Operator Theory - Spectra and Functional Calculi

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Introduction

The principal theme of this course concerns definitions and bounds on functions $f(T)$ of linear operators in Banach spaces $\mathcal{X}$, in particular in Hilbert spaces. Ideally the bounds would be of the form $\|f(T)\| \lesssim \|f\|_\infty$, or better still $\|f(T)\| \le \|f\|_\infty$. The latter happens when T is a self-adjoint operator in a Hilbert space and f is a Borel measurable function on the real line. For example if $\mathcal{X} = \mathbf{C}^N$ and T is represented by the matrix

$$T = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix}, \text{ then } f(T) = \begin{pmatrix} f(\lambda_1) & & & 0 \\ & f(\lambda_2) & & \\ & & \ddots & \\ 0 & & & f(\lambda_N) \end{pmatrix}$$

and so $\|f(T)\| \le \max |f(\lambda_j)| \le \|f\|_\infty$. However such estimates cannot always be obtained, as is indicated by the example $T_n = \begin{pmatrix} \frac{1}{n} & 1 \\ 0 & \frac{1}{n} \end{pmatrix}$ and $f(\zeta) = \sqrt{\zeta}$, because $f(T_n) = {T_n}^{1/2} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{\sqrt{n}}{2} \\ 0 & \frac{1}{\sqrt{n}} \end{pmatrix}$ blows up as $n \to \infty$.

It is assumed that the reader has a basic knowledge of topology, metric spaces, Banach and Hilbert spaces, and measure theory. Suitable references for this material are the books *Real and Complex Analysis* by W. Rudin, *Real Analysis* by H.L. Royden [Rud87], *Introduction to Topology and Modern Analysis* by G.F. Simmons [Sim83], *Functional Analysis* by F. Riesz and B. Sz.-Nagy [RSN90], and *Linear Operators, Part I, General Theory* by N. Dunford and J.T. Schwartz [DS88]. Later, we may also expect some knowledge of Fourier theory and partial differential equations.

Useful books include: Tosio Kato, *Perturbation theory for linear operators* [Kat76]; Paul Halmos, *Introduction to Hilbert space* [Hal98]; Edgar Lorch, *Spectral theory* [Lor62]; Michael Reed and Barry Simon, *Methods of modern mathematical physics. I. Functional analysis* [RS72].

Parts of these lectures are based on the lecture notes *Operator theory and harmonic analysis* by David Albrecht, Xuan Duong and Alan McIntosh [ADM96], which are in turn based on notes taken, edited, typed and refined by Ian Doust and Elizabeth Mansfield, whose willing assistance is gratefully acknowledged.

Overview and Motivation

Let $T \in \mathcal{L}(X)$. We construct a functional calculus which is an algebra homomorphism $\Phi_T : \mathsf{H}(\Omega) \longrightarrow \mathcal{L}(X)$. Here, $\sigma(T) \subset \Omega$, and Ω is open in $\mathbf{C}$. The functional calculus satisfies the following properties:

$$f \mapsto f(T)$$
$$1 \mapsto I$$
$$\mathrm{id} \mapsto T$$

In addition, if the spectrum of T breaks up into two disjoint compact components $\sigma_+(T), \sigma_-(T)$, we want $\chi_+ \mapsto P_+ := \chi_+(T)$ and $\chi_- \mapsto P_- := \chi_-(T)$. In this case, we can write $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$, where $\mathcal{X}_\pm = \mathsf{R}(P_\pm)$. This is the background setting. Naturally, we want various generalisations of this.

The first is to consider the situation when we have some $\lambda \in \sigma(T)$, and $\sigma(T) \setminus \{\lambda\} \subset \Omega$. We want to find a suitable class of operators T to define $f(T) \in \mathcal{L}(X)$ when $f \in \mathsf{H}(\Omega)$ such that $\| f(T) \| \leq C \| f \|_\infty$.

In analogy to the earlier situation, now suppose that $\sigma(T) \setminus \{\lambda\}$ breaks up into two compact sets $\sigma_+(T), \sigma_-(T)$. Then, if $P_\pm$ are bounded, then as before, we have $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$.

Another natural generalisation is to relax the condition that $T \in \mathcal{L}(X)$. For closed operators, we find $\sigma(T)$ is a compact subset of the extended complex plane $\mathbf{C}_\infty$. We want to find suitable unbounded T for each of the following situations:

1. $\sigma(T) \subset \Omega$
2. $\sigma(T) \setminus \{\infty\} \subset \Omega$
3. $\sigma(T) \setminus \{0, \infty\} \subset \Omega$.
4. $\sigma(T) \setminus \{0, \infty\}$ breaks into two components, each contained in a sector on the left and right half planes of $\mathbf{C}$.

The last is a particularly important case, as illustrated by the following example.

Let $\mathcal{X} = L^2(\mathbf{R})$, and let $D = T = \frac{1}{i}\frac{d}{dx}$. This is a unbounded, self adjoint operator $\sigma(D) = \mathbf{R} \cup \{\infty\}$. As before, we can write $\sigma_-(T), \sigma_+(T)$ respectively for the left and right half planes and:

$$\| P_{\pm} \| = \| \chi_{\pm}(D) \| = \| \chi_{\pm} \|_{\infty} = 1$$

Let,

$$\operatorname{sgn}(\lambda) = \begin{cases} -1 & \operatorname{Re}\lambda > 0 \\ 0 & \operatorname{Re}\lambda = 0 \\ +1 & \operatorname{Re}\lambda < 0 \end{cases}$$

Observe that $(\operatorname{sgn}\lambda)\sqrt{\lambda^2} = \lambda$, and $\operatorname{sgn}(\lambda)\lambda = \sqrt{\lambda^2}$. Also, $\operatorname{sgn}(D) = \chi_+(D) - \chi_-(D)$ and $\| \operatorname{sgn}(D) \| = 1$. Here, $H := \operatorname{sgn}(D)$ is the Hilbert transform, and $\sqrt{D^2} = HD, H\sqrt{D^2} = D$.

We can also consider replacing $L^2(\mathbf{R})$ by $L^p(\mathbf{R})$. In fact, we can find that $H = \operatorname{sgn}(D) = \chi_+(D) + \chi_-(D)$ is bounded. The self adjoint theory no longer applies to access such results and instead, dyadic decompositions near 0 and ∞ are needed. These are the beginnings of Harmonic Analysis.

We also want to look for n dimensional analogues. The setting is as follows. Let $\mathcal{H} = L^2(\mathbf{R}^n) \oplus L^2(\mathbf{R}^n, \mathbf{C}^n)$. Here, we have $\nabla : L^2(\mathbf{R}^n) \longrightarrow L^2(\mathbf{R}^n, \mathbf{C}^n)$ and $\operatorname{div} : L^2(\mathbf{R}^n, \mathbf{C}^n) \longrightarrow L^2(\mathbf{R}^n)$. In fact,

$$\langle -\nabla f, u \rangle = \int \sum_j \frac{\partial f}{\partial x_j} \overline{u}_j = \int f \sum \frac{\partial \overline{u}_j}{\partial x_j} = \langle f, \operatorname{div} u \rangle$$

So, each operator is dual to the negative of the other.

Let $A = (A_{jk})_{j,k=1}^n$ where $A_{jk} \in L^\infty(\mathbf{R}^n)$. Also we assume the following ellipticity condition: $\zeta \in \mathbf{C}^n$, and for almost all $x \in \mathbf{R}^n$,

$$\operatorname{Re} \sum A_{jk} \zeta_j \overline{\zeta}_k \geq |x| |\zeta|^2$$

Define the operator $T : \mathcal{H} \to \mathcal{H}$:

$$T = \begin{pmatrix} 0 & \operatorname{div} A \\ -\nabla & 0 \end{pmatrix}$$

We find that $\sigma(T) \subset \{0, \infty\} \cup S^o_{\omega+} \cup S^o_{\omega-}$ where $S^o_{\omega+} = \{\zeta \in \mathbf{C} \setminus \{0\} : |\arg \zeta| < \omega\}$, $S^o_{\omega-} = -S^o_{\omega+}$ and we can formulate a generalised Hilbert transform as $H = \operatorname{sgn}(T)$. A recent deep theorem then gives: T has a bounded functional calculus and in particular, $\operatorname{sgn}(T)$ is bounded.

As a consequence, we have:

$$\begin{aligned} \sqrt{T^2} &= \operatorname{sgn}(T) T \\ T &= \operatorname{sgn}(T) \sqrt{T^2} \end{aligned}$$

and $\left\| \sqrt{T^2} u \right\| \simeq \| Tu \|$.

Furthermore, observe that:

$$\sqrt{T^2} = \begin{pmatrix} \sqrt{-\operatorname{div} A\nabla} & 0 \\ 0 & \sqrt{-\nabla \operatorname{div} A} \end{pmatrix}$$

$$\left\| \begin{pmatrix} \sqrt{-\operatorname{div} A\nabla} & 0 \\ 0 & \sqrt{-\nabla \operatorname{div} A} \end{pmatrix} \begin{pmatrix} f \\ u \end{pmatrix} \right\| \simeq \left\| \begin{pmatrix} 0 & \operatorname{div} A \\ -\nabla & 0 \end{pmatrix} \begin{pmatrix} f \\ u \end{pmatrix} \right\|$$

Setting $u = 0$ we find:

$$\left\| \sqrt{-\operatorname{div} A\nabla} \right\| \simeq \| \nabla f \|$$

This is the Kato Square Root problem [AHL+02]. See [AMK] and [AM] for further details.

Chapter 1

Preliminaries

1.1 Banach Spaces

We work in a Banach Space. This is a normed linear space that is complete. More formally,

Definition 1.1.1 (Banach Space (over $\mathbf{C}$))**.** We say that $\mathcal{X}$ is a Banach space if:

(i) $\mathcal{X}$ is a linear space over $\mathbf{C}$.

(ii) There is a norm $\|\cdot\| : \mathcal{X} \to \mathbf{R}$. That is:

   (a) $\| u \| = 0 \implies u = 0$

   (b) $\| u + v \| \leq \| u \| + \| v \|$ *(Triangle Inequality)*

   (c) $\| \alpha u \| = |\alpha| \, \| u \|$

   For all $\alpha \in \mathbf{C}$ and $u, v \in \mathcal{X}$.

(iii) $\mathcal{X}$ is complete in the metric $\rho(u, v) = \| u - v \|$, or in other words every Cauchy sequence (x_n) in $\mathcal{X}$ converges to some $x \in \mathcal{X}$. That is, if $\rho(x_n, x_m) \to 0$, there exists an $x \in \mathcal{X}$ such that $\rho(x_n, x) \to 0$.

Exercise 1.1.2. Show that $\mathbf{C} \times \mathbf{C} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ given by $(\alpha, \beta, u, v) \mapsto \alpha u + \beta v$ is continuous.

Since a Banach space has a vector space structure, we can talk about infinite series in the same way we treat infinite series of complex numbers.

Definition 1.1.3 (Infinite Series)**.** Let $(x_n)_{i=1}^{\infty}$ be a sequence in $\mathcal{X}$. We define partial sums $S_n := \sum_{i=1}^{n} x_n$. If there is an $S \in \mathcal{X}$ such that $\lim_{n\to\infty} S_n = S$, then we write:

$$S = \sum_{j=1}^{\infty} x_j$$

and we say that the infinite series $\sum_{i=1}^{\infty} x_n$ converges.

Definition 1.1.4 (Absolutely Convergent)**.** We say that a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \| x_n \|$ is convergent.

Proposition 1.1.5. *Every absolutely convergent series is convergent.*

Proof. We show that the partial sums S_n are Cauchy. Let $n > m$. Then by the triangle inequality,

$$\| S_n - S_m \| = \left\| \sum_{j=m+1}^{n} x_j \right\| \leq \sum_{j=m+1}^{n} \| x_j \|$$

The right hand side tends to 0 as $m, n \to \infty$ since $\sum_{j=1}^{m} \| x_j \|$ is convergent and consequently Cauchy. $\square$

Example 1.1.6. (i) $\mathbf{C}^n$ with

$$\| u \|_p = \begin{cases} \sum (|u_j|^p)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup |u_j| = \max |u_j| & p = \infty \end{cases}$$

These norms are all equivalent since $\mathbf{C}^n$ is finite dimensional.

(ii) $\ell^p = \left\{ u = (u_j)_{j=1}^{\infty} : \| u \|_p < \infty \right\}$, with $\| u \|_p$ the same as above. .

(iii) $L^p(\Omega) = \left\{ f : \Omega \to \mathbf{R} : \| f \|_p < \infty \right\}$, where Ω is a σ-finite measure space, where:

$$\| f \|_p = \begin{cases} \left(\int_\Omega |f|^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \text{ess sup}|f| & p = \infty \end{cases}$$

(iv) $C_b(M) = \{ f : M \to \mathbf{C} : \| f \|_\infty < \infty \}$, where $\| f \|_\infty = \sup_{x \in M} |f(x)|$.

(v) $\mathsf{H}^\infty(\Omega) = \{ f : \Omega \to \mathbf{C}$ bounded and holomorphic$\}$, where $\Omega \subset \mathbf{C}$ open and again equipped with $\| \cdot \|_\infty$.

