theorem 4.1. We saw in Section 3 that the operator

$$T \;=\; \begin{bmatrix} 0 & -a\frac{d}{dx} \\ b\frac{d}{dx} & 0 \end{bmatrix}$$

is type S_ω in $\mathcal{H} = L_2(\mathbb{R}, \mathbb{C}^2)$ with a bounded H_∞ functional calculus. Therefore, as shown above, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^\pm = E_\pm(\mathcal{H}) = \mathcal{R}(\,\chi_\pm(T))$, and

$$\left\{ \int_0^\infty ||\psi(tT)u||^2 \;\tfrac{dt}{t} \right\}^{\frac{1}{2}} \;\approx\; \| u \|$$

with, for example, $\psi(\zeta) = \zeta e^{-\zeta}\,\chi_+(\zeta) + \zeta e^{\zeta}\,\chi_-(\zeta)$. Hence, making this choice of ψ, we see that for $u \in \mathcal{H}^+$,

$$\left\{ \int_0^\infty ||\tfrac{d}{dt}e^{-tT}u||^2 \; t\,dt \right\}^{\frac{1}{2}} = \left\{ \int_0^\infty ||tTe^{-tT}u||^2 \;\tfrac{dt}{t} \right\}^{\frac{1}{2}} = \left\{ \int_0^\infty ||\psi(tT)u||^2 \;\tfrac{dt}{t} \right\}^{\frac{1}{2}} \approx \| u \| \,.$$

Suppose that $u \in \mathcal{H}^+$. For all $t>0$, define $\begin{bmatrix} w(x,t) \\ v(x,t) \end{bmatrix} \;=\; e^{-tT}u(x)$, so that

$$\left\{ \int_0^\infty \int_{\mathbb{R}} \left\{ \left|\frac{\partial w}{\partial t}(x,t)\right|^2 + \left|\frac{\partial v}{\partial t}(x,t)\right|^2 \right\} t\,dx dt \right\}^{\frac{1}{2}} \;=\; \left\{ \int_0^\infty ||\tfrac{d}{dt}e^{-tT}u||^2 \; t\,dt \right\}^{\frac{1}{2}} \;\approx\; \| u \|_2 \,.$$

Let us check that w, v is a solution of (GCR$_+$). First, note that

$$\begin{bmatrix} \frac{\partial w}{\partial t}(x,t) \\ \frac{\partial v}{\partial t}(x,t) \end{bmatrix} \;=\; -T \begin{bmatrix} w(x,t) \\ v(x,t) \end{bmatrix} \;=\; \begin{bmatrix} 0 & a(x)\frac{\partial}{\partial x} \\ -b(x)\frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} w(x,t) \\ v(x,t) \end{bmatrix} \;=\; \begin{bmatrix} a(x)\frac{\partial v}{\partial x}(x,t) \\ -b(x)\frac{\partial w}{\partial x}(x,t) \end{bmatrix}$$

as required.

We must next prove the L_2 convergence of $\begin{bmatrix} w(\,.\,,t) \\ v(\,.\,,t) \end{bmatrix}$ to u as $t \to 0$ and to 0 as $t \to \infty$. This is a consequence of standard results from semi–group theory. It also follows from the Convergence Lemma stated below. Thus w, v is a solution of (GCR$_+$) as claimed.

The result for $u \in \mathcal{H}^-$ is proved in a similar way. $\square$

Lemma 4.2. The Convergence Lemma. *([M^c]) Let $0 \le \omega < \mu < \frac{\pi}{2}$. Let T be a one–one operator of type S_ω in a Hilbert space $\mathcal{H}$. Let $\{f_\alpha\}$ be a uniformly bounded net in $H_\infty(S_\mu^0)$ which converges to $f \in H_\infty(S_\mu^0)$ uniformly on compact subsets of S_μ^0, such that $\{f_\alpha(T)\}$ is a uniformly bounded net in $\mathcal{L}(\mathcal{H})$. Then $f(T) \in \mathcal{L}(\mathcal{H})$, $f_\alpha(T)u \to f(T)u$ for all $u \in \mathcal{H}$, and $\| f(T) \| \le \sup_\alpha \| f_\alpha(T) \|$.*

§5. Homogeneous second order differential operators in $L_p(\mathbb{R})$.

In this section we use estimates for the Green's function of $T - \zeta I$ to develop the L_p theory of the operators L and T which were defined in Section 3. Our aim is to establish the following result.

Theorem 5.1. *Let $1 < p < \infty$. Then $\{ w \in H^1(\mathbb{R}) : L^{\frac{1}{2}} w \in L_p(\mathbb{R}) \} = \{ w \in H^1(\mathbb{R}) : \frac{dw}{dx} \in L_p(\mathbb{R}) \}$ with $\left\| L^{\frac{1}{2}} w \right\|_p \approx \left\| \frac{dw}{dx} \right\|_p$.*

A related result of independent interest is the following.

Theorem 5.2. *Let $1 < p < \infty$ and $\mu > \omega$. For each $f \in H_\infty(S_{2\mu+}^0)$,*

$$f(L) : L_p \cap L_2(\mathbb{R}) \to L_p \cap L_2(\mathbb{R}) \quad \textit{with} \quad \| f(L)w \|_p \ \le\ c_{p,\mu} \| f \|_\infty \| w \|_p \ .$$

These results follow from L_p estimates for functions of T.

Theorem 5.3. *Let $1 < p < \infty$ and $\mu > \omega$. For each $f \in H_\infty(S_\mu^0)$,*

$$f(T) : L_p \cap L_2(\mathbb{R}, \mathbb{C}^2) \to L_p \cap L_2(\mathbb{R}, \mathbb{C}^2) \quad \textit{with} \quad \| f(T)u \|_p \ \le\ c_{p,\mu} \| f \|_\infty \| u \|_p \ .$$

Before proving Theorem 5.3, we show that it implies Theorems 5.1 and 5.2.

Proof of Theorem 5.1. It follows from Theorem 5.3 that $\operatorname{sgn}(T)$ maps $L_p \cap L_2(\mathbb{R}, \mathbb{C}^2)$ to itself with $\| \operatorname{sgn}(T)u \|_p \le c \| u \|_p$. Also $(\operatorname{sgn}(T))^{-1} = \operatorname{sgn}(T)$ has the same property, so in fact $\operatorname{sgn}(T)$ is a one–one mapping of $L_p \cap L_2(\mathbb{R}, \mathbb{C}^2)$ onto itself with $\| \operatorname{sgn}(T)u \|_p \approx \| u \|_p$.

Let $w \in H^1(\mathbb{R})$ and define $u = \begin{bmatrix} w \\ 0 \end{bmatrix}$. Then $\frac{dw}{dx} \in L_p(\mathbb{R})$ if and only if $Tu = du \in L_p(\mathbb{R}, \mathbb{C}^2)$, which holds if and only if $\begin{bmatrix} L^{\frac{1}{2}} w \\ 0 \end{bmatrix} = |T|_* \, u = \operatorname{sgn}(T) T u \in L_p(\mathbb{R}, \mathbb{C}^2)$. Moreover $\left\| L^{\frac{1}{2}} w \right\|_p = \| \, |T|_* \, u \|_p \approx \| Tu \|_p = \left\| \frac{dw}{dx} \right\|_p$ as claimed. $\square$

Proof of Theorem 5.2. Given $f \in H_\infty(S^0_{2\mu+})$ and $w \in L_p \cap L_2(\mathbb{R})$, define $g \in H_\infty(S^0_\mu)$ by $g(\zeta) = f(\zeta^2)$ and let $u = \begin{bmatrix} w \\ 0 \end{bmatrix}$. Then, by Theorem 5.3,

$$\| f(L)w \|_p \;=\; \left\| f(T^2)u \right\|_p \;=\; \| g(T)u \|_p \;\le\; c_{p,\mu} \| g \|_\infty \| u \|_p \;=\; c_{p,\mu} \| f \|_\infty \| w \|_p \,. \quad \square$$

We shall see that the operators $f(T)$ are actually Calderón–Zygmund singular integral operators for all $f \in H_\infty(S^0_\mu)$. Thus the operators $f(L)$ are also Calderón–Zygmund operators when $f \in H_\infty(S^0_{2\mu+})$.

Let us turn to the proof of Theorem 5.3. For this purpose, we derive bounds on the Green's function of T. We need the following lemma.

Lemma 5.4. *Let Ω be a real interval, possibly $\mathbb{R}$ itself. Suppose that $\frac{dv}{dx} = h + g$ where $v \in L_2(\Omega)$, $h \in L_2(\Omega)$ and $g \in L_1(\Omega)$, the derivative being taken in the distributional sense. Then $v \in C(\Omega)$ and*

$$\| v \|_\infty \;\le\; \frac{1}{\sqrt{\kappa}} \| v \|_2 \;+\; \sqrt{\kappa} \| h \|_2 \;+\; \| g \|_1$$

for all κ such that $0 < \kappa \le \operatorname{length}(\Omega)$.

Proof. The function v is continuous (or, strictly speaking, equals a continuous function almost everywhere) because its derivative is locally integrable. For each $x \in \Omega$ choose a subinterval $S \subset \Omega$ of length κ with $x \in S$. For almost all $y \in S$,

$$\begin{aligned} v(x) \;&=\; v(y) \;+\; \int_y^x \{ h(\tau) + g(\tau) \}\, d\tau \qquad \text{so} \\ |v(x)| \;&\le\; |v(y)| \;+\; \sqrt{\kappa} \| h \|_2 \;+\; \| g \|_1 \qquad \text{and, averaging over } y \in S, \\ |v(x)| \;&\le\; \frac{1}{\sqrt{\kappa}} \| v \|_2 \;+\; \sqrt{\kappa} \| h \|_2 \;+\; \| g \|_1 \qquad \text{as required.} \quad \square \end{aligned}$$

In the next two results, $L_p(\mathbb{R}, \mathbb{C}^2)$ is abbreviated to $L_p(\mathbb{R})$.

Proposition 5.5. *The operator T satisfies the following properties.*
(i) $\mathcal{D}(T) \subset C(\mathbb{R})$ with

$$\| u \|_\infty \;\le\; c \left\{ |\zeta|^{\frac{1}{2}} \| u \|_2 + |\zeta|^{-\frac{1}{2}} \| (T - \zeta I)u \|_2 \right\}$$

for all $\zeta \in \mathbb{C}$.