Note that: $\mathsf{H}^\infty(\Omega) \subset C_b(\Omega) \subset L^\infty(\Omega)$.

(vi) $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the space of bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$ where $\mathcal{X}$ is normed and $\mathcal{Y}$ is a Banach Space. The latter condition is necessary and sufficient for $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ to be a Banach space. The norm here is the operator norm:

$$\| T \| = \sup_{x \neq 0} \frac{\| Tx \|_{\mathcal{Y}}}{\| x \|_{\mathcal{X}}} = \sup_{\| x \|_{\mathcal{X}} = 1} \| Tx \|_{\mathcal{Y}}$$

(vii) $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$.

1.2 Direct Sums of Banach Spaces

Given two Banach spaces, we can define the notion of a direct sum.

Definition 1.2.1 (Direct sum of Banach spaces)**.** Let $\mathcal{X}_1, \mathcal{X}_2$ be Banach Spaces. Define:

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 = \{(u_1, u_2) : u_i \in \mathcal{X}_i\}$$
$$\| u \| = \| (u_1, u_2) \| = \| u_1 \|_{\mathcal{X}_1} + \| u_2 \|_{\mathcal{X}_2}$$

Operators on the direct sum can be defined as follows.

Definition 1.2.2 (Direct sum of operators)**.** Let $T_j \in \mathcal{L}(\mathcal{X}_j)$, and $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$. Define $T \in \mathcal{L}(\mathcal{X})$:

$$Tu = (T_1 u_1, T_2 u_2)$$

We also write:

$$Tu = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Proposition 1.2.3.

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$$
$$f(T) = f(T_1) \oplus f(T_2)$$

Proof. Exercise. □

By induction, we can extend this to $\mathcal{X} = \bigoplus_{j=1}^{n} \mathcal{X}_j$.

Another notion of direct sums is required to write a Banach space as a sum of subsets.

Definition 1.2.4 (Banach space as a direct sum of subsets)**.** Let $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$. Then writing $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$ means that:

(i) $\mathcal{Y}, \mathcal{Z}$ are linear subspaces

(ii) $\mathcal{Y} \cap \mathcal{Z} = 0$.

(iii) For all $x \in \mathcal{X}$, there exists $y \in \mathcal{Y}$, $z \in \mathcal{Z}$ and a $C > 0$ such that $x = y + z$ with

$$\| y \| + \| z \| \leq C \| x \|$$

Remark 1.2.5. It is worth noting that we have no concept of orthogonality here.

The following consequences are immediate.

Proposition 1.2.6. *(i)* $\| x \| \simeq \| y \| + \| z \|$

(ii) The decomposition is unique

(iii) $\mathcal{Y}, \mathcal{Z}$ are closed

Definition 1.2.7 (Projection)**.** We say $P \in \mathcal{L}(\mathcal{X})$, is a projection if $P^2 = P$.

Proposition 1.2.8. *Given $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$ there exist a projection $P \in \mathcal{L}(\mathcal{X})$ with $\mathsf{R}(P) = \mathcal{Y}$ and $\mathsf{N}(P) = \mathsf{R}(I - P) = \mathcal{Z}$. Conversely given a projection $P \in \mathcal{L}(X)$ we can decompose $\mathcal{X} = \mathsf{N}(P) \oplus \mathsf{R}(P)$.*

This extends to the situation $\mathcal{X} = \bigoplus_{j=1}^{n} \mathcal{X}_j$. We get projections $P_j \in \mathcal{L}(X)$ with $\mathsf{R}(P_i) = \mathcal{X}_i$, $I = \sum_{j=1}^{n} P_j$, $P_j^2 = P_j$ and $P_j P_k = 0$ when $j \neq k$.

1.3 Integration of Banach Valued Functions

As in the case of $\mathbf{R}$, we can define various notions of integration. The following will be sufficient for our purposes.

Definition 1.3.1 (Definite (Riemann) integral of Banach valued functions)**.** Let $f : [a, b] \to \mathcal{X}$ be a continuous function. Define $x_j = \frac{j(b-a)}{n}$ and:

$$\int_a^b f(\tau) d\tau = \lim_{n\to\infty} \sum_{j=1}^{n-1} (x_{j+1} - x_j) f(x_j) = \lim_{n\to\infty} \sum_{j=1}^{n-1} \frac{b-a}{n} f\left(\frac{j(b-a)}{n}\right)$$

Remark 1.3.2. The limit exists since $\| f \| : [a, b] \to \mathbf{R}$ is continuous and so any such partial sum is absolutely convergent.

Remark 1.3.3. It is also possible to easily generalise this to piecewise continuous functions as in the $\mathbf{R}$ case.

Proposition 1.3.4.

$$\left\| \int_a^b f(\tau) d\tau \right\| \leq \int_a^b \| f(\tau) \| d\tau$$

Just as in the real variable case, we also consider indefinite integrals.

Definition 1.3.5 (Indefinite Integral of Banach valued functions)**.** Let $f : (a, b] \to \mathcal{X}$ be continuous, where $-\infty \leq a < b < \infty$. Define:

$$\int_a^b f(\tau) d\tau = \lim_{a' \to a} \int_{a'}^b f(\tau) d\tau$$

whenever the limit exists.

Proposition 1.3.6. *If f is absolutely integrable, then f is integrable and*

$$\left\| \int_a^b f(\tau) d\tau \right\| \leq \int_a^b \| f(\tau) \| d\tau$$

Proof. Exercise. □

Similarly, it is useful to define the contour integral when the domain of a function is **C**.

Definition 1.3.7 (Contour integral of a Banach valued function)**.** Let $f : \Omega \to \mathcal{X}$ be a continuous function, and let $\gamma : [a, b] \to \Omega$ be a continuous curve. Define $x_j = \frac{j(b-a)}{n}$ and:

$$\oint_\gamma f(\zeta)d\zeta = \lim_{n\to\infty} \sum_{j=1}^{n-1} [\gamma(x_{j+1}) - \gamma(x_j)]\ f(x_j)$$

Remark 1.3.8. Again, this can be easily generalised to piecewise continuous curves.

Definition 1.3.9 (Piecewise C^k curve)**.** If $\gamma : [0, n+1] \to \mathbf{C}$ and $\gamma = \sum_{j=1}^{n} \gamma_j$, where each $\gamma_j \in C^k([j, j+1], \mathbf{C})$, then we say that γ is a piecewise C^k curve.

Definition 1.3.10 (Closed contour)**.** Let $\gamma : [0, 1] \to \mathbf{C}$ be a curve such that $\gamma(0) = \gamma(1)$. Then we say that γ is a closed contour.

1.4 Banach Algebras

Informally, a Banach algebra is a Banach space which has multiplication. It has a richer structure that will be useful to us.

Definition 1.4.1 (Banach Algebra)**.** Let $\mathcal{X}$ be a Banach space and further suppose that there is a map $\cdot : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $(u, v) \mapsto uv$ satisfying

(i) $u(vw) = (uv)w$ *(Associative)*

(ii) $u(v + w) = uv + uw$ *(Left distributive)*

(iii) $(v + w)u = vu + wu$ *(Right distributive)*

(iv) $\lambda(uv) = (\lambda u)v = u(\lambda v)$

(v) $\| uv \| \leq \| u \| \| v \|$

for all $u, v \in \mathcal{X}$ and $\lambda \in \mathbf{C}$.

Proposition 1.4.2. *(i) The map $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ which sends $(u, v) \mapsto uv$ is continuous.*

(ii) $\| u^k \| \leq \| u \|^k$.

Proof. Exercise. □

Definition 1.4.3 (Banach Algebra with identity)**.** If $\mathcal{X}$ is a Banach algebra and there exists an element $e \in \mathcal{X}$ such that $eu = ue = u$ for all $u \in \mathcal{X}$, then we call this element an identity (or unit) and the algebra a *Banach algebra with identity* or a *Unital Banach algebra.* Often, we denote the identity simply with 1 or I.

Definition 1.4.4 (Invertible)**.** Let $\mathcal{X}$ be a Banach algebra with identity. We say that an element $u \in \mathcal{X}$ is invertible if there exists $u^{-1} \in \mathcal{X}$ such that $uu^{-1} = 1 = u^{-1}u$. We call u^{-1} the inverse of u.

Many of the spaces that we have seen so far are Banach algebras.

Example 1.4.5. (i) $l^\infty, L^\infty(\Omega), C_b(M), \mathsf{H}^\infty(\Omega), \mathcal{L}(X)$ are all Banach algebras with identity

(ii) Let $C_0(\mathcal{X}) = \{f \in C_b(\mathcal{X}) : f(x) \to 0 \text{ as } \| x \| \to \infty\}$. This is a Banach algebra but it does not have an identity, since $f \equiv 1$ does not decay at ∞.

For a Banach algebra with identity, we can define polynomials in the algebra.

Definition 1.4.6. Let $\mathcal{X}$ be a Banach algebra with identity. For each polynomial $p(\zeta) = \sum_{j=0}^n \alpha_j \zeta^j$, $\alpha_j \in \mathbf{C}$ and $u \in \mathcal{X}$, define:

$$p(u) = \sum_{j=0}^n \alpha_n u^j, \alpha_n \in \mathbf{C}$$

Remark 1.4.7. Note that:

$$\left\| \sum_{i=0}^n \alpha_n u^k \right\| \leq \sum_{i=0}^n |\alpha_n| \left\| u^k \right\|$$

Recall that the following formal expression has a *radius of convergence* $R \in [0, \infty]$ for $\alpha_k \in \mathbf{C}, z \in \mathbf{C}$:

$$f(z) = \sum_{n=0}^\infty \alpha_k z^k$$

This series diverges for $|z| > R$ and converges absolutely when $|z| < R$.

Definition 1.4.8 (Power Series)**.** Suppose $f(z) = \sum_{n=0}^\infty \alpha_k z^k$ is a power series with a radius of convergence $R \in [0, \infty]$. Let $u \in \mathcal{X}$ with $\| u \| < R$. Then, we define:

$$f(u) := \sum_{n=0}^\infty \alpha_k u^k \in \mathcal{X}$$

We can think of the map $u \mapsto f(u)$ as a *functional calculus* of u as it is a Banach algebra homomorphism which sends a complex valued function to a Banach valued function.

In a Banach algebra with identity $\mathcal{X}$, we are also interested in invertibility and stability.

Proposition 1.4.9. *Suppose that $u \in \mathcal{X}$ and that $\| u \| < 1$. Then $(1 - u)$ is invertible and*

$$\| (1 - u)^{-1} \| \leq \frac{1}{1 - \|u\|} .$$

indeed, $(1 - u)^{-1} = \sum_{k=0}^\infty u^k$.

Proof. As $\| u \| < 1$ and $\|u^k\| \leq \|u\|^k$, then

$$\sum_{k=0}^\infty \|u\|^k = \frac{1}{1 - \| u \|} .$$

Thus $\sum_{k=0}^{\infty} u^k = w$ for some $w \in \mathcal{X}$. Now

$$\begin{aligned}(1-u)w &= \lim_{n\to\infty}(1-u)\sum_{k=0}^{n} u^k \qquad \text{(since } (1-u) \text{ is a continuous mapping)}\\ &= \lim_{n\to\infty}(1-u^{n+1})\\ &= 1\ .\end{aligned}$$

The proof that $w(1-u) = 1$ is similar. □

Theorem 1.4.10. *Suppose that $u \in \mathcal{X}$ is invertible and that $w \in \mathcal{X}$ satisfies*

$$\|\, w\,\| < \|u^{-1}\|^{-1}\ .$$

Then $u - w$ is invertible in $\mathcal{X}$ and

$$\left\|\,(u-w)^{-1}\,\right\| \le \frac{\|u^{-1}\|}{1-\|w\|\|u^{-1}\|}\ .$$

Proof. Write $u - w \ = \ (1 - wu^{-1})u$. Now $\|wu^{-1}\| \le \|w\|\|u^{-1}\| < 1$, and so $(1 - wu^{-1})$ is invertible with

$$\|(1-wu^{-1})^{-1}\| \ \le \ \frac{1}{1-\|w\|\|u^{-1}\|}\ .$$

1t follows that $u - w$ is invertible with

$$(u-w)^{-1} \ = \ u^{-1}(1-wu^{-1})^{-1} \ \in \ \mathcal{X}\ .$$

□

Corollary 1.4.11. *The set of invertible elements of $\mathcal{X}$ is an open set.*

1.5 Holomorphic Banach valued functions

We begin by defining a notion of holomorphic for Banach valued functions.

Definition 1.5.1 (Holormorphic)**.** Suppose that Ω is an open subset of $\mathbf{C}$ and that $f : \Omega \to \mathcal{X}$. Then we say that f is holomorphic (or differentiable) if for every $z \in \Omega$ there exists $f'(z) \in X$ such that

$$\left\| \frac{f(z+h)-f(z)}{h} - f'(z) \right\| \to 0 \quad \text{as } h \to 0\ ;$$

The set of all such functions are denoted by $\mathsf{H}(\Omega, \mathcal{X})$, and we write $\mathsf{H}(\Omega) := \mathsf{H}(\Omega, \mathbf{C})$.