Henceforth suppose that $\omega < \nu < \frac{\pi}{2}$ *and that* $\zeta \notin S_\nu$. *(The constants depend on* ν *but not on* ζ *itself.)*

$$
\begin{array}{llll}
(ii) & (T-\zeta I) \text{ is a one–one mapping of } \mathcal{D}(T) & \text{onto} & L_2(\mathbb{R}) \\
 & & \text{with} & \|u\|_2 \le c\,|\zeta|^{-1}\,\|(T-\zeta I)u\|_2 \\
(iii) & (T-\zeta I)^{-1} : L_2(\mathbb{R}) \to C(\mathbb{R}) & \text{with} & \|u\|_\infty \le c\,|\zeta|^{-\frac{1}{2}}\,\|(T-\zeta I)u\|_2 \\
(iv) & (T-\zeta I)^{-1} : L_1 \cap L_2(\mathbb{R}) \to L_2(\mathbb{R}) & \text{with} & \|u\|_2 \le c\,|\zeta|^{-\frac{1}{2}}\,\|(T-\zeta I)u\|_1 \\
(v) & (T-\zeta I)^{-1} : L_1 \cap L_2(\mathbb{R}) \to C(\mathbb{R}) & \text{with} & \|u\|_\infty \le c\,\|(T-\zeta I)u\|_1
\end{array}
$$

(vi) The mapping $\zeta \mapsto (T-\zeta I)^{-1}$ *is continuous from* $\mathbb{C}\backslash S_\omega$ *to* $\mathcal{L}(L_1(\mathbb{R}), C(\mathbb{R}))$ *(where* $(T-\zeta I)^{-1}$ *is extended by continuity to all of* $L_1(\mathbb{R})$*).*

Proof. (i) Let

$$u = \begin{bmatrix} w \\ v \end{bmatrix} \in \mathcal{D}(T) \subset L_2(\mathbb{R}) .$$

Then

$$(T-\zeta I)u = \begin{bmatrix} -av' - \zeta w \\ bw' - \zeta v \end{bmatrix} \in L_2(\mathbb{R})$$

and so

$$
\begin{aligned}
v' &= -\zeta \tfrac{1}{a} w - \tfrac{1}{a}((T-\zeta I)u)_0 \quad \text{and} \\
w' &= \zeta \tfrac{1}{b} v - \tfrac{1}{b}((T-\zeta I)u)_1 .
\end{aligned}
$$

Therefore, applying Lemma 5.4 with $g=0$, we obtain $u \in C(\mathbb{R})$ with

$$\|u\|_\infty \le \tfrac{c}{\sqrt{\kappa}}\|u\|_2 + c\sqrt{\kappa}\,\{|\zeta|\,\|u\|_2 + \|(T-\zeta I)u\|_2\}$$

for all $\kappa > 0$. Hence, on choosing $\kappa = |\zeta|^{-1}$, we conclude that

$$\|u\|_\infty \le c\left\{|\zeta|^{\frac{1}{2}}\,\|u\|_2 + |\zeta|^{-\frac{1}{2}}\,\|(T-\zeta I)u\|_2\right\} .$$

We proved part (ii) in Section 3. Part (iii) is a consequence of (i) and (ii). Part (iv) follows by duality. To prove (v), use the same formula as in (i), this time applying Lemma 5.4 successively with $g = \frac{1}{a}((T-\zeta I)u)_0$, $\frac{1}{b}((T-\zeta I)u)_1 \in L_1(\mathbb{R})$, and making use of (iv). For part (vi), use the resolvent identity $(T-\zeta_1 I)^{-1} - (T-\zeta_2 I)^{-1} = (\zeta_1-\zeta_2)(T-\zeta_1 I)^{-1}(T-\zeta_2 I)^{-1}$. $\square$

Let $G_\zeta(x,y)$ denote the distribution kernel of $(T-\zeta I)^{-1}$. It follows from (v) that, for each $\zeta \notin S_\omega$, $G_\zeta(x,y) \in L_\infty(\mathbb{R}^2, \mathbb{C}^2)$ with $\|G_\zeta\|_\infty = \left\|(T-\zeta I)^{-1}\right\|_{\mathcal{L}(L_1,L_\infty)}$ and that

$$(T-\zeta I)^{-1}u(x) = \int_{\mathbb{R}} G_\zeta(x,y)u(y)\,dy \quad \text{a.e.}$$

for all $u \in L_1 \cap L_2(\mathbb{R})$.

By (vi), the mapping $\zeta \mapsto G_\zeta$ is continuous from $\mathbb{C}\setminus S_\omega$ to $L_\infty(\mathbb{R}^2, \mathbb{C}^2)$. The function $G_\zeta(x,y)$ is called the *Green's function* of $T-\zeta I$. Let us show that it has exponential decay in $|x-y|$.

Proposition 5.6. *If $\zeta \notin S_\nu$ then the Green's function $G_\zeta(x,y)$ of $T-\zeta I$ satisfies*

$$|G_\zeta(x,y)| \;\le\; ce^{-C|\zeta||x-y|}$$

for some c, $C>0$ and almost all x, $y\in\mathbb{R}$. Therefore

$$(T-\zeta I)^{-1} : L_p \cap L_2(\mathbb{R}) \to L_p(\mathbb{R}) \quad \text{with} \quad \|u\|_p \le \frac{c_\nu}{|\zeta|}\,\|(T-\zeta I)u\|_p$$

whenever $1\le p\le\infty$.

Proof. For each $\phi \in C_c^1(\mathbb{R})$, denote by e^ϕ the operator on $L_p(\mathbb{R})$ defined by $(e^\phi)u(x) = e^{\phi(x)}u(x)$, and note that it, along with its inverse $(e^\phi)^{-1} = e^{-\phi}$, maps each of the spaces $L_p(\mathbb{R})$ and $\mathcal{D}(T) = H^1(\mathbb{R})$ to itself, albeit with large norm.

Set $T_\phi = e^{-\phi}Te^\phi$ with $\mathcal{D}(T_\phi) = \mathcal{D}(T)$. A simple calculation using $e^{-\phi}\frac{d}{dx}(e^\phi u) = \frac{du}{dx} + \phi' u$ shows that $T_\phi = T + M$ where $M = \begin{bmatrix} 0 & -a\phi' \\ b\phi' & 0 \end{bmatrix}$.

Suppose that $\|\phi'\|_\infty \le C|\zeta|$ where the constant C will be chosen shortly. All of the statements in Proposition 5.5 remain true when T is replaced by $T_\phi = T+M$ provided (ii) does, namely $(T_\phi - \zeta I)$ is a one–one mapping of $\mathcal{D}(T)$ onto $L_2(\mathbb{R})$ with

$$(\#) \qquad \|u\|_2 \le c|\zeta|^{-1}\,\|(T_\phi - \zeta I)u\|_2 \;.$$

Although $T_\phi = BD + M - \zeta I$ is not of type S_ω, it nevertheless follows from the lemma below that $(\#)$ holds provided C is chosen suitably.

Therefore the kernel $e^{-\phi(x)}G_\zeta(x,y)e^{\phi(y)}$ of T_ϕ is bounded. That is,

$$|G_\zeta(x,y)| \;\le\; ce^{\phi(x)-\phi(y)}\;.$$

For each fixed x, y and ζ it is possible to choose $\phi\in C_c^1(\mathbb{R})$ such that $\phi(x) - \phi(y) = -C|\zeta|\,|x-y|$. Therefore $|G_\zeta(x,y)| \le ce^{-C|\zeta||x-y|}$ as required. □

Lemma 5.7. *(Proposition 8.4 of [AMcN]) Let S be a one–one self–adjoint operator in a Hilbert space $\mathcal{H}$, let B be a bounded invertible ω-accretive operator on $\mathcal{H}$, let $A\in\mathcal{L}(\mathcal{H})$, and let $\nu>\omega$. Denote $\inf\{\,|(B^{-1}u,u)| : \|u\| = 1\,\} = \kappa > 0$. If $\zeta\notin S_\nu$ and $|\zeta| \ge \frac{2}{\kappa}\,\|B^{-1}A\|\operatorname{cosec}(\nu-\omega)$, then $(BS + A - \zeta I)$ has an inverse in $\mathcal{L}(\mathcal{H})$ and*

$$\|(BS + A - \zeta I)^{-1}\| \;\le\; 2\kappa^{-1}\,\|B^{-1}\|\operatorname{cosec}(\nu-\omega)\,|\zeta|^{-1}\;.$$

The same result holds with SB replacing BS provided the condition on $\zeta\notin S_\nu$ is replaced by $|\zeta| \ge \frac{2}{\kappa}\,\|AB^{-1}\|\operatorname{cosec}(\nu-\omega)$.

We turn now to a consideration of the kernels $k_f(x,y)$ of the operators $f(T)$.

These kernels can be defined as follows. Choose θ such that $\omega < \theta < \mu$. Let δ be the unbounded contour consisting of the four rays $\{\zeta \in \mathbb{C} : |\arg\zeta| = \theta$ or $|\arg\zeta| = 2\pi - \theta\}$ parametrised clockwise around S_ω. For each $f \in H_\infty(S^0_\mu)$, define k_f to be the measurable function defined for almost all (x, y), $x \neq y$, by

$$k_f(x,y) \;=\; \frac{1}{2\pi i}\int_\delta G_\zeta(x,y) f(\zeta)\, d\zeta \ .$$

On applying Proposition 5.6 with $\omega < \nu < \theta$, we see that the integral is absolutely convergent with

$$|k_f(x,y)| \;\le\; c' \int_\delta e^{-C|\zeta||x-y|}\, |d\zeta|\, \| f \|_\infty \;=\; \frac{c}{|x-y|}\, \| f \|_\infty \quad \text{a.e.} \quad (x \neq y).$$

(We are not considering the distributional behaviour of k_f when $x = y$.)

Lemma 5.8. *Suppose* $\psi \in H_\infty(S^0_\mu)$ *satisfies* $|\psi(\zeta)| \le c|\zeta|^s(1 + |\zeta|^{2s})^{-1}$ *for some* c *and* $0 < s < 1$. *Then*

$$|k_\psi(x,y)| \;\le\; \frac{c_s}{|x-y|}\, \frac{|x-y|^s}{1 + |x-y|^{2s}} \quad \text{a.e.}$$

and, for all $u \in C^1_c(\mathbb{R}, \mathbb{C}^2)$,

$$\psi(T)u(x) \;=\; \int_{\mathbb{R}} k_\psi(x,y) u(y)\, dy \quad \text{a.e.}$$

Proof. The estimate is straightforward. It allows us to use Fubini's Theorem to obtain

$$\begin{aligned}
\psi(T)u(x) \;&=\; \frac{1}{2\pi i}\int_\delta (T - \zeta I)^{-1} u \psi(\zeta)\, d\zeta\, (x) \\
&=\; \frac{1}{2\pi i}\int_\delta \int_{\mathbb{R}} G_\zeta(x,y) u(y) \psi(\zeta)\, dy\, d\zeta \\
&=\; \int_{\mathbb{R}} k_\psi(x,y) u(y)\, dy \ . \quad \square
\end{aligned}$$

Proposition 5.9. *For all* $f \in H_\infty(S^0_\mu)$, $f(T)$ *has the kernel* k_f *in the following sense. For all* $u \in C^1_c(\mathbb{R}, \mathbb{C}^2)$ *and almost all* $x \notin$ *sppt* u,

$$f(T)u(x) \;=\; \int_{\mathbb{R}} k_f(x,y) u(y)\, dy \ .$$

Moreover $k_f(x,y)B(y)$ *satisfies the Calderón–Zygmund bounds:*

(i) $$|k_f(x,y)B(y)| \;\le\; \frac{c}{|x-y|}\, \| f \|_\infty \quad \text{a.e.} \quad (x \neq y)\,;$$

(ii) $$\left|\tfrac{\partial}{\partial x} k_f(x,y)B(y)\right| + \left|\tfrac{\partial}{\partial y}(k_f(x,y)B(y))\right| \;\le\; \frac{c}{|x-y|^2}\, \| f \|_\infty \quad \text{a.e.} \quad (x \neq y)\,.$$

The derivatives in (ii) exist in $L_{1,\mathrm{loc}}(\{(x,y)\in\mathbb{R}^2 : x\neq y\})$ *in the distributional sense.*

Proof. Choose a uniformly bounded sequence of functions $\psi_n \in H_\infty(S^0_\mu)$ which converges to f uniformly on compact sets, and such that $|\psi_n(\zeta)| \le c_n|\zeta|^s(1+|\zeta|^{2s})^{-1}$ for constants c_n and $0<s<1$. Let $u\in C^1_c(\mathbb{R},\mathbb{C}^2)$. We proved in Lemma 5.8 that

$$\psi_n(T)u(x) \;=\; \int_{\mathbb{R}} k_{\psi_n}(x,y)u(y)\,dy \quad \text{a.e.}$$

By the Convergence Lemma, $\psi_n(T)u$ converges to $f(T)u$ in the L_2 sense. Suppose that $x\notin \operatorname{sppt} u$. It is not hard to check that

$$\operatorname{ess\,sup}\{\,|k_{\psi_n}(x,y)-k_f(x,y)| : y\in \operatorname{sppt}\, u\,\} \;\to\; 0$$

as $n\to\infty$, and hence that

$$\int_{\mathbb{R}} k_{\psi_n}(x,y)u(y)\,dy \;\to\; \int_{\mathbb{R}} k_f(x,y)u(y)\,dy\ .$$

Thus

$$f(T)u(x) \;=\; \int_{\mathbb{R}} k_f(x,y)u(y)\,dy\ .$$

The bound (i) is an immediate consequence of the similar bound already given for $k_f(x,y)$.