The reason we call such functions holomorphic is justified in the following theorem.

Theorem 1.5.2. *Suppose that Ω is an open subset of $\mathbf{C}$ and that $f : \Omega \to \mathcal{X}$. Then the following are equivalent:*

(i) f is differentiable.

(ii) f is a continuously differentiable function of $(x, y) \in \mathbf{R}^2$ and f satisfies the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x}(x, y) = \frac{1}{\imath}\frac{\partial f}{\partial y}(x, y) \ ;$$

(iii) f is analytic in the sense that for all $z \in \Omega$, there exists $r > 0$ and $C_k \in X$ $(k = 0, 1, 2, \dots)$ such that

$$f(\zeta) = \sum_{k=0}^{\infty} (\zeta - z)^k C_k$$

for all ζ such that $|\zeta - z| < r$;

(iv) f is continuous and for all closed piecewise C^1 contours γ in Ω which are null-homotopic in Ω,

$$\oint_\gamma f(z)\, dz = 0$$

where the integral is defined in the Riemann sense as $\oint_\gamma f(z(\zeta)) \frac{dz}{d\zeta}(\zeta)\, d\zeta$.

We shall omit a proof of these equivalences. A suitable reference is Section III.14 of [DS88]. Much of the important results of the theory of a complex variable hold in this setting, including the following important theorem.

Theorem 1.5.3 (Cauchy's Theorem). *Let $f \in \mathcal{H}(\Omega, \mathcal{X})$ and $z \in \Omega$ and γ a closed contour in $\Omega \backslash \{z\}$ null-homotopic to a sufficiently small circle δ_r of radius r which are parametrised anticlockwise. Then,*

$$f(z) = \frac{1}{2\pi\imath} \oint_\gamma \frac{f(w)}{w - z} dw$$

Corollary 1.5.4. *Let $N \in \mathbf{Z}$. Then,*

$$\frac{1}{2\pi\imath} \oint_\gamma (w - z)^N dw = \begin{cases} 0 & N \neq -1 \\ 1 & N = -1 \end{cases}$$

Proof. (i) $N \geq 0$ is easy since $(w - z)^N$ is analytic.

(ii) $N = -1$, apply Theorem 1.5.3 to $f \equiv 1$.

(iii) $N < -1$, set $w - z = re^{\imath\theta}$, for a small circle δ_r. Then,

$$\begin{aligned} \oint_{\delta_r} (w - z)^N dw &= \int_0^{2\pi} r^N e^{\imath N\theta} \imath r e^{\imath\theta} d\theta \\ &= \int_0^{2\pi} \imath r^{N+1} e^{\imath(N+1)\theta} d\theta \\ &= \frac{r^{N+1}}{\imath(N+1)} (e^{\imath(N+1)2\pi} - e^{\imath 0}) \\ &= 0 \end{aligned}$$

□

Theorem 1.5.5 (Liouville's Theorem)**.** *Suppose that $f : \mathbf{C} \to X$ is holomorphic, and bounded in the sense that there exists $M > 0$ such that $\| f(z) \| < M$ for all $z \in \mathbf{C}$. Then f is constant.*

Definition 1.5.6 (Bounded holomorphic functions)**.** We denote the space of bounded holomorphic functions from Ω into $\mathcal{X}$ by $\mathsf{H}^\infty(\Omega, \mathcal{X})$.

Proposition 1.5.7. *$\mathsf{H}^\infty(\Omega, \mathcal{X})$ is a Banach space with respect to the $\| \cdot \|_\infty$ norm.*

Proof. Let $\{f_n\} \subset \mathsf{H}^\infty(\Omega, \mathcal{X}) \subset C_b(\Omega, \mathcal{X})$ be Cauchy. So, there exists $f \in C_b(\Omega, \mathcal{X})$ such that $\| f_n - f \|_\infty \to 0$. By the characterisation of differentiability, $\oint_\gamma f_n = 0$ for all n and all null homotopic curves γ in Ω. We can pass this limit through the integral sign to find $\oint_\gamma f = 0$ giving $f \in \mathsf{H}^\infty(\Omega, \mathcal{X})$. □

Proposition 1.5.8. *$f \in \mathsf{H}(\Omega, \mathcal{X}) \implies f$ infinitely differentiable.*

Proof. We use Cauchy's Theorem:

$$\begin{aligned} f(z) &= \frac{1}{2\pi\imath} \oint_\gamma \frac{f(w)}{w - z} dw \\ f'(z) &= \frac{1}{2\pi\imath} \oint_\gamma \frac{f(w)}{(w - z)^2} dw \\ \vdots &= \vdots \\ f^{(n)}(z) &= \frac{n!}{2\pi\imath} \oint_\gamma \frac{f(w)}{(w - z)^{n+1}} dw \end{aligned}$$

□

We'll now fix $\mathsf{H}(\Omega) = \mathsf{H}(\Omega, \mathbf{C})$ and leave it to the reader to question which results hold for $\mathsf{H}(\Omega, \mathcal{X})$.

Proposition 1.5.9. *Suppose $\lambda \in \Omega$ and $f \in \mathsf{H}(\Omega \setminus \{\lambda\})$ and $f : \Omega \to \mathbf{C}$ continuous. Then, $f \in \mathsf{H}(\Omega)$.*

Proof. Exercise. □

Corollary 1.5.10 (Factorisation)**.** *If $f \in \mathsf{H}(\Omega)$ and $\lambda \in \Omega$, then there exists $g \in \mathsf{H}(\Omega)$ such that:*

$$f(z) - f(\lambda) = (z - \lambda) g(z)$$

Proof. We define g:

$$g(z) := \begin{cases} \frac{f(z) - f(\lambda)}{z - \lambda}, z \neq \lambda \\ f'(z), z = \lambda \end{cases}$$

Then g is continuous on Ω and $g \in \mathsf{H}(\Omega \setminus \{\lambda\})$. The result follows from the previous proposition. □

Proposition 1.5.11. *$r \in H(\Omega)$ is rational if and only if poles of r lie outside Ω if and only if there exist polynomials p, q such that $r = \frac{p}{q}$ and q has no zeros in Ω.*

We also have the following important theorem:

Theorem 1.5.12 (Runge's Theorem)**.** *Let K be a compact subset of $\mathbf{C}$, and let E be a subset of $\mathbf{C}_\infty$ which contains at least one point of every component of $\mathbf{C}_\infty \setminus K$. Let $f \in H(\Omega)$ where Ω is an open set containing K. Then there exists a sequence of rational functions r_n with poles in E which converge uniformly to f on K.*

Runge also proved [Rem98], p292:

Theorem 1.5.13 (Approximation by rationals)**.** *Let Ω be an open subset of $\mathbf{C}$, and let $f \in H(\Omega)$. Then there exists a sequence of rational functions $r_j \in H(\Omega)$ such that $r_j \to f$ uniformly on all compact subsets of Ω.*

Chapter 2

Functional Calculus of bounded operators

2.1 Spectral Theory of bounded operators

Let $\mathcal{X}$ be a Banach space. In this section, we fix our attention to operators $T \in \mathcal{L}(\mathcal{X})$.

Definition 2.1.1 (Eigenvalue, Eigenvector)**.** We say that $\lambda \in \mathbf{C}$ is an *eigenvalue* of T is there exists $u \in \mathcal{X}$, $u \neq 0$ such that $Tu = \lambda u$. We call the corresponding u an *eigenvector*.

Definition 2.1.2 (Resolvent, Resolvent Set)**.** We write $\rho(T)$ for the set of all values $\zeta \in \mathbf{C}$ such that $(\zeta I - T)$ is one-one, onto and for which $\mathsf{R}_T(\zeta) = (\zeta I - T)^{-1} \in \mathcal{L}(X)$. The map $\mathsf{R}_T : \rho(T) \to \mathcal{L}(X)$ is called the *resolvent operator*.

Definition 2.1.3 (Spectrum)**.** We denote the spectrum of T by $\sigma(T) := \mathbf{C} \setminus \rho(T)$.

Remark 2.1.4. If λ is an eigenvalue for T, then $\lambda \in \sigma(T)$. Typically, $\sigma(T)$ is larger than the set of eigenvalues, except for the finite dimensional case of $\mathcal{X} = \mathbf{C}^N$.

Example 2.1.5. Let ℓ^p denote the Banach space of all sequences

$$x = (x_1, x_2, \ldots, x_n, \ldots)$$

of complex numbers, with finite norm

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} .$$

Given a sequence $d = (d_1, d_2, \ldots)$ such that $\sup_n |d_n| < \infty$, define the operator $D = \operatorname{diag}(d)$ on ℓ^p by $(Dx)_n = d_n x_n$ for every n. Then $D \in \mathcal{L}(\ell^p)$ and $\sigma(D)$ is the closure of the set $\{d_1, d_2, \ldots\}$ in $\mathbf{C}$. Moreover

$$\mathsf{R}_D(\zeta) = \operatorname{diag}\left(\frac{1}{\zeta - d_1}, \frac{1}{\zeta - d_2}, \ldots \right)$$

and

$$\| \mathsf{R}_D(\zeta) \| = \frac{1}{\zeta, \sigma(D)} = \sup_n \frac{1}{\operatorname{dist}(\zeta, d_n)}$$

for all $\zeta \in \rho(D)$.

Theorem 2.1.6 (Properties of the Resolvent)**.** *For all $\zeta, \mu \in \rho(T)$,*

(i) $R_T(\zeta)R_T(\mu) = R_T(\mu)R_T(\zeta)$*.*

(ii) $R_T(\zeta)T = TR_T(\zeta) = \zeta R_T(\zeta) - I$*.*

(iii) $R_T(\zeta) - R_T(\mu) \ = \ (\mu - \zeta)\, R_T(\zeta)\, R_T(\mu)$*. (the resolvent equation)*

(iv) If $|\zeta| > \| T \|$*, then* $\zeta \in \rho(T)$ *and* $R_T(\zeta) \to 0$ *as* $|\zeta| \to \infty$*.*

(v) $\rho(T)$ *is an open set and if* $\zeta \in \rho(T)$ *and* $|h| < \|R_T(\zeta)\|^{-1}$*, then*

$$\|R_T(\zeta + h)\| \ \leq \ \frac{\|R_T(\zeta)\|}{1 - |h|\|R_T(\zeta)\|} \ .$$

(vi) $R_T : \rho\,(T) \to \mathcal{L}\,(X)$ *is a continuous map.*

(vii) $R_T \in H(\rho(T), \mathcal{L}\,(X))$ *and*

$$\frac{d}{d\zeta} R_T\zeta = -R_T(\zeta)^2 \ .$$

Proof. The proofs of (i) - (iii) are straightforward.

(iv) For $|\zeta| > \| T \|$, $\mathsf{R}_T(\zeta) = \sum_{i=n}^{\infty} \zeta^{-n} T^{n-1}$ so that

$$\| \mathsf{R}_T(\zeta) \| \leq \frac{1}{|\zeta| - \| T \|} \to 0 \text{ as } |\zeta| \to \infty \ .$$

(v) Apply Theorem 1.4.10 with $u = \zeta I - T$ and $w = -hI$, and hence with $u^{-1} = \mathsf{R}_T(\zeta)$ and $(u - w)^{-1} = \mathsf{R}_T(\zeta + h)$.

(vi) By the resolvent equation

$$\begin{aligned} \|\mathsf{R}_T(\zeta) - \mathsf{R}_T(\zeta + h)\| \ &\leq \ |h|\|\mathsf{R}_T(\zeta)\|\|\mathsf{R}_T(\zeta + h)\| \\ &\leq \ |h|\|\mathsf{R}_T(\zeta)\| \frac{\|\mathsf{R}_T(\zeta)\|}{1 - |h|\|\mathsf{R}_T(\zeta)\|} \\ &\to 0 \quad \text{as } h \to 0 \ . \end{aligned}$$

(vii) By the resolvent equation

$$\begin{aligned} \left\| \frac{\mathsf{R}_T(\zeta + h) - \mathsf{R}_T(\zeta)}{h} + \mathsf{R}_T(\zeta)^2 \right\| \ &\leq \ \| - \mathsf{R}_T(\zeta + h)\mathsf{R}_T(\zeta) + \mathsf{R}_T(\zeta)^2\| \\ &\leq \ \|\mathsf{R}_T(\zeta + h) - \mathsf{R}_T(\zeta)\|\|\mathsf{R}_T(\zeta)\| \\ &\to 0 \quad \text{as } h \to 0 \ . \end{aligned}$$

□

Exercise 2.1.7. Show that for $n = 0, 1, 2, 3, \ldots,$

$$\frac{d^n}{d\zeta^n} R_T(\zeta) \;=\; (-1)^n n!\, R_T(\zeta)^{n+1} \;.$$

We also have the following important theorem:

Theorem 2.1.8 (Compactness of $\sigma(T)$)**.**

$\sigma(T)$ is a compact subset of $\overline{B_{\| T \|}(0)}$ and if $\mathcal{X} \neq \{0\}$, then $\mathcal{X}$ is nonempty.

Proof. Since $\rho(T)$ is open, $\sigma(T)$ is closed. Also, $\sigma(T) \subset \overline{B_{\| T \|}(0)}$, and so it is compact.

Suppose that $\sigma(T) = \varnothing$. Then $\mathsf{R}_T \in \mathsf{H}(\mathbf{C}, \mathcal{L}(X))$ and $\| \mathsf{R}_T \| \to 0$. By Liouville's Theorem 1.5.5, $\mathsf{R}_T = 0$ which is impossible. $\square$

2.2 Functions of operators

We initially define a way to take functions of a polynomial.

Definition 2.2.1 (Polynomial functional calculus)**.** Let $\mathcal{P}$ denote the algebra of polynomials. Define $\Phi_T : \mathcal{P} \to \mathcal{L}(X)$:

$$\Phi_T(p) = \sum_{k=1}^{n} \alpha_k T^k$$

where

$$p(\zeta) = \sum_{k=1}^{n} \alpha_k \zeta^k \in \mathcal{P}$$

and $T^0 = I$. We write $p(T) = \Phi_T(p)$.

This is in fact an algebra homomorphism. We generalise this to the algebra of rational functions.

Definition 2.2.2 (Algebra of rational functions)**.** We denote the algebra of rational functions with no poles in a compact set K by $\mathcal{R}_K$.

Remark 2.2.3. If $r \in \mathcal{R}_K$, then there exists polynomials p, q such that

$$r(\zeta) = \frac{p(\zeta)}{q(\zeta)}$$

$$q(\zeta) = \prod_{k=1}^{n} (\alpha_k - \zeta)$$

where α_k are zeros of q and $\alpha_n \notin K$.

Definition 2.2.4 (Rational functional calculus)**.** We extend $\Phi_T : \mathcal{R}_{\sigma(T)} \to \mathcal{L}(X)$ by:

$$\Phi_T(r) = p(T) \prod_{k=1}^{n} \mathsf{R}_T(\alpha_k)$$

where

$$r(\zeta) = \frac{p(\zeta)}{\prod_{k=1}^{n}(\alpha_k - \zeta)} \in \mathcal{R}_{\sigma(T)}$$

As before, we write $r(T) = \Phi_T(r)$.

Exercise 2.2.5. Show that Φ_T is well defined and that it is an algebra homomorphism.

We can make another generalisation to the power series.

Definition 2.2.6 (Power series algebra)**.** We denote the algebra of power series with radius of convergence greater than R by $\mathcal{P}_R$.

Definition 2.2.7 (Power series functional calculus)**.** We extend $\Phi_T : \mathcal{P}_{\| T \|} \to \mathcal{L}(X)$ by:

$$\Phi_T(s) = \sum_{k=0}^{\infty} \alpha_k T^k$$

where

$$s(\zeta) = \sum_{k=0}^{\infty} \alpha_k \zeta^k \in \mathcal{P}_{\| T \|}$$

We write $s(T) = \Phi_T(s)$.

Remark 2.2.8. Note that

$$\| s(T) \| \leq \sum_{k=0}^{\infty} \alpha_k \| T \|^k < \infty$$

since the radius of convergence of s is greater than $\| T \|$.

Exercise 2.2.9. Show that Φ_T is an algebra homomorphism.

Example 2.2.10. Let $\mathcal{X} = \mathbf{C}^2$, and

$$T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Then, $\sigma(T) = \{\lambda_1, \lambda_2\}$, and $\rho(T) = \mathbf{C} \setminus \{\lambda_1, \lambda_2\}$. Then the resolvent is given by:

$$\mathsf{R}_T(\zeta) = (\zeta I - T)^{-1} = \begin{pmatrix} \frac{1}{\zeta - \lambda_1} & 0 \\ 0 & \frac{1}{\zeta - \lambda_2} \end{pmatrix} \in \mathsf{H}(\Omega, \mathcal{L}(C^2))$$

The functional calculus for $f \in \mathcal{P}, \mathcal{R}_{\sigma(T)}, \mathcal{P}_{\| T \|}$ is:

$$f(T) = \begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix}$$

For this particular operator, f could be *any* function defined only on $\sigma(T)$.

Example 2.2.11. Let $\mathcal{X} = \mathbf{C}^2$, and

$$T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

and

$$T^2 = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

Then,

$$f(T) = \begin{pmatrix} f(\lambda) & f'(\lambda) \\ 0 & f(\lambda) \end{pmatrix}$$

Here, $\sigma(T) = \{\lambda\}$, but unlike the previous example, it is not enough to know f just at λ. We need f' at λ so f needs to be defined in a neighbourhood of $\sigma(T)$ and differentiable at λ.

In fact, the product rule becomes a consequence of the algebra homomorphism property:

$$(fg)(T) = \begin{pmatrix} (fg)(\lambda) & (fg)'(\lambda) \\ 0 & (fg)(\lambda) \end{pmatrix} = \begin{pmatrix} f(\lambda)g(\lambda) & (f'g)(\lambda) + (fg')(\lambda) \\ 0 & f(\lambda)g(\lambda) \end{pmatrix} = f(T)g(T)$$

In general, replacing $\mathcal{X} = \mathbf{C}^n$ and

$$T = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Taking n multiples of T, we find that to define $f(T)$, we need n derivatives of f. In particular, $f(T)$ is defined when f is holomorphic in a neighbourhood of λ. This highlights that in general, we need stronger conditions on functions f than simply being defined on the spectrum.

2.3 Holomorphic functional calculus

Throughout this section, we fix $\mathcal{X}$ to be a Banach space and $T \in \mathcal{L}(X)$.

We have already seen how to define $f(T)$ when f is a polynomial, rational function or a power series with a radius of convergence larger than $\| T \|$.

We want to generalise these three situations simultaneously. While we could consider defining a generalised functional calculus via a power series expansion, this would in fact give bad bounds: the functions would have to be holomorphic on B_R with $R > \| T \|$. Rather, we define a functional calculus for functions which are simply holomorphic on some neighbourhood Ω of the spectrum.

Definition 2.3.1 ($\mathsf{H}(\Omega)$ functional calculus)**.** Let Ω be an open set and $\sigma(T) \subset \Omega$. We call $\Phi_T : \mathsf{H}(\Omega) \to \mathcal{L}(X)$ an $\mathsf{H}(\Omega)$ *functional calculus* provided:

(i) Φ_T is an algebra homomorphism:

(a) $\Phi_T(\alpha f + \beta g) = \alpha\Phi_T(f) + \beta\Phi_T(g)$.

(b) $\Phi_T(fg) = \Phi_T(f)\Phi_T(g)$.

(ii) $\Phi_T(p) = p(T)$ when $p \in \mathcal{P}$.

(iii) If $f_n \to f$ uniformly on all $K \Subset \Omega$ (compact subset of Ω), then $\Phi_T(f_n) \to \Phi_T(f)$ in $\mathcal{L}(X)$.

Proposition 2.3.2. *Provided condition (ii) holds, then condition (ii) is equivalent to each of the following:*

(ii') $\Phi_T(1) = I$, $\Phi_T(\mathrm{id}) = T$.

(ii'') $\Phi_T(1) = I$, $\Phi_T(R_\alpha) = -R_T(\alpha)$ for some $\alpha \in \rho(T)$, where $R_\alpha(\zeta) = (\zeta - \alpha)^{-1}$.

Moreover, $\Phi_T(r) = r(T)$ for all $r \in \mathcal{R}_{\sigma(T)} \cap H(\Omega)$.

Proof. (ii') is an easy consequence of (ii).

We prove (ii') $\implies$ (ii''). We note that $(\zeta - \alpha)\mathsf{R}_\alpha(\zeta) = 1 = \mathsf{R}_\alpha(\zeta)(\zeta - \alpha)$. It follows that $(T - \alpha I)\Phi_T(\mathsf{R}_\alpha) = \Phi_T(\mathsf{R}_\alpha)(T - \alpha I)$, giving $\Phi_T(\mathsf{R}_\alpha) = -\mathsf{R}_T(\alpha)$.

We leave as an exercise that (ii'') implies (ii). ☐

While we have axiomatically defined an $\mathsf{H}(\Omega)$ functional calculus, we still to prove existence. In fact, we can do better by proving that such a calculus is unique.

Theorem 2.3.3 (Existence and Uniqueness of $\mathsf{H}(\Omega)$ functional calculus)**.** *An $H(\Omega)$ functional calculus exists and it is unique.*

Proof. Uniqueness follows from Runge's Theorem (1.5.12): Given $f \in \mathsf{H}(\Omega)$, there exists rational functions r_n such that $r_n \to f$ uniformly on every $K \Subset \Omega$. So, by (iii), $\Phi_T(f) = \lim_{n\to\infty} r_n(T)$.

We prove existence. Let γ be a closed contour which envelops $\sigma(T)$ in Ω. Define:

$$f(T) := \frac{1}{2\pi\imath}\oint_\gamma f(\zeta)\mathsf{R}_T(\zeta)d\zeta$$

First, observe that given some other such curve δ,

$$\oint_{\gamma-\delta} f(\zeta)\mathsf{R}_T(\zeta)d\zeta = 0$$

since R_T holomorphic on $\rho(T)$. So, this definition is independent of the curve γ and we define $\Phi_T(f) := f(T)$.

We check (i), (ii') and (iii)

(i) Linearity is clear by the linearity of $\oint$. For the product:

$$\begin{aligned}
f(T)g(T) &= \frac{1}{2\pi\imath}\oint_\gamma f(\zeta)R_T(\zeta)\,d\zeta\,\frac{1}{2\pi\imath}\oint_\delta g(w)R_T(w)\,dw \\
&= \frac{1}{(2\pi\imath)^2}\oint_\gamma\oint_\delta f(\zeta)g(w)R_T(\zeta)R_T(w)\,dw\,d\zeta \\
&= \frac{1}{(2\pi\imath)^2}\oint_\gamma f(\zeta)R_T(\zeta)\oint_\delta \frac{g(w)}{w-\zeta}\,dw\,d\zeta \\
&\quad - \frac{1}{(2\pi\imath)^2}\oint_\delta g(w)R_T(w)\oint_\gamma \frac{f(\zeta)}{w-\zeta}\,d\zeta\,dw \\
&= 0\ +\ \frac{1}{2\pi\imath}\oint_\delta f(w)g(w)R_T(w)\,dw \\
&= (fg)(T)\ .
\end{aligned}$$

(ii')

$$\begin{aligned}
1(T) &= \frac{1}{2\pi\imath}\oint_\gamma(\zeta I-T)^{-1}\,d\zeta \\
&= \frac{1}{2\pi\imath}\oint_\gamma\sum_{n=0}^{\infty}\frac{T^n}{\zeta^{n+1}}\,d\zeta \\
&= \sum_{n=0}^{\infty}T^n\frac{1}{2\pi\imath}\oint_\gamma\frac{d\zeta}{\zeta^{n+1}} \\
&= I
\end{aligned}$$

Using the same argument:

$$\mathrm{id}(T)\ =\ \frac{1}{2\pi\imath}\oint_\gamma\zeta(\zeta I-T)^{-1}\,d\zeta = \sum_{n=0}^{\infty}T^n\frac{1}{2\pi\imath}\oint_\gamma\frac{d\zeta}{\zeta^n}\ =\ T\ .$$

(ii) Suppose $f_n \to f$ uniformly on all $K \Subset \Omega$. Then since γ is compact,

$$\begin{aligned}
\|\,f_n(T)-f(T)\,\| &\le \frac{1}{2\pi}l(\gamma)\sup_{\zeta\in\gamma}|f_n(\zeta)-f(\zeta)|\sup_{\zeta\in\gamma}\|\,\mathsf{R}_T(\zeta)\,\| \\
&\le C\sup_{\zeta\in\gamma}|f_n(\zeta)-f(\zeta)| \\
&\to 0 \text{ as } n\to\infty \text{ in } \mathcal{L}(X)
\end{aligned}$$

□

Remark 2.3.4. This definition of $f(T)$ is also called *the* Dunford-Riesz functional calculus.

We list some important properties of the $\mathsf{H}(\Omega)$ functional calculus:

Theorem 2.3.5 (Properties of the $\mathsf{H}(\Omega)$ functional calculus)**.** *(i) Suppose Ω_1, Ω_2 are open sets and $\sigma(T) \subset \Omega_1 \cap \Omega_2$. Let $f \in \mathsf{H}(\Omega_1 \cup \Omega_2)$. Let $\Phi_T^{\Omega_i}$ be the functional calculus with respect to Ω_i. Then,*

$$\Phi_T^{\Omega_1}(f) = \Phi_T^{\Omega_2}(f) \ .$$

(ii) $\Phi_T|_{\mathsf{H}^\infty(\Omega)} : \mathsf{H}^\infty(\Omega) \to \mathcal{L}(X)$ is a bounded algebra homomorphism. That is, there exists a constant $C > 0$ such that

$$\| \Phi_T(f) \| \leq C \| f \|_\infty$$

whenever $f \in \mathsf{H}^\infty(\Omega)$.

(iii) If $s \in \mathcal{P}_{\| T \|}$ with radius of convergence $R > \| T \|$, then taking $\Omega = B_R(0)$,

$$\Phi_T(s) = s(T)$$

where $s(T)$ is defined via the power series calculus.