To prove (ii), use the identities

$$\begin{aligned} D(T-\zeta I)^{-1}B \;&=\; I \,+\, \zeta B^{-1}(T-\zeta I)^{-1}B \quad \text{and}\\ (T-\zeta I)^{-1}BD \;&=\; I \,+\, \zeta(T-\zeta I)^{-1} \end{aligned}$$

to obtain, in the distributional sense,

$$\begin{aligned} \tfrac{\partial}{\partial x}J(G_\zeta(x,y)B(y)) \;&=\; \delta(x-y) \,+\, \zeta B^{-1}(x)G_\zeta(x,y)B(y) \quad \text{and}\\ \tfrac{\partial}{\partial y}(G_\zeta(x,y)B(y))J \;&=\; \delta(x-y) \,+\, \zeta G_\zeta(x,y) \end{aligned}$$

where $J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$. Therefore

$$\begin{aligned} \left|\tfrac{\partial}{\partial x}k_f(x,y)B(y)\right| \,+\, \left|\tfrac{\partial}{\partial y}(k_f(x,y)B(y))\right| \;&\le\; c'\int_{\delta}|\zeta|\,e^{-C|\zeta||x-y|}\;|d\zeta|\;\|f\|_\infty\\ &\le\; \frac{c}{|x-y|^2}\,\|f\|_\infty \quad \text{a.e.}\quad (x\neq y) \end{aligned}$$

as required. $\square$

Proof of Theorem 5.3. For each $f \in H_\infty(S_\mu^0)$, $f(T)B \in \mathcal{L}(L_2(\mathbb{R},\mathbb{C}^2))$ with $\| f(T)B \| \le c \| f \|_\infty$. Moreover $f(T)B$ has a kernel $k_f(x,y)b(y)$ which satisfies the Calderón–Zygmund bounds described above. Therefore the estimate

$$f(T)B : L_p \cap L_2(\mathbb{R},\mathbb{C}^2) \to L_p \cap L_2(\mathbb{R},\mathbb{C}^2) \quad \text{with} \quad \| f(T)Bu \|_p \le c_{p,\mu} \| f \|_\infty \| u \|_p$$

follows on applying standard Calderón–Zygmund theory. (This is described in many places. One good reference is by Meyer [M].) □

We conclude this section with some further remarks on the L_2 theory. Now that we have the kernel bounds of the operators $f(T)B$ when $f \in H_\infty(S_\mu^0)$, we find that we can prove that T has a bounded H_∞ functional calculus in $L_2(\mathbb{R},\mathbb{C}^2)$ directly, without recourse to Theorem 2.1. Indeed it is really no harder to prove that T has a bounded H_∞ functional calculus, than it is to prove that $-ib\frac{d}{dx}$ does. Let us outline two approaches.

First method. Apply the $T(b)$ Theorem to each component of the operator $\psi(T)B$ when $\psi \in H_\infty(S_\mu^0)$ satisfies $|\psi(\zeta)| \le c\,|\zeta|\,(1+|\zeta|^2)^{-1}$, to obtain the bound $\| \psi(T)B \| \le c \| \psi \|_\infty$. Then use the Convergence Lemma to obtain the same bound for all $f \in H_\infty(S_\mu^0)$. The $T(b)$ Theorem was proved by David, Journé and Semmes in [DJS].

The $T(b)$ Theorem can be applied because of the following facts.
(i) Each component $(\psi(T)B)_{j,k}$, $j = 1,2$, $k = 1,2$ satisfies Calderón–Zygmund bounds;

(ii) $$(\psi(T)B)_{j,1}(\tfrac{1}{a}) = 0 \text{ and } (\psi(T)B)_{j,2}(\tfrac{1}{b}) = 0\,, \quad j = 1,2\,,$$

or, in other words,

$$\int_{\mathbb{R}} (k_\psi(x,y)B(y))B^{-1}(y)\,dy = \int_{\mathbb{R}} k_\psi(x,y)\,dy = 0\,;$$

(iii) $$\int_{\mathbb{R}} B^{-1}(x)(k_\psi(x,y)B(y))\,dx = 0\,;$$

(iv) $B^{-1}(\psi(T)B)B^{-1} = B^{-1}\psi(T)$ satisfies the weak boundedness property with bound $\le c \| \psi \|_\infty$, because, when $u, v \in C^\infty(\mathbb{R})$ with support in a compact interval Ω, then, letting δ be the contour specified before Lemma 5.8, $\delta_1 = \{\zeta \in \delta : |\zeta| \le \text{length}(\Omega)^{-1}\}$

and $\delta_2 = \{\zeta \in \delta : |\zeta| \geq \mathrm{length}(\Omega)^{-1}\}$, we have

$$\begin{aligned}
\left|(B^{-1}\psi(T)u,v)\right| &= \left|\int_\Omega\int_\Omega (B^{-1}(x)k_\psi(x,y)u(y),v(x))\,dxdy\right| \\
&= \left|\frac{1}{2\pi i}\int_\Omega\int_\Omega\int_\delta (B^{-1}(x)G_\zeta(x,y)\psi(\zeta)u(y),v(x))\,d\zeta dxdy\right| \\
&\leq \left|\frac{1}{2\pi i}\int_\Omega\int_\Omega\int_{\delta_1} (B^{-1}(x)G_\zeta(x,y)\psi(\zeta)u(y),v(x))\,d\zeta dxdy\right| \\
&\quad + \left|\frac{1}{2\pi i}\int_\Omega\int_\Omega\int_{\delta_2} \tfrac{1}{\zeta}(JG_\zeta(x,y)\psi(\zeta)u(y),\tfrac{\partial v}{\partial x}(x))\,d\zeta dxdy\right| \\
&\quad + \left|\frac{1}{2\pi i}\int_\Omega\int_{\delta_2} \tfrac{1}{\zeta}(B^{-1}(y)\psi(\zeta)u(y),v(y))\,d\zeta dy\right| \\
&\leq c\,\mathrm{length}(\Omega)\,\|\psi\|_\infty\,\|u\|_\infty\,\|v\|_\infty \\
&\quad + c\left|\int_\Omega\int_\Omega\int_{\delta_2}\tfrac{1}{|\zeta|}e^{-C|\zeta||x-y|}\,d\zeta dxdy\right|\,\|\psi\|_\infty\,\|u\|_\infty\,\|v'\|_\infty \\
&\quad + c\,\mathrm{length}(\Omega)\left|\int_{\delta_2}\tfrac{1}{\zeta}\psi(\zeta)\,d\zeta\right|\,\|u\|_\infty\,\|v\|_\infty \\
&\leq c'\,\mathrm{length}(\Omega)\,\|\psi\|_\infty\,\|u\|_\infty\,\{\|v\|_\infty + \mathrm{length}(\Omega)\,\|v'\|_\infty\}\ .
\end{aligned}$$

In the above estimate we have used the formula for the distribution derivative of the Green's function derived in the proof of Proposition 5.9, and the estimate for the Green's function stated in Proposition 5.6. The bound

$$\left|\int_{\delta_2}\tfrac{1}{\zeta}\psi(\zeta)\,d\zeta\right| \leq 4\theta\,\|\psi\|_\infty$$

is obtained by using the analyticity of ψ and its decay at infinity to replace the contour δ_2 by the two arcs $\{|\zeta|=\beta,\ |\arg\zeta|\leq\theta\}$ and $\{|\zeta|=\beta,\ |\arg(-\zeta)|\leq\theta\}$.

Second method. Use the theorem of Semmes in [S], together with Lemma 5.8, Proposition 5.9 and the above cancellation properties (ii) and (iii), to prove the quadratic estimate

$$\left\{\int_0^\infty \|\psi(tT)Bu\|_2^2\ \tfrac{dt}{t}\right\}^{\frac{1}{2}} \leq c\,\|u\|_2\ , \quad \text{and hence}$$

$$\left\{\int_0^\infty \|\psi(tT)u\|_2^2\ \tfrac{dt}{t}\right\}^{\frac{1}{2}} \leq c\left\|B^{-1}u\right\|_2 \leq c'\,\|u\|_2\ ,$$

together with a dual estimate. Thus T has a bounded H_∞ functional calculus in $L_2(\mathbb{R},\mathbb{C}^2)$ [Mc].

§6. General second order differential operators in $L_2(\Omega)$.

Our aim now is to extend the preceding results to the case of operators with lower order terms defined by boundary conditions on intervals.

Henceforth in this paper, Ω denotes an interval of $\mathbb{R}$, which may be either $\mathbb{R}$ or a half–line or a bounded interval. Also $\mathcal{V}$ denotes a Hilbert space with norm $\|u\|_{\mathcal{V}} = \|u\|_{H^1} = \{\|u\|_2^2 + \|\frac{du}{dx}\|_2^2\}^{\frac{1}{2}}$ such that $\overset{o}{H}{}^1(\Omega) \subset \mathcal{V} \subset H^1(\Omega)$. Here $H^1(\Omega)$ is the Sobolev space $H^1(\Omega) = \{u \in L_2(\Omega) : \frac{du}{dx} \in L_2(\Omega)\}$ with norm $\|u\|_{H^1}$, and $\overset{o}{H}{}^1(\Omega) = \{u \in H^1(\Omega) : u_{|\partial\Omega} = 0\}$ where $\partial\Omega$ denotes the end–points of Ω.

Let $b, \alpha, \beta, \gamma \in L_\infty(\Omega)$ such that

$$B(x) = \begin{bmatrix} b(x) & \alpha(x) \\ \beta(x) & \gamma(x) \end{bmatrix}$$

is a bounded invertible ω-accretive matrix for some $\omega < \frac{\pi}{2}$. (This holds for some $\omega < \frac{\pi}{2}$ if and only if there exists $\kappa > 0$ such that $\mathrm{Re}\,[\,\overline{\zeta_1}\ \overline{\zeta_2}\,]\, B(x) \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \geq \kappa\,|\zeta|^2$ for all $\zeta \in \mathbb{C}^2$ and almost all $x \in \Omega$. So long as $\mathrm{Re}\, b \geq 2\kappa > 0$, this condition can always be achieved by adding a large positive constant to γ.)

Then the sesquilinear form J defined on $\mathcal{V} \times \mathcal{V}$ by

$$J[u, v] = \int_\Omega \left\{ b(x)\tfrac{du}{dx}(x)\overline{\tfrac{dv}{dx}(x)} + \alpha(x)u(x)\overline{\tfrac{dv}{dx}(x)} + \beta(x)\tfrac{du}{dx}(x)\overline{v(x)} + \gamma(x)u(x)\overline{v(x)} \right\} dx$$

satisfies $\mathrm{Re}\, J[u,u] \approx \|u\|_{H^1}$ and $|\arg J[u,u]| \leq \omega$, so that the associated maximal accretive operator L_J is one–one with numerical range in $S_{\omega+}$. Here L_J is the operator in $L_2(\Omega)$ with largest domain which satisfies $J[u, v] = (L_J u, v)$ for all $u \in \mathcal{D}(L_J)$, $v \in \mathcal{V}$. See Chapter VI of Kato's book [K].

Further, let $a \in L_\infty(\Omega)$ be a bounded invertible ω-accretive function on Ω. We now define the principal operator of our investigation, namely $L = aL_J$ with $\mathcal{D}(L) = \mathcal{D}(L_J)$. By Proposition 2.2(ii), L is type 2ω in $L_2(\Omega)$. It is given by

$$Lu(x) = a(x)\left\{ -\tfrac{d}{dx}\{ b(x)\tfrac{du}{dx}(x) + \alpha(x)u(x) \} + \beta(x)\tfrac{du}{dx}(x) + \gamma(x)u(x) \right\}$$

with appropriate boundary conditions, which we now make precise.

Let $\mathcal{W}$ be the largest subspace of $H^1(\Omega)$ such that $(\frac{du}{dx}, v) = -(u, \frac{dv}{dx})$ for all $u \in \mathcal{V}$, $v \in \mathcal{W}$. Then $\mathcal{D}(L) = \{u \in \mathcal{V} : b\frac{du}{dx} + \alpha u \in \mathcal{W}\}$.

For example, if $\mathcal{V} = \overset{o}{H}{}^1(\Omega)$ then $\mathcal{W} = H^1(\Omega)$ and $\mathcal{D}(L) = \{u \in \overset{o}{H}{}^1(\Omega) : b\frac{du}{dx} + \alpha u \in H^1(\Omega)\}$, which corresponds to Dirichlet boundary conditions. If $\mathcal{V} = H^1(\Omega)$, then $\mathcal{W} = \overset{o}{H}{}^1(\Omega)$, and we have natural boundary conditions. If $\Omega = (0,1)$, then $\mathcal{V} = \{\, u \in H^1(\Omega) : u(0) = u(1) \,\} = \mathcal{W}$ corresponds to the periodic boundary conditions, $u(0) = u(1)$ and $b(0)\frac{du}{dx}(0) + \alpha(0)u(0) = b(1)\frac{du}{dx}(1) + \alpha(1)u(1)$. All possibilities for $\mathcal{V}$ and $\mathcal{W}$ are listed at the end of this section.

Let us now state our main result about square roots of differential operators. In the case when $a = 1$, it was derived previously by Auscher and Tchamitchian by constructing suitable wavelets on Ω [AT1].

Theorem 6.1. *Each operator L defined above is one–one of type 2ω, and $\mathcal{D}(L^{\frac{1}{2}}) = \mathcal{V}$ with $\left\| L^{\frac{1}{2}} u \right\|_2 \approx \| u \|_{H^1}$.*

This clearly extends Theorem 3.1 for $-a\frac{d}{dx}b\frac{d}{dx}$ on $\mathbb{R}$. A related result of independent interest is the following.

Theorem 6.2. *The operator L has a bounded H_∞ functional calculus in $L_2(\Omega)$.*

The proofs of both theorems rely on the holomorphic functional calculus of the first order system T which we now introduce.

Let $\Lambda^0 = \mathbb{C}$, $\Lambda^1 = \mathbb{C}^2$, $\Lambda^2 = \mathbb{C}$, and set $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2$. Therefore $L_p(\Omega, \Lambda) = L_p(\Omega, \Lambda^0) \oplus L_p(\Omega, \Lambda^1) \oplus L_p(\Omega, \Lambda^2)$. In particular, $L_2(\Omega, \Lambda)$ is a Hilbert space.

Introduce the unbounded operators $d = d_{1,0} + d_{2,1}$ and $\delta = \delta_{0,1} + \delta_{1,2}$ in $L_2(\Omega, \Lambda)$ as indicated in the pair of exact sequences below, where f is another bounded invertible ω-accretive function. Let $D = d + \delta$ be the corresponding Dirac operator with $\mathcal{D}(D) = \mathcal{D}(d) \cap \mathcal{D}(\delta)$.

$$
\begin{tikzcd}
\{0\} \arrow[r] & L_2(\Omega,\Lambda^0) \arrow[r, "d_{1,0}"] \arrow[d, "\frac{1}{a}"'] & L_2(\Omega,\Lambda^1) \arrow[r, "d_{2,1}"] \arrow[d, "B"'] & L_2(\Omega,\Lambda^2) \arrow[r, "0"] \arrow[d, "f"'] & \{0\} \\
\{0\} & L_2(\Omega,\Lambda^0) \arrow[l, "0"'] & L_2(\Omega,\Lambda^1) \arrow[l, "\delta_{0,1}"'] & L_2(\Omega,\Lambda^2) \arrow[l, "\delta_{1,2}"'] & \{0\} \arrow[l]
\end{tikzcd}
$$

$$
\begin{aligned}
d_{1,0} &= \begin{bmatrix} \frac{d}{dx} \\ I \end{bmatrix}, & \mathcal{D}(d_{1,0}) &= \mathcal{V} \\
d_{2,1} &= [\, I \quad -\tfrac{d}{dx} \,], & \mathcal{D}(d_{2,1}) &= \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \in L_2(\Omega, \Lambda^1) : v \in \mathcal{V} \right\} \\
\delta_{0,1} &= [\, -\tfrac{d}{dx} \quad I \,], & \mathcal{D}(\delta_{0,1}) &= \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \in L_2(\Omega, \Lambda^1) : w \in \mathcal{W} \right\} \\
\delta_{1,2} &= \begin{bmatrix} I \\ \frac{d}{dx} \end{bmatrix}, & \mathcal{D}(\delta_{1,2}) &= \mathcal{W}
\end{aligned}
$$

Lemma 6.3. *(i)* δ *and* d *are closed operators with closed ranges in* $L_2(\Omega,\Lambda)$ *;*
(ii) $\delta_{0,1} = {d_{1,0}}^*$ *and* $\delta_{1,2} = {d_{2,1}}^*$ *, so that* $\delta = d^*$ *and* $d = \delta^*$ *;*
(iii) $\mathcal{R}(d) = \mathcal{N}(d)$ *and* $\mathcal{R}(\delta) = \mathcal{N}(\delta) = \mathcal{R}(d)^\perp$ *;*
(iv) $D = d + \delta$ *is a one–one self adjoint operator in* $L_2(\Omega)$ *with* $\overset{o}{H}{}^1(\Omega,\Lambda) \subset \mathcal{D}(D) \subset H^1(\Omega,\Lambda)$ *.*

Proof. Parts (i) and (ii) are easy to prove, as is (iii) once we have the following identity. If $u = \begin{bmatrix} w \\ v \end{bmatrix} \in \mathcal{D}(d_{2,1}) \cap \mathcal{D}(\delta_{0,1})$, then $w \in \mathcal{W}$ and $v \in \mathcal{V}$, so that

$$\begin{aligned} \| d_{2,1} u \|_2^2 + \| \delta_{0,1} u \|_2^2 &= \| w - \tfrac{dv}{dx} \|_2^2 + \| \tfrac{dw}{dx} - v \|_2^2 \\ &= \| w \|_{H^1}^2 + \| v \|_{H^1}^2 - 2(w, \tfrac{dv}{dx}) - 2(\tfrac{dw}{dx}, v) = \| u \|_{H^1}^2 . \end{aligned}$$

Therefore $\mathcal{N}(d_{2,1}) \cap \mathcal{N}(\delta_{0,1}) = \{0\}$. Part (iv) also follows from this. □

Let $\mathcal{B}$ denote the bounded invertible ω -accretive operator defined on $L_p(\Omega,\Lambda)$ by $\mathcal{B}(u_0,u_1,u_2) = (\frac{1}{a}u_0, Bu_1, fu_2)$ where $u_k \in L_p(\Omega,\Lambda^k)$, and set

$$T = d + \delta_{\mathcal{B}} = d + \mathcal{B}^{-1}\delta\mathcal{B}$$

with $\mathcal{D}(\delta_{\mathcal{B}}) = \mathcal{B}^{-1}\mathcal{D}(\delta)$ and $\mathcal{D}(T) = \mathcal{D}(d) \cap \mathcal{D}(\delta_{\mathcal{B}})$. Then

$$Lu = a\delta_{0,1} B d_{1,0} u = \delta_{\mathcal{B}} du = T^2 u$$

$$\begin{aligned} \text{for all} \quad u \in \mathcal{D}(L) &= \{ u \in \mathcal{V} : b\tfrac{du}{dx} + \alpha u \in \mathcal{W} \} \\ &= \{ u \in \mathcal{D}(d_{1,0}) : Bd_{1,0}u \in \mathcal{D}(\delta_{0,1}) \} = \mathcal{D}(T^2) \cap L_2(\Omega,\Lambda^0) . \end{aligned}$$

Theorem 6.4. *The operator* T *is one–one of type* S_ω *in* $L_2(\Omega,\Lambda)$ *and has a bounded* H_∞ *functional calculus.*

Before proving this, let us show how it implies Theorem 6.1 and Theorem 6.2.

Proof of Theorems 6.1 and 6.2. By Theorem 2.3 and the above result, T^2 is a one–one operator of type 2ω with a bounded H_∞ functional calculus in $L_2(\Omega,\Lambda)$, and moreover, $\mathcal{D}(|T|_*) = \mathcal{D}(T)$, so that $\mathcal{D}(|T|_*) \cap L_2(\Omega,\Lambda^0) = \mathcal{D}(T) \cap L_2(\Omega,\Lambda^0) = \mathcal{V}$ and

$$\| \, |T|_* u \|_2 \approx \| Tu \|_2 = \| du \|_2 = \| u \|_{H^1}$$

for all $u \in \mathcal{V}$.

Now T^2 is diagonal on $\sum_{j=0}^2 L_2(\Omega,\Lambda^j)$ and, for $u \in \mathcal{D}(T^2) \cap L_2(\Omega,\Lambda^0) = \mathcal{D}(L)$, $T^2 u = Lu$. Therefore L is one–one of type 2ω with a bounded H_∞ functional calculus in $L_2(\Omega)$.

Further, $|T|_* u = L^{\frac12} u$ when $u \in \mathcal{D}(|T|_*) \cap L_2(\Omega, \Lambda^0) = \mathcal{D}(L^{\frac12})$, so $\mathcal{D}(L^{\frac12}) = \mathcal{V}$ with $\left\| L^{\frac12} u \right\|_2 \approx \| u \|_{H^1}$. $\square$

Our aim for the remainder of this section is to prove Theorem 6.4. To this end we introduce the operators E, V, W, U, $E_{\mathcal{B}}$, $V_{\mathcal{B}} = V^{-1}$, $W_{\mathcal{B}} = W^{-1}$, P, $\mathcal{A} \in \mathcal{L}(L_2(\Omega, \Lambda))$.