(iv) $\mathsf{R}(\Phi_T)$ is a commutative subalgebra. That is $[\Phi_T(f), \Phi_T(g)] = 0$, for all $f, g \in \mathsf{H}(\Omega)$, where $[a, b] = ab - ba$ is the commutator.

(v) $\Phi_T(f)$ belongs to the bicommutator of T. That is, for all $f \in \mathsf{H}(\Omega)$, $[\Phi_T(f), S] = 0$ whenever $[S, T] = 0$.

From now on, we will typically write $f(T)$ rather than $\Phi_T(f)$ as there is no chance of confusion.

Theorem 2.3.6 (Spectral mapping theorem)**.** *If $f \in H(\Omega)$ then $f(\sigma(T)) = \sigma(f(T))$.*

Proof. (i) $f(\sigma(T)) \subset \sigma(f(T))$: Let $\lambda \in \sigma(T)$. By Proposition 1.5.10, there exists $g \in H(\Omega)$ such that $f(\zeta) - f(\lambda) = g(\zeta)(\zeta - \lambda)$ for all $\zeta \in \Omega$, so $f(T) - f(\lambda)I = g(T)(T - \lambda I)$. If $f(\lambda) \in \rho(f(T))$ then $f(T) - f(\lambda)I$ would have a bounded inverse, and hence so would $(T - \lambda I)$, which contradicts the assumption that $\lambda \in \sigma(T)$. Therefore $f(\lambda) \in \sigma(f(T))$.

(ii) $\sigma(f(T)) \subset f(\sigma(T))$: If $\mu \notin f(\sigma(T))$ then $h(\zeta) = (f(\zeta) - \mu)^{-1}$ is holomorphic on a neighbourhood of $\sigma(T)$, say Ω'. Applying the above results to $H(\Omega')$, we get $h(T)(f(T) - \mu I) = I$, which implies $\mu \notin \sigma(f(T))$. □

Theorem 2.3.7 (Composition)**.** *Let $T \in \mathcal{L}(X)$, $\sigma(T) \subset \Omega$, $f \in \mathsf{H}(\Omega)$ and $g \in \mathsf{H}(\Omega')$, $f(\sigma(T)) \subset \Omega'$, so that $g \circ f \in \mathsf{H}(\Omega \cap f^{-1}(\Omega'))$. Then,*

$$g(f(T)) = (g \circ f)(T) \ .$$

Proof. This result can be proved in two ways. We leave both methods as exercises.

(i) Check:

$$\oint_\gamma (g \circ f)(\zeta) \mathsf{R}_T(\zeta) \ d\zeta = \oint_\delta g(\zeta) \mathsf{R}_{f(T)}(\zeta) \ d\zeta$$

(ii) Use uniqueness. Define: $\Phi' : \mathsf{H}(\Omega) \to \mathcal{L}(X)$ via $g \mapsto (g \circ f)(T)$, and show that this satisfies the axioms for a functional calculus. Then uniqueness yields the result.

$\square$

2.4 Spectral decomposition

Suppose that T is an operator whose spectrum $\sigma(T)$ is a pairwise disjoint union of non–empty compact sets

$$\sigma(T) \;=\; \sigma_1(T) \cup \sigma_2(T) \cup \cdots \cup \sigma_n(T) \;.$$

Proposition 2.4.1. *We can choose pairwise open sets Ω_k such that $\sigma_k(T) \subset \Omega_k$ and $\sigma(T) \subset \Omega = \cup_{k=1}^{N} \Omega_k$. Define χ_k on Ω:*

$$\chi_k(\zeta) = \begin{cases} 1 & \zeta \in \Omega_k \\ 0 & \zeta \in \Omega \setminus \Omega_k \end{cases}$$

Then $\chi_k : \Omega \to [0,1]$ is holomorphic.

Theorem 2.4.2 (Family of spectral projections)**.** *There exist a family of spectral projections $E_k \in \mathcal{L}(X)$ satisfying:*

(i) $E_k^2 = E_k$, $E_k E_j = 0$ *when* $j \neq k$.

(ii) $I = \sum_{k=1}^{n} E_k$.

(iii) $E_k T = T E_k$.

Proof. Define $E_k = \chi_k(T)$. The result follows by the algebra homomorphism property and by the observation that $\chi_k^2 = \chi_k$, $\chi_j \chi_k = 0$ when $j \neq k$, $1 = \sum_{i=1}^{n} \chi_k$ on Ω, and $\mathrm{id}\chi_k = \chi_k \mathrm{id}$. $\square$

Applying the results of §1.2, we get the following consequence.

Corollary 2.4.3 (Spectral decomposition)**.** *(i) There exists a collection of linear subspaces $\mathcal{X}_k$ such that $\mathcal{X} = \bigoplus_{k=1}^{n} \mathcal{X}_k$.*

(ii) There are operators $T_k \in \mathcal{L}(\mathcal{X}_k)$ such that $T = \bigoplus_{k=1} T_k$.

(iii) $\sigma(T_k) = \sigma_k(T)$.

(iv) $f(T_k) = f(T)|_{\mathcal{X}_k}$.

Proof. (i) Put $\mathcal{X}_k = E_k(\mathcal{X}) = \mathsf{R}(E_k)$ and the result follows from property (ii).

(ii) Define $T_k = T|_{\mathcal{X}_k}$. Then by property (iii), $T_k x_k = T E_k x = E_k T x \in \mathcal{X}_k$.

(iii) We note that $\sigma(T) = \cup_{k=1}^{n} \sigma(T_k)$. Let us show that if $\lambda \notin \Omega_k$ then $\lambda \in \rho(T_k)$.

We note that by Proposition 1.5.10, there exists a $g_k \in \mathsf{H}(\Omega)$ such that $(\zeta - \lambda) g_k = \chi_k$, and it is given by

$$g_k(\zeta) = \begin{cases} \frac{1}{\zeta-\lambda} & \zeta \in \Omega_k \\ 0 & \zeta \in \Omega \setminus \Omega_k \end{cases}$$

It follows that $(T - \lambda I) g_k(T) = E_k$ and on $\mathcal{X}_k$, $(T_k - \lambda I) g_k(T) = E_k$ and so $\lambda \in \rho(T_k)$. Therefore, $\sigma(T_k) \subset \Omega_k \cap \sigma(T) = \sigma_k(T)$. If $\sigma(T_k) \neq \sigma_k(T)$, then $\sigma(T) \neq \cup_{k=1}^{n} \sigma(T_k)$ which is a contradiction.

(iv) Exercise.

□

Remark 2.4.4. Again as the discussion in §1.2, we can represent T as:

$$T = \begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{pmatrix}$$

So, T has been diagonalised and this decomposition can be thought of as a generalisation of the Jordan canonical form.

2.5 Exponentials and Fractional powers

We are interested in focusing on particular holomorphic functions that form a family of operators.

Definition 2.5.1 (Exponential family)**.** Let $f_t(\zeta) = \mathrm{e}^{-t\zeta}$, with $f_t : \Omega \to \mathbf{C}$ and $t \in \mathbf{R}$. Define:

$$\mathrm{e}^{-tT} = f_t(T)$$

Theorem 2.5.2 (Properties of the Exponential family)**.** *(i)* e^{-tT} *forms a group. That is:*

$$\mathrm{e}^{-(t+s)T} = \mathrm{e}^{-tT} \mathrm{e}^{-sT} \ .$$

(ii) $\lim_{t \to 0} \mathrm{e}^{-tT} = I$.

(iii) $\frac{d}{dt} \mathrm{e}^{-tT} = -T \mathrm{e}^{-tT}$.

(iv) If $\sigma(T) \subset \{\zeta \in \mathbf{C} : \mathrm{Re}\, \zeta > 0\}$, *there exists a constant* C_T *such that*

$$\| e^{-tT} \| \leq C_T$$

for all $t \geq 0$.

(v) If $\sigma(T) \subset \{\zeta \in \mathbf{C} : \mathrm{Re}\, \zeta > 0\}$, *then*

$$\lim_{t \to \infty} \mathrm{e}^{-tT} = 0 \ .$$

Proof. (i) By the properties of the functional calculus.

(ii) Follows from the fact that $f_t \to 1$ uniformly on compacts subsets of Ω.

(iii) Exercise.

(iv) By the condition on $\sigma(T)$, we can find an $\Omega \subset \{\zeta \in \mathbf{C} : \operatorname{Re}\zeta > 0\}$. Then,

$$\begin{aligned}\| \mathrm{e}^{-tT} \| &= \frac{1}{2\pi} \left\| \oint_\gamma \mathrm{e}^{-t\zeta} \mathsf{R}_T\zeta \; d\zeta \right\| \\ &\le \frac{1}{2\pi}\, l(\gamma)\ \sup_\gamma \| \mathsf{R}_T\zeta \| \ \sup_{\zeta\in\Omega} |\mathrm{e}^{-t\zeta}| \\ &\le C_T \sup_{\zeta\in\Omega} |\mathrm{e}^{-t\operatorname{Re}\zeta}| \\ &\le C_T.\end{aligned}$$

(v) Fix $t > 0$. We can choose Ω such that $\overline{\Omega} \subset \{\zeta \in \mathbf{C} : \operatorname{Re}\zeta > 0\}$ and compact. Then, $\operatorname{Re}\overline{\Omega} \subset [a, b]$ where $a > 0$ and

$$\| \mathrm{e}^{-tT} \| \le C_T \sup_{\zeta\in\Omega} |\mathrm{e}^{-t\operatorname{Re}\zeta}| = C_T \mathrm{e}^{-ta}$$

since e^{-t} is a decreasing function. The result follows by letting $t \to \infty$.

□

Remark 2.5.3. Since $\mathrm{e}^{-t\zeta}$ is analytic on the entire complex plane, we could have used the power series to define the functional calculus. However, this method gives bad bounds:

$$\| \mathrm{e}^{-tT} \| \le 1 + t\, \| T \| + \frac{t^2 \| T \|^2}{2} + \ldots \le \mathrm{e}^{t\| T \|}$$

Remark 2.5.4. The last two conditions illustrates that for such families of operators, it is often of importance to know where the spectrum lies in the complex plane.

In particular, if $\sigma(T) \cap \{\zeta \in \mathbf{C} : \operatorname{Re}\zeta < 0\} \neq \varnothing$ then the limit $\lim_{t\to\infty} \mathrm{e}^{-tT}$ does not exist. We can ask what happens when $\sigma(T) \setminus \{0\} \subset \{\zeta \in \mathbf{C} : \operatorname{Re}\zeta > 0\}$. If $0 \in \sigma_p(T)$ (the point spectrum), then $Tu = 0$ for some $u \neq 0$ and $\mathrm{e}^{-tT} = u + 0 + \ldots$. So, $\lim_{t\to\infty} e^{-tT} u = u$. We return to this later.

We can also consider taking fractional powers of an operator.

Definition 2.5.5 (Fractional powers)**.** Let $\sigma(T) \subset S^0_\pi$, where $S^0_\pi = \left\{ re^{\imath\theta} : -\pi < \theta < \pi, r > 0 \right\}$. Then for $\alpha \in [0, 1]$, let $g_\alpha(\zeta) = \zeta^\alpha$. Define:

$$T^\alpha = g_\alpha(T)$$

Remark 2.5.6. We impose this condition on the spectrum since ζ^α fails to be holomorphic everywhere. There is nothing special about S^0_π. We could "cut" the plane using a curve γ connecting 0 and ∞.

Remark 2.5.7. We cannot (in general) take a power series expansion of T^α, since on the cut plane, the spectrum may not lie inside a disc.

It is easy to check that $T^\alpha T^\beta = T^{\alpha+\beta}$. In particular, $T^{\frac{1}{2}} T^{\frac{1}{2}} = T$. We write $\sqrt{T} = T^{\frac{1}{2}}$. We can define families of operators such that $\cos(tT)$, $\sin(tT)$ similarly. Study of operators such as $\cos(t\sqrt{T})$ are of importance in hyperbolic PDEs.

2.6 Spectral Projections

So far, all the results on the functional calculus of $T \in \mathcal{L}(X)$ would hold for $T \in \mathcal{A}$ where $\mathcal{A}$ is any Banach Algebra.

However, we will now access results that are particular to $\mathcal{L}(X)$ by working with a weaker topology.

Definition 2.6.1 (Strong convergence)**.** Let $S_n, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Write $\text{s-}\lim_{n\to\infty} S_n = S$ if for all $u \in \mathcal{X}$,

$$\| S_n u - S u \|_{\mathcal{Y}} \to 0$$

Remark 2.6.2. Uniform (or Norm) convergence implies strong convergence.

Proposition 2.6.3. *Let (S_n) be a uniformly bounded sequence of operators $S_n \in \mathcal{L}(\mathcal{X})$, and suppose $\mathcal{Y} \subset \mathcal{X}$.*
(i) If $S_n u \to 0$ for all $u \in \mathcal{Y}$, then $S_n u \to 0$ for all $u \in \overline{\mathcal{Y}}$.
(ii) If $S_n u$ converges for all $u \in \mathcal{Y}$, then $S_n u$ converges for all $u \in \overline{\mathcal{Y}}$.