(i) Recalling from Lemma 6.3 that $L_2(\Omega,\Lambda) = \mathcal{R}(d) \oplus \mathcal{R}(\delta)$ where the decomposition is orthogonal, let E denote the orthogonal projection of $L_2(\Omega,\Lambda)$ onto $\mathcal{R}(d)$, so that $E^\perp = I - E$ is the orthogonal projection onto $\mathcal{R}(\delta)$. Note that $d = Ed = dE^\perp = EdE^\perp$ and $\delta = E^\perp\delta = \delta E = E^\perp \delta E$.

(ii) Let $V = E + \mathcal{B}^{-1}E^\perp$, $W = E^\perp + E\mathcal{B}$ and $U = WV = E\mathcal{B}E + E^\perp\mathcal{B}^{-1}E^\perp$. Note that U is a bounded invertible ω-accretive operator on $L_2(\Omega,\Lambda)$, because $(Uu, u) = (\mathcal{B}Eu, Eu) + (\mathcal{B}^{-1}E^\perp u, E^\perp u)$.

(iii) Because $\mathcal{B}$ is a bounded invertible ω-accretive operator, we can also decompose $L_2(\Omega,\Lambda) = \mathcal{R}(d) \oplus \mathcal{B}^{-1}\mathcal{R}(\delta) = \mathcal{R}(d) \oplus \mathcal{R}(\delta_{\mathcal{B}})$, where the decomposition is typically not orthogonal. Let $E_{\mathcal{B}}$ be the projection of $L_2(\Omega,\Lambda)$ onto $\mathcal{R}(d)$ with $\mathcal{N}(E_{\mathcal{B}}) = \mathcal{R}(\delta_{\mathcal{B}})$. Clearly $(I-E)E_{\mathcal{B}} = 0$ and $E\mathcal{B}(I - E_{\mathcal{B}}) = 0$.

(iv) Let $V_{\mathcal{B}} = E_{\mathcal{B}} + \mathcal{B}(I - E_{\mathcal{B}})$ and $W_{\mathcal{B}} = (I - E_{\mathcal{B}}) + E_{\mathcal{B}}\mathcal{B}^{-1}$. Then $VV_{\mathcal{B}} = I$, $V_{\mathcal{B}}V = I$, $WW_{\mathcal{B}} = I$ and $W_{\mathcal{B}}W = I$, so that V and W are invertible in $\mathcal{L}(L_2(\Omega,\Lambda))$ with $V^{-1} = V_{\mathcal{B}}$ and $W^{-1} = W_{\mathcal{B}}$. Let us just check the first identity.

$$\begin{aligned} VV_{\mathcal{B}} &= (E + \mathcal{B}^{-1}(I-E))\,(E_{\mathcal{B}} + \mathcal{B}(I - E_{\mathcal{B}})) \\ &= EE_{\mathcal{B}} \;+\; E\mathcal{B}(I-E_{\mathcal{B}}) \;+\; \mathcal{B}^{-1}(I-E)E_{\mathcal{B}} \;+\; \mathcal{B}^{-1}(I-E)\mathcal{B}(I-E_{\mathcal{B}}) \\ &= E_{\mathcal{B}} \;+\; 0 \;+\; 0 \;+\; (I - E_{\mathcal{B}}) \;=\; I\,. \end{aligned}$$

The reason for introducing these operators and checking their invertibility, is to write

$$T \;=\; d \;+ \mathcal{B}^{-1}\delta\mathcal{B} \;=\; EdE^\perp \;+\; \mathcal{B}^{-1}E^\perp\delta E\mathcal{B} \;=\; VDW \;=\; V(DU)V^{-1}$$

where $D = d + \delta$.

(v) Define $P(u_0, \begin{bmatrix} w \\ v \end{bmatrix}, u_2) = (0, \begin{bmatrix} w \\ 0 \end{bmatrix}, u_2)$. Then $D(E - P) \in \mathcal{L}(L_2(\Omega,\Lambda))$ too. In fact,

$$D(E-P)u \;=\; \delta(I-P)u \;-\; dPu \;=\; \delta(u_0, \begin{bmatrix} 0 \\ v \end{bmatrix}, 0) \;-\; d(0, \begin{bmatrix} w \\ 0 \end{bmatrix}, u_2) \;=\; (v, 0, -w)\,.$$

(vi) Define $\mathcal{A}(u_0, \begin{bmatrix} w \\ v \end{bmatrix}, u_2) = (au_0, \begin{bmatrix} bw \\ \tilde{b}v \end{bmatrix}, fu_2)$ where $\tilde{b} = b(\det B)^{-1}$, or in other words,

$$\mathcal{A} \;=\; P\mathcal{B}P + (I-P)\mathcal{B}^{-1}(I-P)\,.$$

Then, writing $Z = E - P$ and $P^{\perp} = (I - P)$, we have

$$\begin{aligned} U &= E\mathcal{B}E + E^{\perp}\mathcal{B}^{-1}E^{\perp} \\ &= (P\mathcal{B}P + P^{\perp}\mathcal{B}^{-1}P^{\perp}) + Z(\mathcal{B}E - \mathcal{B}^{-1}E^{\perp}) + (P\mathcal{B} - P^{\perp}\mathcal{B}^{-1})Z \\ &= \mathcal{A} + X + Y \end{aligned}$$

where $X = Z(\mathcal{B}E - \mathcal{B}^{-1}E^{\perp})$, $Y = (P\mathcal{B} - P^{\perp}\mathcal{B}^{-1})Z \in \mathcal{L}(L_2(\Omega,\Lambda))$ satisfy DX, $DY^* \in \mathcal{L}(L_2(\Omega,\Lambda))$ as follows from (v) above.

We need to introduce one more operator in $L_2(\Omega,\Lambda)$, namely $\Delta = D^2$. Note that, on its domain $\mathcal{D}(\Delta)$,

$$\Delta(u_0, \begin{bmatrix} w \\ v \end{bmatrix}, u_2) = ((I - \tfrac{d^2}{dx^2})u_0, \begin{bmatrix} (I - \frac{d^2}{dx^2})w \\ (I - \frac{d^2}{dx^2})v \end{bmatrix}, (I - \tfrac{d^2}{dx^2})u_2) .$$

This domain satisfies $\overset{o}{H}{}^2(\mathbb{R},\Lambda) \subset \mathcal{D}(\Delta) \subset H^2(\mathbb{R},\Lambda)$ where these Sobolev spaces are defined as usual. See, for example, [LM].

Lemma 6.5. *Let $r > 0$. The operators T, DU and $D\mathcal{A}$ are all one–one operators of type S_ω in $L_2(\Omega,\Lambda)$, while $\Delta\mathcal{A}$ and $\Delta^r\mathcal{A}$ are one–one operators of type ω in $L_2(\Omega,\Lambda)$. The following statements about them are equivalent.*

(1) T has a bounded H_∞ functional calculus;
(2) DU has a bounded H_∞ functional calculus;
(3) $D\mathcal{A}$ has a bounded H_∞ functional calculus;
(4) $\Delta\mathcal{A}$ has a bounded H_∞ functional calculus;
(5) $\Delta^r\mathcal{A}$ has a bounded H_∞ functional calculus.

Proof. The fact that the operators are one-one of type ω or type S_ω is a consequence of Proposition 2.2, except for T, in which case it follows from the similarity $T = V(DU)V^{-1}$ noted in (iv) above. The equivalence of (1) and (2) is also an immediate consequence of this similarity and the fact that $f(T) = Vf(DU)V^{-1}$.

The equivalence of (2) and (3) follows from (vi) above and Theorem 11.4 of [AM^cN], which is stated below for convenience. To apply this theorem, note that

$$\| Du \|_2 = \{ (du, du) + (\delta u, \delta u) \}^{\frac{1}{2}} = \| u \|_{H^1} \geq \| u \|_2$$

for all $u \in \mathcal{D}(D)$.

The equivalence of (3), (4) and (5) follows from Theorems 7.3 and 8.5 of [AM^cN]. □

Let us record the three results from [AM^cN] which we have just used, together with a fourth to be used shortly. In each case, A denotes a bounded invertible ω-accretive operator on a Hilbert space $\mathcal{H}$. Some editorial changes have been made.

Theorem 11.4 of [AM^{c}N]. *Let* S *be a self–adjoint operator in* $\mathcal{H}$ *such that* $\| Su \| \geq \| u \|$ *for all* $u \in \mathcal{D}(S)$. *Suppose that* SA *has a bounded* H_∞ *functional calculus in* $\mathcal{H}$. *If* U *is a bounded invertible* ω*-accretive operator on* $\mathcal{H}$, *such that* $U = A + X + Y$ *where* X, Y, SX *and* SY^* *are all bounded on* $\mathcal{H}$, *then* SU *has a bounded* H_∞ *functional calculus in* $\mathcal{H}$.

Theorem 7.3 of [AM^{c}N]. *Let* S *be a positive self–adjoint operator in* $\mathcal{H}$, *and let* $r > 0$. *Then* $S^r A$ *is a one–one operator of type* ω *which has a bounded* H_∞ *functional calculus if and only if* SA *does.*

Theorem 8.5 of [AM^{c}N]. *Let* S *be a one–one self–adjoint operator in* $\mathcal{H}$. *Then* $S^2 A$ *is a one–one operator of type* ω *with a bounded* H_∞ *functional calculus if and only if* SA *is a one–one operator of type* S_ω *with a bounded* H_∞ *functional calculus.*

Proposition 11.2 of [AM^{c}N]. *Let* S *be a positive self–adjoint operator in* $\mathcal{H}$. *If* SA *has a bounded* H_∞ *functional calculus, then so does* $(S + I)A$.

Let us now treat the special case $\Omega = \mathbb{R}$.

Proof of Theorem 6.4 when $\Omega = \mathbb{R}$. We have seen in Lemma 6.5 that the problem of showing that T has a bounded H_∞ functional calculus is equivalent to showing that $\Delta\mathcal{A}$ has. When $\Omega = \mathbb{R}$ and hence $\mathcal{V} = \mathcal{W} = H^1(\mathbb{R})$, then $\mathcal{D}(\Delta) = H^2(\mathbb{R}, \Lambda)$ and Δ acts on each component separately, as does $\mathcal{A}$. So our problem reduces to showing that $(I - \frac{d^2}{dx^2})h$ has a bounded H_∞ functional calculus in $L_2(\mathbb{R}) = L_2(\mathbb{R}, \mathbb{C})$ where h denotes one of a, b, f or $\tilde{b} = b(\det B)^{-1}$.