Proof. let us just prove (ii), as (i) is a touch simpler. By assumption, there exists C such that $\|S_n\| \leq C$ for all n. Suppose u is in the closure of $\mathcal{Y}$. It suffices to show that $S_n u$ is a Cauchy sequence. Let $\varepsilon > 0$. There exists $v \in \mathcal{Y}$ such that $\|u - v\| < \frac{\varepsilon}{3C}$. Moreover, there exists N such that $\| S_n v - S_m v \| < \frac{\varepsilon}{3}$ for all $n, m \geq N$. Therefore $\|S_n u - S_m u\| \leq \|S_n(u - v)\| + \|S_n v - S_m v\| + \|S_m(u - v)\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ for all $n, m \geq N$, as required. □

Our aim is to extend the functional calculus to the situation where $\sigma(T) \setminus \{\lambda\} \subset \Omega$. We will need resolvent bounds near λ.

Definition 2.6.4 (Resolvent bound)**.** Let $\lambda \in \sigma(T)$. We say that we have *resolvent bounds near* λ if there exists $C > 0$ such that

$$|\lambda - \zeta| \, \| \mathsf{R}_T(\zeta) \| \leq C$$

for some $\zeta \in \rho(T)$ near λ.

Also, we want splitting of the space $\mathcal{X} = \mathsf{N}(T - \lambda I) \oplus \overline{\mathsf{R}(T - \lambda I)}$.

Example 2.6.5. Let $\mathcal{X} = \mathbf{C}^N$, $\lambda_j \neq 0$ for $j > 1$ and define T:

$$T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

So, $\sigma(T) = \{\lambda_1, \dots, \lambda_N\}$. Take $\lambda = \lambda_1$. Then,

$$\mathsf{R}_T(\zeta) = \begin{pmatrix} \frac{1}{\zeta - \lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\zeta - \lambda_N} \end{pmatrix}$$

and

$$\| \mathsf{R}_T(\zeta) \| \leq \frac{1}{\operatorname{dist}(\zeta, \sigma(T))} \ .$$

If ζ is near λ, then $|\lambda - \zeta| \, \| \mathsf{R}_T(\zeta) \| = 1$. Also,

$$T = \begin{pmatrix} 0 & & & \\ & \lambda_2 - \lambda & & \\ & & \ddots & \\ & & & \lambda_N - \lambda \end{pmatrix}$$

Then $\mathsf{N}(T - \lambda I) = \left\{ (\alpha, 0, \ldots, 0) \in \mathbf{C}^N \right\}$ and $\overline{\mathsf{R}(T - \lambda I)} = \mathsf{R}(T - \lambda I) = \left\{ (0, \beta_1, \ldots, \beta_{n-1}) \in \mathbf{C}^N \right\}$. So, $\mathbf{C}^N = \mathsf{N}(T - \lambda I) \oplus \overline{\mathsf{R}(T - \lambda I)}$. We get both resolvent bounds and a splitting.

Example 2.6.6. Let $\mathcal{X} = \mathbf{C}^2$, and define T:

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then, $\sigma(T) = \{0\}$. The resolvent is given by:

$$\mathsf{R}_T(\zeta) = \begin{pmatrix} \frac{1}{\zeta} & \frac{1}{\zeta^2} \\ 0 & \frac{1}{\zeta} \end{pmatrix}$$

We do not get resolvent bounds:

$$|\zeta - 0| \, \| \mathsf{R}_T(\zeta) \| = |\zeta| \, \| \mathsf{R}_T(\zeta) \| \simeq \frac{1}{|\zeta|}$$

Also we do not have a splitting of the space: $\mathsf{N}(T) = \left\{ (\alpha, 0) \in \mathbf{C}^2 \right\} = \mathsf{R}(T)$ so $\mathbf{C}^2 \neq \mathsf{N}(T) \oplus \overline{\mathsf{R}(T)}$.

Theorem 2.6.7 (Spectral Decomposition)**.** *Let $T \in \mathcal{L}(\mathcal{X})$ and $\lambda \in \mathbf{C}$ and suppose there exist $\zeta_n \in \rho(T)$ and C such that $\zeta_n \to \lambda$ and $|\zeta_n - \lambda| \, \| R_T(\zeta_n) \| \leq C$ for all n. Then,*

$$\mathcal{X} \supset \tilde{\mathcal{X}} = N(T - \lambda I) \oplus \overline{R(T - \lambda I)}$$

where

$$\tilde{\mathcal{X}} \;=\; \{u \in \mathcal{X}; (\zeta_n - \lambda) R_T(\zeta_n) u \text{ converges in } \mathcal{L}(\mathcal{X})\}$$

and the projection $P : \tilde{\mathcal{X}} \to N(T - \lambda I)$ with $N(P) = \overline{R(T - \lambda I)}$ is given by

$$Pu \;=\; \lim_{n \to \infty} (\zeta_n - \lambda) R_T(\zeta_n) u \ , \quad u \in \tilde{\mathcal{X}} \ .$$

Proof. Define

$$\begin{aligned} \mathcal{X}_\lambda &= \{u \in \mathcal{X}; (\zeta_n - \lambda) R_T(\zeta_n) u \to u\} \\ \mathcal{X}^\lambda &= \{u \in \mathcal{X}; (\zeta_n - \lambda) R_T(\zeta_n) u \to 0\} \end{aligned}$$

and define $P : \tilde{\mathcal{X}} \to \mathcal{X}$ by $Pu = \lim_{n\to\infty}(\zeta_n - \lambda)R_T(\zeta_n)u$. All three subspaces are linear, and, by the preceding proposition, are closed linear subspaces of $\mathcal{X}$. By the hypothesis on the resolvent bounds, P is a bounded linear operator with $\|P\| \le C$. We shall show that $\tilde{\mathcal{X}}$ is closed and that P is a projection in $\tilde{\mathcal{X}}$ with

$$\begin{aligned} \mathsf{N}(P) &= \mathcal{X}^\lambda = \overline{\mathsf{R}(T-\lambda I)} \\ \mathsf{R}(P) &= \mathcal{X}_\lambda = \mathsf{N}(T-\lambda I)\ . \end{aligned}$$

This will prove the theorem.

It suffices to consider the case when $\lambda = 0$, as the general case can be obtained from this on replacing T by $T - \lambda I$. Note then that $(\zeta_n - 0)R_T(\zeta_n) = \zeta_n(\zeta_n I - T)^{-1}$, and the assumption becomes $\left\| \zeta_n(\zeta_n I - T)^{-1} \right\| \le C$ for all n. By using the identity $T(\zeta_n I - T)^{-1} = \zeta_n(\zeta_n I - T)^{-1} - I$, we also have the bound $\left\| T(\zeta_n I - T)^{-1} \right\| \le C + 1$.

(i) $\mathsf{N}(T) = \mathcal{X}_0 = \mathsf{R}(P)$: It is straightforward to see that $\mathsf{N}(T) \subset \mathcal{X}_0 \subset \mathsf{R}(P)$. Let us prove $\mathsf{R}(P) \subset \mathsf{N}(T)$. Let $w \in \mathsf{R}(P)$. Then there exists $u \in \tilde{\mathcal{X}}$ such that $\zeta_n(\zeta_n I - T)^{-1}u \to w$. Now

$$\left\| T\zeta_n(\zeta_n I - T)^{-1}u \right\| \ \le\ |\zeta_n|(C+1)\|u\| \ \to\ 0$$

as $n \to \infty$. Therefore $Tw = 0$.

(ii) P is a projection: Note that $Pv = v$ for all $v \in \mathcal{X}_0 = \mathsf{R}(P)$. Therefore $P^2u = Pu$ for all $u \in \tilde{\mathcal{X}}$.

(iii) $\mathsf{R}(T) \subset \mathcal{X}^0 = \mathsf{N}(P)$: Let $u = Tw \in \mathsf{R}(T)$. Then $\zeta_n(\zeta_n I - T)^{-1}u = \zeta_n T(\zeta_n I - T)^{-1}w \to 0$ as $n \to \infty$, so $u \in \mathcal{X}^0$.

(iv) $\overline{\mathsf{R}(T)} \subset \mathcal{X}^0 = \mathsf{N}(P)$.

(v) $\mathsf{N}(P) \subset \overline{\mathsf{R}(T)}$: Exercise.

$\square$

We also have the following important result.

Proposition 2.6.8. *T preserves the spaces $\mathsf{N}(T - \lambda I)$ and $\overline{\mathsf{R}(T - \lambda I)}$.*

Proof. Suppose $u \in \mathsf{N}(T - \lambda I)$, so then $Tu = \lambda u$ and $T^2u = \lambda Tu = \lambda^2 u \in \mathsf{N}(T - \lambda I)$. Now let $u \in \overline{\mathsf{R}(T - \lambda I)}$. Then, $u = (T - \lambda I)u + \lambda u \in \overline{\mathsf{R}(T - \lambda I)}$. $\square$

Remark 2.6.9. Define $\tilde{T} \in \mathcal{L}(\tilde{\mathcal{X}})$ by $\tilde{T}u = Tu$. Here, $\mathsf{R}(\tilde{T} - \lambda I)$ will in general be *smaller* than $\mathsf{R}(T - \lambda I)$, but the splitting will not change: $\tilde{\mathcal{X}} = \mathsf{N}(\tilde{T} - \lambda I) \oplus \overline{\mathsf{R}(\tilde{T} - \lambda I)}$. Also, since $\mathsf{N}(\tilde{T} - \lambda I) = \mathsf{N}(T - \lambda I)$, we find $\overline{\mathsf{R}(T - \lambda I)} = \overline{\mathsf{R}(\tilde{T} - \lambda I)}$. This shows that while $\mathsf{R}(\tilde{T} - \lambda I)$ is in generally smaller than $\mathsf{R}(T - \lambda I)$, it is still dense in $\overline{\mathsf{R}(T - \lambda I)}$.

The following example highlights that in general $\tilde{\mathcal{X}} \ne \mathcal{X}$.

Example 2.6.10. Let $\mathcal{X} = C[-1,1]$, with $\| u \| = \| u \|_\infty$. Define $\varphi \in \mathcal{X}$:

$$\varphi(x) := \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

Define $T \in \mathcal{L}(\mathcal{X})$ by $Tu = M_\varphi u(x) = \varphi(x)u(x)$. Then, $(\zeta - T)u(x) = (\zeta - \varphi(x))u(x)$, and

$$\mathsf{R}_T(\zeta)u(x) = \frac{u(x)}{\zeta - \varphi(x)}$$

when $\zeta \in \rho(M_\varphi) = \mathbf{C} \setminus \sigma(M_\varphi)$ with $\sigma(M_\varphi) = \{\varphi(x) : x \in [-1,1]\} = [0,1]$.

Now, putting $\zeta = 0$ and $\zeta_n = -\frac{1}{n}$, we have the resolvent bound $\| \zeta_n \mathsf{R}_T(\zeta_n) \| = 1$.

We find $\mathsf{N}(M_\varphi) = \{u \in C[-1,1] = \varphi u = 0\} = \{u \in C[-1,1] : \text{sppt } u \subset [-1,0]\}$. Similarly, $\mathsf{R}(M_\varphi) = \{u = \varphi w : w \in C[-1,1]\}$, so $\overline{\mathsf{R}(M_\varphi)} = \{w \in C[-1,1] : \text{sppt } f \subset [0,1]\}$.

It follows that $\tilde{\mathcal{X}} = \mathsf{N}(M_\varphi) \oplus \overline{\mathsf{R}(M_\varphi)} = \{f \in C[-1,1] : f(0) = 0\} \neq \mathcal{X}$.

Remark 2.6.11. If the previous example was repeated for $\mathcal{X} = L^p[-1,1]$, $1 \leq p < \infty$, we would find $\tilde{\mathcal{X}} = \mathcal{X}$, since functions can jump at 0. However, this does not work for $p = \infty$.

2.7 Duality

Definition 2.7.1 (Dual space)**.** Let $\mathcal{X}$ be a Banach space over $\mathbf{C}$. Then the *dual space* of $\mathcal{X}$ is defined to be $\mathcal{X}' := \mathcal{L}(\mathcal{X}, \mathbf{C})$. That is, the Banach space of all bounded linear functionals from $\mathcal{X}$ to $\mathbf{C}$.

Remark 2.7.2. The norm of this space is the usual norm on the space of bounded linear functions. That is, when $f \in \mathcal{X}'$ then the norm is given by

$$\| f \| = \| f \|_\infty = \sup_{v \neq 0} \frac{|f(v)|}{\| v \|}.$$

Definition 2.7.3 (Bilinear pairing)**.** Define $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X}' \to \mathbf{C}$ by:

$$\langle v, U \rangle = U(v).$$

We also write $\langle \mathcal{X}, \mathcal{X}' \rangle$ to denote that the spaces $\mathcal{X}$ and $\mathcal{X}'$ are dual pairs.