In Section 3 we drew attention to the fact that $-ih\frac{d}{dx}$ has a bounded H_∞ functional calculus in $L_2(\mathbb{R})$. Therefore $-i\frac{d}{dx}h = \frac{1}{h}(-ih\frac{d}{dx})h$ does too, and therefore, by Theorem 8.5 of [AM^{c}N], so does $-\frac{d^2}{dx^2}h$. Now apply Proposition 11.2 of [AM^{c}N] to obtain the desired result. $\square$

Note that, by Lemma 6.5, we have also shown that $\Delta^r\mathcal{A}$ has a bounded H_∞ functional calculus in $L_2(\mathbb{R}, \Lambda)$. Let us also record the facts that $\mathcal{D}(\Delta^r\mathcal{A}) = \mathcal{A}^{-1}H^{2r}(\mathbb{R}, \Lambda)$ with $\| \Delta^r\mathcal{A}u \|_2 \approx \| \mathcal{A}u \|_{H^{2r}}$, and $\mathcal{R}(\Delta^r\mathcal{A}) = L_2(\mathbb{R}, \Lambda)$ with $\| (\Delta^r\mathcal{A})^{-1}u \|_2 \approx \| u \|_{H^{-2r}}$. Here we are writing $H^s(\mathbb{R}, \Lambda)$ for the usual Sobolev space with norm $\| u \|_{H^s}$.

Let us turn our attention to the general case of an interval $\Omega \subset \mathbb{R}$. Our aim is to use the results on $\mathbb{R}$ in the preceding paragraph, together with Theorem 2.1, to prove that $\Delta^r\mathcal{A}$ has a bounded H_∞ functional calculus in $L_2(\Omega, \Lambda)$ when $0 < r < \frac{1}{4}$. The required result follows on once more applying Lemma 6.5.

In doing this, we need notations which differentiate between Ω and $\mathbb{R}$, so write $L_2(\Omega) = L_2(\Omega, \Lambda)$, $H^s(\Omega) = H^s(\Omega, \Lambda)$, D_Ω and Δ_Ω when we are working on Ω, and reserve the symbols $L_2 = L_2(\mathbb{R}, \Lambda)$, $H^s = H^s(\mathbb{R}, \Lambda)$, D and Δ for the case of $\mathbb{R}$. Of course D_Ω and Δ_Ω depend on $\mathcal{V}$ as well.

Proof of Theorem 6.4. By complex interpolation, $\mathcal{D}({\Delta_\Omega}^r) = H^{2r}(\Omega)$ provided $0 < r < \frac{1}{4}$. Therefore $\mathcal{D}({\Delta_\Omega}^r\mathcal{A}) = \mathcal{A}^{-1}H^{2r}(\Omega)$ with $\| {\Delta_\Omega}^r\mathcal{A}u \|_2 \approx \| \mathcal{A}u \|_{H^{2r}(\Omega)}$, and $\mathcal{R}({\Delta_\Omega}^r\mathcal{A}) = L_2(\Omega)$ with $\left\| ({\Delta_\Omega}^r\mathcal{A})^{-1}u \right\|_2 \approx \| u \|_{H^{-2r}(\Omega)}$.

Let $\mathcal{E} : H^s(\Omega) \to H^s$ be the operator of extension by zero, and let $\mathcal{F} : H^s \to H^s(\Omega)$ be the restriction operator, which are both bounded and satisfy $\mathcal{F}\mathcal{E} = I$ provided $-\frac{1}{2} < s < \frac{1}{2}$. Extend $\mathcal{A}$ to a matrix valued function, still called $\mathcal{A}$, on $\mathbb{R}$ which has the same properties as $\mathcal{A}$ on Ω. Note that $\mathcal{A}\mathcal{E} = \mathcal{E}\mathcal{A}$ and $\mathcal{A}\mathcal{F} = \mathcal{F}\mathcal{A}$.

The hypotheses of Theorem 2.1 are satisfied by these operators when $\mathcal{H} = L_2$, $\mathcal{K} = L_2(\Omega)$, $S = \Delta^r\mathcal{A}$ and $T = {\Delta_\Omega}^r\mathcal{A}$ provided $0 < r < \frac{1}{4}$.

Indeed, $\mathcal{E}(\mathcal{D}({\Delta_\Omega}^r\mathcal{A})) \subset \mathcal{D}(\Delta^r\mathcal{A})$ with

$$\| \Delta^r\mathcal{A}\mathcal{E}u \|_2 \approx \| \mathcal{E}\mathcal{A}u \|_{H^{2r}} \leq c\| \mathcal{A}u \|_{H^{2r}(\Omega)} \approx \| {\Delta_\Omega}^r\mathcal{A}u \|_2$$

and $\mathcal{E}(\mathcal{R}({\Delta_\Omega}^r\mathcal{A})) = \mathcal{E}(L_2(\Omega)) \subset L_2 = \mathcal{R}(\Delta^r\mathcal{A})$ with

$$\left\| (\Delta^r\mathcal{A})^{-1}\mathcal{E}u \right\|_2 \approx \| \mathcal{E}u \|_{H^{-2r}} \leq c\| u \|_{H^{-2r}(\Omega)} \approx \left\| ({\Delta_\Omega}^r\mathcal{A})^{-1}u \right\|_2 .$$

Also $\mathcal{F}(\mathcal{D}(\Delta^r\mathcal{A})) \subset \mathcal{D}({\Delta_\Omega}^r\mathcal{A})$ with

$$\| {\Delta_\Omega}^r\mathcal{A}\mathcal{F}u \|_2 \approx \| \mathcal{F}\mathcal{A}u \|_{H^{2r}(\Omega)} \leq c\| \mathcal{A}u \|_{H^{2r}} \approx \| \Delta^r\mathcal{A}u \|_2$$

and $\mathcal{F}(\mathcal{R}(\Delta^r\mathcal{A})) = \mathcal{F}(L_2(\Omega)) \subset L_2 = \mathcal{R}({\Delta_\Omega}^r\mathcal{A})$ with

$$\left\| ({\Delta_\Omega}^r\mathcal{A})^{-1}\mathcal{F}u \right\|_2 \approx \| \mathcal{F}u \|_{H^{-2r}(\Omega)} \leq c\| u \|_{H^{-2r}} \approx \left\| (\Delta^r\mathcal{A})^{-1}u \right\|_2 .$$

Therefore, by Theorem 2.1, ${\Delta_\Omega}^r\mathcal{A}$ has a bounded H_∞ functional calculus in $L_2(\Omega)$ when $0 < r < \frac{1}{4}$. We conclude, once more applying Lemma 6.5, that T has a bounded H_∞ functional calculus in $L_2(\Omega)$ as required. □

Remark. The boundedness of the H_∞ functional calculus of $\Delta\mathcal{A}$ holds for all bounded invertible ω-accretive matrix valued functions, not just the diagonal ones. It remains true in higher dimensions in $L_2(\mathbb{R}^n, \mathbb{C}^m)$ when Δ is the usual Laplacian [McN].

Let us conclude this section by listing all possible choices of Ω, $\mathcal{V}$ and $\mathcal{W}$.

<u>Case I</u>

(*i*) $\Omega = \mathbb{R}$, $\mathcal{V} = H^1(\mathbb{R})$, $\mathcal{W} = H^1(\mathbb{R})$
(*ii*) Ω a half–line, $\mathcal{V} = \overset{o}{H}{}^1(\Omega)$, $\mathcal{W} = H^1(\Omega)$ and vice–versa
(*iii*) $\Omega = (x_1, x_2)$, $\mathcal{V} = \overset{o}{H}{}^1(\Omega)$, $\mathcal{W} = H^1(\Omega)$ and vice–versa
(*iv*) $\Omega = (x_1, x_2)$, $\mathcal{V} = \{u \in H^1(\Omega) : u(x_1) = 0\}$, $\mathcal{W} = \{u \in H^1(\Omega) : u(x_2) = 0\}$ and v-v

Case II

$\Omega = (x_1, x_2)$, $\mathcal{V} = \{u \in H^1(\Omega) : \alpha_1 u(x_1) = \alpha_2 u(x_2)\}$, $\mathcal{W} = \{u \in H^1(\Omega) : \overline{\alpha_2} u(x_1) = \overline{\alpha_1} u(x_2)\}$ where $\alpha_1 \neq 0$, $\alpha_2 \neq 0$

§7. General second order differential operators in $L_p(\Omega)$.

In this section we develop the L_p theory of the operators L and T which were defined in Section 6. This is based on estimates for the Green's function of $T - \zeta I$, or, in other words, on estimates for solutions of certain systems of first order differential equations.

Recall that $T = d + \delta_{\mathcal{B}} = d + \mathcal{B}^{-1}\delta\mathcal{B}$ with $\mathcal{D}(T) = \mathcal{D}(d) \cap \mathcal{B}^{-1}\mathcal{D}(\delta) \subset L_2(\Omega, \Lambda)$ where d and δ are operators in $L_2(\Omega, \Lambda)$, and that the bounded invertible ω -accretive operator $\mathcal{B}$ is already defined on $L_p(\Omega, \Lambda)$, where $0 \leq \omega < \frac{\pi}{2}$. We assume henceforth that length$(\Omega) \geq 1$. The results remain true for a short interval Ω , though the constants in some of the estimates may then depend on its length.

Our aim is to establish the following result, in which $W_p^1(\Omega)$ denotes the Sobolev space $W_p^1(\Omega) = \{ u \in L_p(\Omega) : \frac{du}{dx} \in L_p(\Omega) \}$ with norm $\| u \|_{W_p^1} = \| \frac{du}{dx} \|_p + \| u \|_p$.

Theorem 7.1. *Let* $1 < p < \infty$. *Then* $\{ u \in \mathcal{V} : L^{\frac{1}{2}} u \in L_p(\Omega) \} = \mathcal{V} \cap W_p^1(\Omega)$ *with* $\left\| L^{\frac{1}{2}} u \right\|_p \approx \| u \|_{W_p^1}$.

A related result of independent interest is the following.

Theorem 7.2. *Let* $1 < p < \infty$ *and* $\mu > \omega$. *For each* $f \in H_\infty(S^0_{2\mu+})$,

$$f(L) : L_p \cap L_2(\Omega) \to L_p \cap L_2(\Omega) \quad \textit{with} \quad \| f(T)u \|_p \ \leq \ c_{p,\mu} \| f \|_\infty \| u \|_p \ .$$

These results follow from L_p estimates for functions of T .

Theorem 7.3. *Let* $1 < p < \infty$ *and* $\mu > \omega$. *For each* $f \in H_\infty(S^0_{\mu})$,

$$f(T) : L_p \cap L_2(\Omega, \Lambda) \to L_p \cap L_2(\Omega, \Lambda) \quad \textit{with} \quad \| f(T)u \|_p \ \leq \ c_{p,\mu} \| f \|_\infty \| u \|_p \ .$$

Before proving Theorem 7.3, we show that it implies Theorems 7.1 and 7.2.

Proof of Theorem 7.1. It follows from Theorem 7.3 that $\operatorname{sgn}(T)$ maps $L_p \cap L_2(\Omega, \Lambda)$ to itself with $\| \operatorname{sgn}(T)u \|_p \leq c \| u \|_p$. Also $(\operatorname{sgn}(T))^{-1} = \operatorname{sgn}(T)$ has the same property, so in fact $\operatorname{sgn}(T)$ is a one–one mapping of $L_p \cap L_2(\Omega, \Lambda)$ onto itself with $\| \operatorname{sgn}(T)u \|_p \approx \| u \|_p$.

Let $u \in \mathcal{V} \subset L_2(\Omega, \Lambda^0)$. Then $u \in W_p^1(\Omega)$ if and only if $Tu = du \in L_p(\Omega, \Lambda)$, which holds if and only if $L^{\frac{1}{2}}u = |T|_* u = \operatorname{sgn}(T)Tu \in L_p(\Omega, \Lambda)$. Moreover $\left\| L^{\frac{1}{2}} u \right\|_p \approx \| Tu \|_p \approx \| u \|_{W_p^1}$. $\square$

Proof of Theorem 7.2. Given $f \in H_\infty(S^0_{2\mu+})$, define $g \in H_\infty(S^0_\mu)$ by $g(\zeta) = f(\zeta^2)$. Then $f(L)u = f(T^2)u = g(T)u$ for all $u \in L_2(\Omega, \Lambda^0)$. The result follows from Theorem 7.3. $\square$

Let us turn to the proof of Theorem 7.3. For this purpose, we derive bounds on the Green's function of T . In the next two results, $L_p(\Omega, \Lambda)$ is abbreviated to $L_p(\Omega)$.

Proposition 7.4. *The operator $T = d + \delta_{\mathcal{B}}$ satisfies the following properties.*
(i) $\mathcal{D}(T) \subset L_\infty(\Omega)$ with

$$\| u \|_\infty \;\le\; c\left\{(1+|\zeta|)^{\frac{1}{2}} \| u \|_2 + (1+|\zeta|)^{-\frac{1}{2}} \| (T-\zeta I)u \|_2 \right\}$$

for all $\zeta \in \mathbb{C}$.

Henceforth suppose that $\omega < \mu < \frac{\pi}{2}$ and that $\zeta \notin S_\mu$ or that $\zeta = 0$. (The constants depend on μ but not on ζ itself.)

(ii)	*$(T-\zeta I)$ is a one–one mapping of*	*$\mathcal{D}(T)$*	*onto $L_2(\Omega)$*
		with	$\| u \|_2 \le c(1+|\zeta|)^{-1} \| (T-\zeta I)u \|_2$
(iii)	$(T-\zeta I)^{-1} : L_2(\Omega) \to L_\infty(\Omega)$	*with*	$\| u \|_\infty \le c(1+|\zeta|)^{-\frac{1}{2}} \| (T-\zeta I)u \|_2$
(iv)	$(T-\zeta I)^{-1} : L_1 \cap L_2(\Omega) \to L_2(\Omega)$	*with*	$\| u \|_2 \le c(1+|\zeta|)^{-\frac{1}{2}} \| (T-\zeta I)u \|_1$
(v)	$(T-\zeta I)^{-1} : L_1 \cap L_2(\Omega) \to L_\infty(\Omega)$	*with*	$\| u \|_\infty \le c \, \| (T-\zeta I)u \|_1$

In particular, $T^{-1} : L_1 \cap L_2(\Omega) \to L_\infty(\Omega)$ with $\| u \|_\infty \le c \| Tu \|_1$.

Proof. (i) Let

$$u \;=\; (\, u_0 \,,\, \begin{bmatrix} w \\ v \end{bmatrix} \,,\, u_2 \,) \in \mathcal{D}(T) \subset L_2(\Omega)\ .$$

Recall that $\mathcal{B}$ is defined on $L_p(\Omega)$ by $\mathcal{B}(u_0, u_1, u_2) = (\frac{1}{a}u_0, Bu_1, fu_2)$ where

$$B(x) \;=\; \begin{bmatrix} b(x) & \alpha(x) \\ \beta(x) & \gamma(x) \end{bmatrix}$$

is a bounded invertible ω-accretive matrix, and a and f are bounded invertible ω-accretive functions. Let us write the inverse B^{-1} of B as

$$B^{-1}(x) \;=\; \begin{bmatrix} \tilde{\gamma}(x) & \tilde{\beta}(x) \\ \tilde{\alpha}(x) & \tilde{b}(x) \end{bmatrix}$$

and note that b and $\tilde{b}$ are invertible functions. Then

$$\begin{aligned}
(T-\zeta I)u &= (d+\mathcal{B}^{-1}\delta\mathcal{B}-\zeta I)u \\
&= \left(a\delta_{0,1}B\begin{bmatrix} w \\ v \end{bmatrix} - \zeta u_0\,,\ d_{1,0}u_0 + B^{-1}\delta_{1,2}(fu_2) - \zeta\begin{bmatrix} w \\ v \end{bmatrix}\,,\ d_{2,1}\begin{bmatrix} w \\ v \end{bmatrix} - \zeta u_2 \right) \\
&= \Big(-a(bw+\alpha v)' + a(\beta w+\gamma v) - \zeta u_0\,, \\
&\qquad \begin{bmatrix} u_0{}' + \tilde{\gamma}fu_2 + \tilde{\beta}(fu_2)' - \zeta w \\ u_0 + \tilde{\alpha}fu_2 + \tilde{b}(fu_2)' - \zeta v \end{bmatrix}\,,\ w - v' - \zeta u_2 \Big) \in L_2(\Omega)
\end{aligned}$$

and so

$$\begin{cases}
(bw+\alpha v)' = (\beta w+\gamma v) - \zeta\frac{1}{a}u_0 - \frac{1}{a}((T-\zeta I)u)_0 \\
\begin{bmatrix} u_0{}' + \tilde{\beta}(fu_2)' \\ \tilde{b}(fu_2)' \end{bmatrix} = -\begin{bmatrix} \tilde{\gamma}fu_2 - \zeta w \\ u_0 + \tilde{\alpha}fu_2 - \zeta v \end{bmatrix} + ((T-\zeta I)u)_1 \\
v' = w - \zeta u_2 - ((T-\zeta I)u)_2\ .
\end{cases}$$

Therefore, applying Lemma 5.4 with $g=0$, we obtain $bw+\alpha v,\ fu_2,\ u_0,\ v\in C(\Omega)$ with

$$\begin{aligned}
&\|\, bw+\alpha v\,\|_\infty + \|\, fu_2\,\|_\infty + \|\, u_0\,\|_\infty + \|\, v\,\|_\infty \\
&\qquad \le \frac{c}{\sqrt{\kappa}}\|\, u\,\|_2 + c\sqrt{\kappa}\,\{(1+|\zeta|)\,\|\, u\,\|_2 + \|\,(T-\zeta I)u\,\|_2\}
\end{aligned}$$

for all $\kappa\le 1$. Hence, on choosing $\kappa=(1+|\zeta|)^{-1}$, we conclude that $u\in L_\infty(\Omega)$ and

$$\|\, u\,\|_\infty \le c\left\{(1+|\zeta|)^{\frac{1}{2}}\,\|\, u\,\|_2 + (1+|\zeta|)^{-\frac{1}{2}}\,\|\,(T-\zeta I)u\,\|_2\right\}\ .$$

(ii) We saw in Section 3 that T is a one–one operator of type S_ω in $L_2(\Omega)$, so we know the result when $\zeta\neq 0$ with the estimate $\|\, u\,\|_2\le c\,|\zeta|^{-1}\,\|\,(T-\zeta I)u\,\|_2$. We also proved that $T=VDW$ where V and W are isomorphisms and D is a one–one mapping of $\mathcal{D}(D)$ onto $L_2(\Omega)$ with $\|\, Du\,\|_2\ge\|\, u\,\|_2$, so $0\in\rho(T)$. Since the resolvent set is open, the estimate $\|\, u\,\|_2\le c\,\|\,(T-\zeta I)u\,\|_2$ holds for $|\zeta|$ small enough. The result follows.

Part (iii) is a consequence of (i) and (ii). Part (iv) follows by duality. To prove (v), use the same formula as in (i), this time applying Lemma 5.4 successively with $g=\frac{1}{a}((T-\zeta I)u)_0$, $\frac{1}{\tilde{b}}\begin{bmatrix} \tilde{b} & -\tilde{\beta} \\ 0 & 1 \end{bmatrix}((T-\zeta I)u)_1$, $((T-\zeta I)u)_2\in L_1(\Omega)$, and making use of (iv). $\square$

It is a consequence of (v) that $T-\zeta I$ has a Green's function $G_\zeta(x,y)\in L_\infty(\Omega\times\Omega)$. That is, $(T-\zeta I)^{-1}u(x)=\int_\Omega G_\zeta(x,y)u(y)\,dy$ for all $u\in L_1\cap L_2(\Omega)$ and almost all $x\in\Omega$.

The Green's function has exponential decay with respect to the metric ρ on Ω defined as follows. Set $\rho(x,y)=|x-y|$ in Case I specified at the end of Section 6, and set $\rho(x,y)=\min\{\,|x-y|\,,\ x_2-x_1-|x-y|\,\}$ in Case II.

Proposition 7.5. *If $\zeta \notin S_\mu$ or if $\zeta = 0$, then the Green's function $G_\zeta(x,y)$ of $T-\zeta I$ satisfies*

$$|G_\zeta(x,y)| \ \le \ ce^{-C(1+|\zeta|)\rho(x,y)}$$

for some c, $C>0$ and almost all x, $y \in \Omega$. Therefore

$$(T-\zeta I)^{-1} : L_p \cap L_2(\Omega) \to L_p(\Omega) \quad \text{with} \quad \| u \|_p \ \le \ \frac{c_\mu}{1+|\zeta|} \, \| (T-\zeta I)u \|_p$$

whenever $1 \le p \le \infty$.

Proof. We follow the method of Davies. Define a real valued C^1 function to be *admissible* if its support is a compact subset of the closed interval $\overline{\Omega}$ and, in Case II, it has the additional property that $\phi(x_1) = \phi(x_2)$. For such a ϕ, denote by e^ϕ the operator on $L_p(\Omega)$ defined by $(e^\phi)u(x) = e^{\phi(x)}u(x)$, and note that it, along with its inverse $(e^\phi)^{-1} = e^{-\phi}$, maps each of the spaces $L_p(\Omega)$, $W_p^1(\Omega)$ and $\mathcal{D}(T)$ to itself.

Set $T_\phi = e^{-\phi} T e^\phi$ with $\mathcal{D}(T_\phi) = \mathcal{D}(T)$. A simple calculation using $e^{-\phi} \frac{d}{dx}(e^\phi u) = \frac{du}{dx} + \phi' u$ shows that $T_\phi = T + M$ where M denotes multiplication by a matrix M with $\| M \|_\infty \le c \| \phi' \|_\infty$.

Suppose that $\| \phi' \|_\infty \le 2C(1+|\zeta|)$ where the constant C will be chosen shortly. All of the statements in Proposition 7.4 remain true when T is replaced by $T_\phi = T+M$ provided (ii) does, namely $(T_\phi - \zeta I)$ is a one–one mapping of $\mathcal{D}(T)$ onto $L_2(\Omega)$ with

$$(\#) \qquad \| u \|_2 \le c(1+|\zeta|)^{-1} \| (T_\phi - \zeta I)u \|_2 \ .$$

If $\| \phi' \|_\infty$ and $|\zeta|$ are both small enough, then $T_\phi - \zeta I$ is invertible, because it is a small perturbation of the invertible operator T, and thus $(\#)$ holds.

For larger values of $|\zeta|$ write

$$T_\phi - \zeta I \ = \ V(DU + V^{-1}MV - \zeta I)V^{-1} \ .$$

Although T_ϕ is not of type S_ω, it nevertheless follows from Lemma 5.7 that $(\#)$ holds provided C is chosen suitably.