Example 2.7.4. Fix $1 < p < \infty$. Then $(\ell^p)' \cong l^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. For $p = 1$, $(l^1)' \cong l^\infty$. This isomorphism $\Phi : v \mapsto V$ is in fact an isometry and the pairing is then given by

$$\langle u, V \rangle = \sum_{j=1}^{\infty} v_j u_j.$$

Similarly, for a σ-finite measure space Ω, we find that $(L^p(\Omega))' \cong L^{p'}(\Omega)$ for $1 < p < \infty$ and $(L^1(\Omega))' \cong L^\infty(\Omega)$. Let $\Phi : f \mapsto F$, then the pairing is

$$\langle u, F \rangle = \int_\Omega fu \; d\mu.$$

Definition 2.7.5. Define $j : \mathcal{X} \to \mathcal{X}''$ by:

$$j(u)(V) = \langle u, V \rangle$$

whenever $V \in \mathcal{X}'$.

Definition 2.7.6 (Reflexive Banach space)**.** A Banach space $\mathcal{X}$ is called reflexive if $j : \mathcal{X} \to \mathcal{X}''$ is an isometric isomorphism.

Example 2.7.7. ℓ^p and $L^p(\Omega)$ $1 < p < \infty$ are reflexive.

Definition 2.7.8 (Annhilator)**.** Let $\mathcal{Y} \subset \mathcal{X}$ with $\mathcal{Y}$ linear. Define the *annihilator* $\mathcal{Y}^\perp \subset \mathcal{X}'$:

$$\mathcal{Y}^\perp = \left\{ V \in X' : \langle u, V \rangle = 0, \ \forall u \in \mathcal{Y} \right\}.$$

There is a dual concept.

Definition 2.7.9 (Annihilator of the dual)**.** Let $\mathcal{Z} \subset \mathcal{X}'$ with $\mathcal{Z}$ linear. Define the *dual annihilator* $\mathcal{Z}^\perp \subset \mathcal{X}$:

$$\mathcal{Z}^\perp = \{ u \in X : \langle u, V \rangle = 0, \ \forall V \in \mathcal{Z} \}.$$

Proposition 2.7.10 (Properties of the annihiliator and dual annihilator)**.** *Let $\mathcal{Y}$ be a linear subset of $\mathcal{X}$. Then:*

(i) $\mathcal{Y}^\perp$ *is a closed subspace.*

(ii) $\mathcal{Y}^\perp = \overline{\mathcal{Y}}^\perp$.

(iii) $\mathcal{Y}^{\perp\perp\perp} = \mathcal{Y}^\perp$.

(iv) $\mathcal{Y} \subset \overline{\mathcal{Y}} \subset \mathcal{Y}^{\perp\perp}$.

The properties hold if we replace $\mathcal{Y}$ with $\mathcal{Z} \subset \mathcal{X}'$.

Remark 2.7.11. We can think of $^{\perp\perp}$ as a *weak* closure with respect to duality.

Definition 2.7.12 (Hanh-Banach property)**.** We say that $\mathcal{Y} \subset \mathcal{X}$ has the *Hahn-Banach property* with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$ if for all $v \notin \overline{\mathcal{Y}}$ there exists $U \in \mathcal{Y}^\perp$ such that $\langle v, U \rangle \neq 0$. Similarly, $\mathcal{Z} \subset \mathcal{X}'$ has the *Hahn-Banach property* with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$ means that for all $U \notin \overline{\mathcal{Z}}$ there exists a $v \in \mathcal{Z}^\perp$ such that $\langle v, U \rangle \neq 0$.

Remark 2.7.13. This is a separation property. For instance, given the Hahn-Banach property for $\mathcal{X}$, we can find a functional to separate $\overline{\mathcal{Y}}$ from $u \notin \mathcal{Y}$.

Theorem 2.7.14. $\overline{\mathcal{Y}} = \mathcal{Y}^{\perp\perp}$ *if and only if $\mathcal{Y}$ has the Hahn-Banach property. Similarly for $\mathcal{Z} \subset \mathcal{X}'$.*

Proof. Suppose $\overline{\mathcal{Y}} \subsetneq \mathcal{Y}^{\perp\perp}$. Then there exists $v \in \mathcal{Y}^{\perp\perp}$ with $v \notin \overline{\mathcal{Y}}$. So there exists $U \in \mathcal{Y}^\perp$ such that $\langle v, U \rangle = 0$.

For the other direction, fix $v \notin \overline{\mathcal{Y}}$ and suppose that for all $U \in \mathcal{Y}^\perp$, $\langle v, U \rangle = 0$. By definition $v \in \mathcal{Y}^{\perp\perp}$ but $\mathcal{Y}^{\perp\perp} = \overline{\mathcal{Y}}$. So, $v \in \overline{\mathcal{Y}}$ which is a contradiction. The proof for $\mathcal{Z} \subset \mathcal{X}'$ is similar. $\square$

Definition 2.7.15 (Hanh-Banach property on the whole space)**.** We say that $\mathcal{X}$ or $\mathcal{X}'$ has the Hahn-Banach property with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$ if every linear subspace has the Hahn-Banach property with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$.

Theorem 2.7.16. *(i) If $\mathcal{X}$ is separable then $\mathcal{X}$ has the Hahn-Banach property. Here, separable means that there is a countable dense subset.*

(ii) Using the axiom of choice, a general space $\mathcal{X}$ has the Hahn-Banach property. (Hahn-Banach Theorem).

Remark 2.7.17. Note that for the space $\mathcal{X}'$, the Hahn-Banach property is still taken with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$, not $\langle \mathcal{X}', \mathcal{X}'' \rangle$. Assuming the axiom of choice the previous theorem tells us that the $\mathcal{X}'$ has the Hahn-Banach property with respect to $\langle \mathcal{X}', \mathcal{X}'' \rangle$.

The situation generalises the special case when $\mathcal{X}$ is reflexive, where $\langle \mathcal{X}', \mathcal{X}'' \rangle$ is equivalent to $\langle \mathcal{X}, \mathcal{X}' \rangle$. In fact, it is to generalise the reflexive condition why a "dual" formulation of the annihilator is necessary.

Theorem 2.7.18. *Let Ω be a separable σ-finite measure space and $1 \leq p < \infty$. (Here Ω separable means that the σ-algebra of measurable sets is generated by a countable sequence of measurable sets). Then $L^p(\Omega)$ is separable.*

Definition 2.7.19 (Dual operator with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$)**.** Let $S \in \mathcal{L}(\mathcal{X})$. Define $S' \in \mathcal{L}(\mathcal{X}')$ by:

$$\langle v, S'U \rangle = \langle Sv, U \rangle$$

for all $v \in \mathcal{X}$ and $U \in \mathcal{X}'$.

Proposition 2.7.20. $(S_1 S_2)' = S_2' S_1'$

Let $T \in \mathcal{L}(X)$ and note that $(T - \lambda I)' = (T' - \lambda I)$. We have:

Proposition 2.7.21. *$\rho(T) \subset \rho(T')$, $\sigma(T') \subset \sigma(T)$ and*

$$\mathsf{R}_T(\zeta) = (\mathsf{R}_T(\zeta))'$$

We now relate this back to spectral projections. We are interested in determining when $\mathcal{X} = \mathsf{N}(T - \lambda I) \oplus \mathsf{R}(T - \lambda I)$.

Proposition 2.7.22. *(i) $\overline{\mathsf{R}(T - \lambda I)}^{\perp} = \mathsf{N}(T' - \lambda I)$.*

(ii) If $\mathcal{X}$ has the Hahn-Banach property, then $\mathsf{N}(T' - \lambda I) = \overline{\mathsf{R}(T - \lambda I)}^{\perp}$.

(iii) If $\mathcal{X}$ has the Hahn-Banach property, $\overline{\mathsf{R}(T' - \lambda I)}^{\perp} = \mathsf{N}(T - \lambda I)$.

(iv) If $\mathcal{X}'$ has the Hahn-Banach property, then $\mathsf{N}(T - \lambda I)^{\perp} = \overline{\mathsf{R}(T' - \lambda I)}$.

This leads immediately to the following theorem.

Theorem 2.7.23 (Spectral decomposition of the space)**.** *Let $T \in \mathcal{L}(X)$, and $\lambda \in \mathbf{C}$. Suppose there exists $\zeta_n \in \rho(T)$ such that $\zeta_n \to \lambda$ and there exists a $C > 0$ such that $|\zeta_n - \lambda| \, \| \mathsf{R}_T(\zeta_n) \| \leq C$. If both $\mathcal{X}$ and $\mathcal{X}'$ have the Hahn-Banach property with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$, then:*

$$\mathcal{X} = \mathsf{N}(T - \lambda I) \oplus \overline{\mathsf{R}(T - \lambda I)}.$$

Proof. By Theorem (2.6.7), we already have

$$\mathcal{X} \supset \tilde{\mathcal{X}} = \mathsf{N}(T - \lambda I) \oplus \overline{\mathsf{R}(T - \lambda I)}.$$

We have shown that $\zeta_n \in \rho(T')$, so $\zeta_n \to \lambda$ with the same resolvent bounds for T'. It follows that

$$\mathcal{X}' \supset \tilde{\mathcal{X}}' = \mathsf{N}(T' - \lambda I) \oplus \overline{\mathsf{R}(T' - \lambda I)}.$$

Fix $u \in \tilde{X}^{\perp}$. So, $u \in \mathsf{N}(T - \lambda I)^{\perp} \cap \mathsf{R}(T - \lambda I)^{\perp}$. By the previous proposition, we have:

$$\mathsf{N}(T' - \lambda I) = \overline{\mathsf{R}(T - \lambda I)}^{\perp}$$
$$\mathsf{N}(T - \lambda I)^{\perp} = \overline{\mathsf{R}(T' - \lambda I)}$$

since $\mathcal{X}$ and $\mathcal{X}'$ have the Hahn Banach property respectively. But this means that

$$u \in \mathsf{N}(T' - \lambda I) \cap \overline{\mathsf{R}(T' - \lambda I)} = \{0\}$$

which makes $\tilde{\mathcal{X}}^{\perp} = \{0\}$. It follows that $\tilde{\mathcal{X}}^{\perp\perp} = \mathcal{X}$. Further, since $\mathcal{X}$ has the Hahn-Banach property, $\tilde{\mathcal{X}}^{\perp\perp} = \tilde{\mathcal{X}}$ which establishes the result. □

Remark 2.7.24. In particular, this decomposition holds if $\mathcal{X}$ is reflexive. For instance, for $L^p(\Omega)$ when $1 < p < \infty$ and $\Omega \subset \mathbf{R}^n$.

Note, that this is *not* true for $C[-1, 1]$ and $L^\infty[-1, 1]$. What about for $L^1[-1, 1]$?

Chapter 3

Unbounded Operators

We have so far considered operators that are bounded. But there are many useful operators that are unbounded. The following is an important example to highlight this.

Example 3.0.25. Let $\mathcal{X} = L^2(B)$, where B is an open ball in $\mathbf{R}^n$. Define the *Laplacian*:

$$\Delta u = \sum_j \frac{\partial^2}{\partial x_j^2} u.$$

We can define various domains in $L^2(B)$:

(i) $\Delta_0 : C_c^\infty(B) \to L^2(B)$.

(ii) $\Delta_1 : W^{2,2}(B) \to L^2(B)$, where $W^{2,2}(B) = \left\{ u \in L^2(B) : \frac{\partial^2 u}{\partial x_j^2} \in L^2(B) \right\}$.

(iii) $\Delta_D : \{ u \in W^{2,2}(B) : u = 0 \text{ on } \partial B \} \to L^2(B)$ *(Dirichlet Laplacian).*

(iv) $\Delta_N = \{ u \in W^{2,2}(B) : \frac{\partial u}{\partial x} = 0 \text{ on } \partial B \} \to L^2(B)$ *(Neumann Laplacian).*

These are four totally different operators. The first two operators do not even admit a functional calculus.

3.1 Closed Operators

Definition 3.1.1 (General operator)**.** Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. A *linear operator* T from $\mathcal{X}$ to $\mathcal{Y}$ is a linear mapping $T : \mathsf{D}(T) \to \mathcal{Y}$ where $\mathsf{D}(T)$ is a linear subspace of $\mathcal{X}$.

Remark 3.1.2. For a general operator, it is important to specify the domain $\mathsf{D}(T)$.

Definition 3.1.3 (Closed operator)**.** T is a *closed operator* means that the graph

$$\mathcal{G}(T) = \{(u, Tu) \in \mathcal{X} \times \mathcal{Y} : u \in \mathsf{D}(T)\}$$

is closed in $\mathcal{X} \times \mathcal{Y}$. We usually simply say that T is closed.

Proposition 3.1.4. *The following are equivalent:*

(i) T *is closed.*

(ii) *If* $u_n \in \mathsf{D}(T)$, $u_n \to u \in \mathcal{X}$ *and* $Tu_n \to v \in \mathcal{Y}$ *then* $u \in \mathsf{D}(T)$ *and* $Tu = v$.

(iii) $\mathsf{D}(T)$ *is a Banach space under the graph norm given by:*

$$\| u \|_{\mathsf{D}(T)} = \| u \| + \| Tu \| .$$

Remark 3.1.5. Note that the graph norm looks similar to a Sobolev norm.

Definition 3.1.6 (Set of closed operators)**.** We denote the collection of closed linear operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. We write $\mathcal{C}(\mathcal{X}) := \mathcal{C}(\mathcal{X}, \mathcal{X})$.