Therefore the kernel $e^{-\phi(x)} G_\zeta(x,y) e^{\phi(y)}$ of T_ϕ is bounded. That is,

$$|G_\zeta(x,y)| \ \le \ ce^{\phi(x)-\phi(y)} \ .$$

For each fixed x, y and ζ it is possible to choose an admissible ϕ such that $\phi(x) - \phi(y) = -C(1+|\zeta|)\rho(x,y)$. Therefore $|G_\zeta(x,y)| \le ce^{-C(1+|\zeta|)\rho(x,y)}$ as required. □

We come now to the proof of Theorem 7.3. The usual Calderón–Zygmund theory does not apply owing to the lack of Hölder bounds on G_ζ. Nevertheless we can proceed without them by applying the following result of Duong and Robinson. Let us state a variant of a special case of Theorem 3.1 of [DR].

Theorem 7.6. *Let T be a one–one operator of type S_ω in $L_2(X)$ where (X,ρ,m) is a space of homogeneous type (with the doubling property) with metric ρ and measure m. Suppose that T has a bounded H_∞ functional calculus in $L_2(X)$, that $\omega<\nu<\mu<\frac{\pi}{2}$, and that*

$$|G_\zeta(x,y)| \;\le\; ce^{-C|\zeta|\rho(x,y)}$$

for all $\zeta\notin S^0_\nu$ and almost all $x,\,y\in X$. If $1<p<\infty$, then

$$f(T): L_p\cap L_2(X)\to L_p\cap L_2(X) \quad \textit{with} \quad \|\, f(T)u\,\|_p \;\le\; c_p\,\|\, f\,\|_\infty\,\|\, u\,\|_p$$

for all $f\in H_\infty(S^0_\mu)$.

Proof of Theorem 7.3. This follows from the above result, since T has a bounded H_∞ functional calculus in $L_2(\Omega)$ (Theorem 6.4) and its Green's function satisfies suitable bounds (Proposition 7.5). $\square$

Remark. Extend the operators $f(T)$ in Theorem 7.3 to operators $f(T)_{(p)}\in\mathcal{L}(L_p(\Omega,\Lambda))$ when $1<p<\infty$. Then $f(T)_{(p)}=f(T_{(p)})$ where the operator $T_{(p)}$ is defined on the subspace of $W^1_p(\Omega,\Lambda)$ determined by the same boundary conditions as those determining $\mathcal{D}(T)\subset W^1_2(\Omega,\Lambda)=H^1(\Omega,\Lambda)$. Thus $T_{(p)}$ is a one–one operator of type S_ω in $L_p(\Omega,\Lambda)$ which has a bounded $H_\infty(S^0_\mu)$ functional calculus for all $\mu>\omega$. Moreover $L_{(p)}={T_{(p)}}^2|_{L_p(\Omega,\Lambda^0)}$ is a one–one operator of type 2ω in $L_p(\Omega)$ with the bounded $H_\infty(S^0_{2\mu+})$ functional calculus defined by $f(L_{(p)})u=f(L)u$ for all $f\in H_\infty(S^0_{2\mu+})$ when $u\in L_p\cap L_2(\Omega)$. Details are left to the reader.

Similar comments apply to the operators $f(T)$ in Theorem 5.3.

§8. The Green's function of L.

Let us turn to some results of independent interest.

Proposition 8.1. *The operators $E,\,V,\,W,\,U,\,E_{\mathcal{B}},\,V_{\mathcal{B}}=V^{-1},\,W_{\mathcal{B}}=W^{-1}$ introduced in Section 6 all map $L_p\cap L_2(\Omega,\Lambda)$ to $L_p(\Omega,\Lambda)$ with*

$$\max\Big\{\,\|\,Eu\,\|_p\,,\,\|\,Vu\,\|_p\,,\,\|\,Wu\,\|_p\,,\,\|\,Uu\,\|_p\,,\,\|\,E_{\mathcal{B}}u\,\|_p\,,\,\|\,V_{\mathcal{B}}u\,\|_p\,,\,\|\,W_{\mathcal{B}}u\,\|_p\Big\}\;\le\; c\,\|\,u\,\|_p$$

for $1\le p\le\infty$.

Proof. Let us introduce another projection on $L_p(\Omega,\Lambda)$, namely

$P_{\mathcal{B}}(u_0, \begin{bmatrix} w \\ v \end{bmatrix}, u_2) = (0, \begin{bmatrix} w + \frac{\alpha}{b}v \\ 0 \end{bmatrix}, u_2)$. Then

$$\begin{aligned}
T(E_{\mathcal{B}} - P_{\mathcal{B}})u &= \delta_{\mathcal{B}}(I - P_{\mathcal{B}})u \;-\; dP_{\mathcal{B}}u \\
&= \mathcal{B}^{-1}\delta\mathcal{B}(u_0, \begin{bmatrix} -\frac{\alpha}{b}v \\ v \end{bmatrix}, 0) \;-\; d(0, \begin{bmatrix} w + \frac{\alpha}{b}v \\ 0 \end{bmatrix}, u_2) \\
&= \mathcal{B}^{-1}\delta(\tfrac{1}{a}u_0, \begin{bmatrix} 0 \\ (-\frac{\alpha\beta}{b} + \gamma)v \end{bmatrix}, 0) \;-\; (0, 0, w + \tfrac{\alpha}{b}v) \\
&= \mathcal{B}^{-1}((-\tfrac{\alpha\beta}{b} + \gamma)v, 0, 0) \;-\; (0, 0, w + \tfrac{\alpha}{b}v) \\
&= (\tfrac{a}{b}(b\gamma - \alpha\beta)v, 0, -w - \tfrac{\alpha}{b}v) \qquad \text{so that} \\
E_{\mathcal{B}} &= P_{\mathcal{B}} \;+\; T^{-1}M
\end{aligned}$$

where M denotes multiplication by an L_∞ matrix. Thus, on applying Proposition 7.5 with $\zeta = 0$, we have the required result for $E_{\mathcal{B}}$, and hence for $V_{\mathcal{B}}$ and $W_{\mathcal{B}}$. The result for the remaining operators is the special case $\mathcal{B} = I$. $\square$

Theorem 8.2. *Let L be the operator defined in Section 6. Then, for $1 \le p \le \infty$,*

$$\begin{aligned}
L^{-1} : L_p \cap L_2(\Omega) &\to L_p(\Omega) \quad \textit{with} \quad \|L^{-1}u\|_p \le c\|u\|_p \;; \\
\tfrac{d}{dx}L^{-1} : L_p \cap L_2(\Omega) &\to L_p(\Omega) \quad \textit{with} \quad \|\tfrac{d}{dx}(L^{-1}u)\|_p \le c\|u\|_p \;; \\
L^{-1}a\tfrac{d}{dx} : L_p(\Omega) \cap \mathcal{W} &\to L_p(\Omega) \quad \textit{with} \quad \|L^{-1}a\tfrac{du}{dx}\|_p \le c\|u\|_p \;; \\
\tfrac{d}{dx}L^{-1}a\tfrac{d}{dx} : L_p(\Omega) \cap \mathcal{W} &\to L_p(\Omega) \quad \textit{with} \quad \|\tfrac{d}{dx}(L^{-1}a\tfrac{du}{dx})\|_p \le c\|u\|_p \;.
\end{aligned}$$

Further,

$$\begin{aligned}
L^{-1} : L_1 \cap L_2(\Omega) &\to L_\infty(\Omega) \quad \textit{with} \quad \|L^{-1}u\|_\infty \le c\|u\|_1 \;; \\
\tfrac{d}{dx}L^{-1} : L_1 \cap L_2(\Omega) &\to L_\infty(\Omega) \quad \textit{with} \quad \|\tfrac{d}{dx}(L^{-1}u)\|_\infty \le c\|u\|_1 \;; \\
L^{-1}a\tfrac{d}{dx} : L_1(\Omega) \cap \mathcal{W} &\to L_\infty(\Omega) \quad \textit{with} \quad \|L^{-1}a\tfrac{du}{dx}\|_\infty \le c\|u\|_1 \;; \\
\tfrac{d}{dx}L^{-1}a\tfrac{d}{dx} + \tfrac{1}{b}I : L_1(\Omega) \cap \mathcal{W} &\to L_\infty(\Omega) \quad \textit{with} \quad \|\tfrac{d}{dx}(L^{-1}a\tfrac{du}{dx}) + \tfrac{1}{b}u\|_\infty \le c\|u\|_1 \;.
\end{aligned}$$

Moreover L has a Green's function $g(x,y)$ which satisfies

$$\sup\{|g(x,y)|, \left|\tfrac{\partial}{\partial x}g(x,y)\right|, \left|\tfrac{\partial}{\partial y}(g(x,y)a(y))\right|, \left|\tfrac{\partial^2}{\partial x\partial y}(g(x,y)a(y)) + \tfrac{1}{b}\delta(x-y)\right|\} \le ce^{-C\rho(x,y)}$$

for some constants c, $C > 0$

Proof. The projection $E_{\mathcal{B}}$ can be represented as $E_{\mathcal{B}}u = dT^{-2}\delta_{\mathcal{B}}u$ when $u \in \mathcal{D}(\delta_{\mathcal{B}})$. That is, $dT^{-2}\mathcal{B}^{-1}\delta = E_{\mathcal{B}}\mathcal{B}^{-1}|_{\mathcal{D}(\delta)}$. Therefore, by the previous result,

$$dT^{-2}\mathcal{B}^{-1}\delta : L_p(\Omega, \Lambda) \cap \mathcal{D}(\delta) \to L_p(\Omega, \Lambda) \quad \text{with} \quad \|dT^{-2}\mathcal{B}^{-1}\delta u\|_p \le c\|u\|_p \;.$$

In particular,

$$\left\| \begin{bmatrix} \frac{d}{dx} \\ I \end{bmatrix} L^{-1} a \left[-\frac{d}{dx} \quad I \right] \begin{bmatrix} w \\ v \end{bmatrix} \right\|_p \leq c \left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_p$$

for all $w \in L_p(\Omega) \cap \mathcal{W}$ and $v \in L_p \cap L_2(\Omega)$. The L_p estimates follow.

A little more care needs to be taken with the next set of estimates because $P_{\mathcal{B}}$ does not map L_1 to L_∞. So here we use the facts that

$$\| (E_{\mathcal{B}}\mathcal{B}^{-1} - P_{\mathcal{B}}\mathcal{B}^{-1})u \|_\infty = \| T^{-1} M \mathcal{B}^{-1} u \|_\infty \leq c \| u \|_1$$

(which follows from Proposition 7.4) and that $P_{\mathcal{B}}\mathcal{B}^{-1}(u_0, \begin{bmatrix} w \\ v \end{bmatrix}, u_2) = (0, \begin{bmatrix} \frac{1}{b} w \\ 0 \end{bmatrix}, \frac{1}{f} u_2)$.

Similar estimates are satisfied by $L_\phi = e^{-\phi} L e^{\phi}$ where ϕ is an admissible function as defined in the proof of Proposition 7.5. The kernel estimates follow from this by again applying the method of Davies. See, for example, the appendix of [AM^cT]. □

This result was proved in [AM^cT] in the case when $\Omega = \mathbb{R}$ and $a = 1$. Though first order systems were not used in that work, nevertheless there are many features in common between the two approaches. The aim there was to prove heat kernel and resolvent bounds for the operator L in this case, along with higher dimensional results.

It may be of interest to continue this work and see whether all the one–dimensional results in [AM^cT] have analogues on bounded intervals, at least when $a = 1$.

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