Remark 3.1.7. $\mathcal{L}(\mathcal{X}, \mathcal{Y}) \subset \mathcal{C}(\mathcal{X}, \mathcal{Y})$.

Definition 3.1.8. Let $S, T \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$. We write $S \subset T$ to mean $\mathcal{G}(S) \subset \mathcal{G}(T)$. That is, $\mathsf{D}(S) \subset \mathsf{D}(T)$ and $Su = Tu$ for all $u \in \mathsf{D}(S)$.

Remark 3.1.9. $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ is *not* a linear space. Note that $\Delta_0 - \Delta_0 \neq 0$ since 0 is defined everywhere. However, $\Delta_0 - \Delta_0 \subset 0$. It is in fact important to compute domains when operators are added and multiplied. The domains taken are the largest for which the constructions make sense.

Proposition 3.1.10 (Domains)**.** *(i)* $(S+T)u = Su+Tu$, $u \in \mathsf{D}(S+T) = \mathsf{D}(S) \cap \mathsf{D}(T)$.

(ii) $(ST)u = S(Tu)$, $u \in \mathsf{D}(ST) = \{u \in \mathsf{D}(T) : Tu \in \mathsf{D}(S)\}$.

(iii) *If* $T : \mathsf{D}(T) \to \mathsf{R}(T)$ *is bijective, then* $\mathsf{D}(T^{-1}) = \mathsf{R}(T)$.

We list properties of operators in $\mathcal{C}(\mathcal{X}, \mathcal{Y})$.

Theorem 3.1.11 (Algebraic properties of closed operators)**.** *(i)* $S + T = T + S$.

(ii) $(S + T) + U = S + (T + U)$.

(iii) $S + 0 = S$.

(iv) $0S \subset S0 = 0$.

(v) $S - S \subset 0$.

(vi) $(ST)U = S(TU)$.

(vii) $S(T + U) \supset ST + SU$.

(viii) $(S + T)U = SU + TU$

(ix) *If* S *is bijective,* $S^{-1}S \subset I$ *and* $SS^{-1} \subset I$.

Proof. Exercise. □

Also,

Proposition 3.1.12. *Suppose $B \in \mathcal{L}(\mathcal{X})$ and $T \in \mathcal{C}(\mathcal{X})$. Then,*

(i) $TB \in \mathcal{C}(\mathcal{X})$.

(ii) If in addition B is bijective and $B^{-1} \in \mathcal{L}(\mathcal{X})$, then $BT \in \mathcal{C}(\mathcal{X})$.

(iii) If T is bijective, then $T^{-1} \in \mathcal{C}(\mathcal{X})$.

Proof. (i) Let $u_n \in \mathsf{D}(TB)$, with $u_n \to u$ and $TBu_n \to v$. Now, $Bu_n \to Bu$ by the bounded hypothesis. Also, $w_n = Bu_n \in \mathsf{D}(T)$ and $w_n \to Bu$ and $Tw_n \to v$. So, $v = TBu$.

We leave (ii) and (iii) as an exercise. □

We want to start examining the spectral properties of closed operators. We start with a notion of invertibility.

Definition 3.1.13 (Invertible closed operator)**.** $T \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ is invertible means:

(i) T is bijective with $\mathsf{R}(T) = \mathcal{Y}$.

(ii) $T^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$.

Proposition 3.1.14. *Let $S \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ with S invertible. Let $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that*

$$\| B \| < \frac{1}{\| S^{-1} \|}.$$

Then, $(S - B)$ is invertible and

$$\| (S-B)^{-1} \| \leq \frac{\| S^{-1} \|}{1 - \| B \| \, \| S^{-1} \|}.$$

Proof. We note $BS^{-1} \in \mathcal{L}(Y)$ and $\| BS^{-1} \| \leq \| B \| \, \| S^{-1} \| < 1$. Then,

$$\| I - BS^{-1} \| \leq \frac{\| S^{-1} \|}{1 - \| BS^{-1} \|} \leq \frac{\| S^{-1} \|}{1 - \| B \| \, \| S^{-1} \|}.$$

So, $S - B = S - BS^{-1}S = (I - BS^{-1})$ and $(I - BS^{-1})$ is bounded and invertible and $(S-B)^{-1} = S^{-1}(I - BS^{-1})^{-1}$. □

Definition 3.1.15 (Resolvent operator and resolvent set of a closed operator)**.** If $T \in \mathcal{C}(\mathcal{X}) \setminus \mathcal{L}(\mathcal{X})$. Then $\zeta \in \rho(T)$ means that $(\zeta I - T)$ is invertible. As before, we call $\rho(T)$ the resolvent set of T and $\mathsf{R}_T(\zeta) = (\zeta I - T)^{-1} \in \mathcal{L}(X)$ the resolvent operator. If further $T \in \mathcal{L}(\mathcal{X})$ then $\infty \in \rho(T)$.

Proposition 3.1.16. *$\mathsf{R}_T \in \mathsf{H}(\rho(T), \mathcal{L}(X))$.*

Proof. As before, except for the case when $T \in \mathcal{L}(X)$ where we get $\infty \in \rho(T)$. We leave this as an exercise. □

Corollary 3.1.17. *$\rho(T)$ is an open subset of* $\mathbf{C}$.

Proof. Let $\zeta \in \rho(T)$ and fix $\mu \in \mathbf{C}$ such that

$$|\mu - \zeta| < \frac{1}{\| (\zeta I - T)^{-1} \|}.$$

Note that $(\mu I - T) = (\zeta I - T) + (\mu - \zeta)I$, so apply previous proposition with $B = (\mu - \zeta)I$ and $S = (\zeta I - T)$ to find that $(\mu I - T)$ is invertible. □

In order to discuss the spectrum of a closed operator, we need to define the extended complex plane.

Definition 3.1.18 (Extended Complex plane)**.** We write $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$ for the extended complex plane. We topologise this by saying that U is open in $\mathbf{C}_\infty$ if either

(i) U is open in $\mathbf{C}$.

(ii) $U = \mathbf{C}_\infty \setminus K$, where $K \Subset \mathbf{C}$.

Definition 3.1.19 (Spectrum of a closed operator)**.** The definition is unchanged if $T \in \mathcal{L}(X)$. When $T \in \mathcal{C}(X) \setminus \mathcal{L}(X)$, define

$$\sigma(T) = \mathbf{C}_\infty \setminus \rho(T).$$

Remark 3.1.20. The spectrum of a $T \in \mathcal{L}(\mathcal{X})$ is consistent with the previous definition.

Proposition 3.1.21. *$\sigma(T)$ is nonempty.*

Proof. We have already proved this for $T \in \mathcal{L}(X)$. When $T \in \mathcal{C}(X) \setminus \mathcal{L}(X)$, $\infty \in \sigma(T)$. □

3.2 Functional calculus for closed operators

We have seen that while $T \in \mathcal{C}(X)$, the resolvent operator $\mathsf{R}_T(\alpha) \in \mathcal{L}(X)$. The idea is to map $\sigma(T)$ to some $\sigma(S)$ with S bounded and define an $\mathsf{H}(\Omega)$ functional calculus by the previous Dunford-Riesz functional calculus for the operator S.

In general, closed operators may have $\sigma(T) = \mathbf{C}_\infty$, but we will not be discussing these. Unless otherwise stated, we assume that $\rho(T) \neq \varnothing$.

Definition 3.2.1. Fix $\alpha \in \mathbf{C}$. Define $\mathsf{R}_\alpha : \mathbf{C}_\infty \to \mathbf{C}_\infty$ by:

$$\mathsf{R}_\alpha(\mu) = \begin{cases} \frac{1}{\mu - \alpha} & \mu \neq \alpha, \mu \neq \infty \\ \infty & \mu = \alpha \\ 0 & \mu = \infty \end{cases}$$

Proposition 3.2.2. *R_α is a homeomorphism.*

Proof. Exercise. $\square$

Proposition 3.2.3. *Let $\alpha \in \rho(T)$. Then,*

$$-R_\alpha(\sigma(T)) = \sigma(R_T(\alpha))$$

or equivalently,

$$-R_\alpha(\rho(T)) = \rho(R_T(\alpha)).$$

Also, for $\zeta \in \rho(T)$ where $\zeta \notin \{\alpha, \infty\}$, setting $\mu = -R_\alpha(\zeta)$, we have

$$\mu(R_{R_T(\alpha)}(\mu) = (\alpha - \zeta)R_T(\zeta) + I.$$

Proof. The equivalence is a direct consequence of the previous proposition that R_α is a homeomorphism.

Fix $\zeta \in \rho(T)$. If $\zeta = \alpha$, then $\mu = \infty$ and so $\mu \in \rho(\mathsf{R}_T(\zeta))$ since $\mathsf{R}_T(\zeta)$ is bounded.

Suppose $\zeta = \infty$. This means that T is bounded and $(\alpha I - T) \in \mathcal{L}(X)$ which holds if and only if $\mathsf{R}_T(\alpha)$ is invertible, that is, if and only if $\mu = 0 \in \rho(\mathsf{R}_T(\alpha))$.

Now, suppose $\zeta \in \rho(T) \setminus \{\alpha, \infty\}$. Then, $\mu \notin \{0, \infty\}$. So,

$$W = \frac{1}{\mu}\left[(\alpha - \zeta)\mathsf{R}_T(\zeta) + I\right] \in \mathcal{L}(\mathcal{X}).$$

We claim that

$$[\mu - \mathsf{R}_T(\alpha)]W = W[\mu - \mathsf{R}_T(\alpha)] = I.$$

which we this as an exercise.

We also leave the proof of

$$-\mathsf{R}_\alpha(\sigma(T)) \subset \sigma\left(\mathsf{R}_{\mathsf{R}_T(\alpha)}\right)$$

as an exercise. $\square$

Definition 3.2.4 (Holomorphic functional calculus for closed operators)**.** Let $T \in \mathcal{C}(\mathcal{X})$ and Ω an open set in $\mathbf{C}_\infty$ such that $\sigma(T) \subset \Omega \subsetneq \mathbf{C}_\infty$. An $\mathsf{H}(\Omega)$ functional calculus of T is a mapping $\Phi_T : \mathsf{H}(\Omega) \to \mathcal{L}(\mathcal{X})$ such that

(i) Φ_T is an algebra homomorphism and $\Phi_T(1) = I$.

(ii) There exists $\alpha \in \mathbf{C}_\infty$ such that:

$$\Phi_T(\mathsf{R}_\alpha) = -\mathsf{R}_T(\alpha)$$
$$\Phi_T(\mathrm{id}) = T.$$

if $\alpha \neq \infty$.

(iii) If $f_n \to f$ uniformly on compact subsets of Ω, then $f_n(T) \to f(T)$ in $\mathcal{L}(\mathcal{X})$.

Remark 3.2.5. Note that the statement $\Phi_T(\mathrm{id}) = T$ along with $\Phi_T(1) = \mathrm{id}$ is equivalent to the definition of an $\mathsf{H}(\Omega)$ functional calculus when T is bounded.

Theorem 3.2.6 (Existence and uniqueness of a functional calculus for closed operators)**.** *Such a functional calculus exists and it is unique.*

Proof. Fix $f \in \mathsf{H}(\Omega)$. Then, $(f \circ (-\mathsf{R}_\alpha)^{-1}) \in \mathsf{H}(-\mathsf{R}_\alpha(\Omega))$. Define Φ_T by:

$$\Phi_T(f) := (f \circ (-\mathsf{R}_\alpha)^{-1})(\mathsf{R}_T(\alpha)) = (f \circ \mathsf{R}_\alpha^{-1})(-\mathsf{R}_T(\alpha)) = \Phi_{-\mathsf{R}_T(\alpha)}(f \circ \mathsf{R}_\alpha^{-1}).$$

Since $\Phi_{-\mathsf{R}_T}$ is the previous bounded functional calculus, the only thing we need to show is $\Phi_T(\mathsf{R}_\alpha) = -\mathsf{R}_T(\alpha)$. But this follows easily:

$$\Phi_T(\mathsf{R}_\alpha) = (\mathsf{R}_\alpha \circ \mathsf{R}_\alpha^{-1})(-\mathsf{R}_T(\alpha)) = \mathrm{id}(-\mathsf{R}_T(\alpha)) = -\mathsf{R}_T(\alpha).$$

□

As before, we typically write $f(T) := \Phi_T(f)$ in the light of the previous theorem. We list the following important properties of the $\mathsf{H}(\Omega)$ functional calculus we have defined.

Theorem 3.2.7 (Properties of $\mathsf{H}(\Omega)$ functional calculus for closed operators)**.** *(i)* $\Phi_T(r) = r(T)$ *where* $r \in \mathsf{H}(\Omega)$ *is a rational function.*

(ii) When $T \in \mathcal{L}(\mathcal{X})$, Φ_T *agrees with the* $\mathsf{H}(\Omega)$ *functional calculus defined in §2.3.*

(iii) Let Ω_1, Ω_2 *be open sets in* $\mathbf{C}_\infty$ *and suppose* $\sigma(T) \subset \Omega = \Omega_1 \cap \Omega_2$. *If* $f_1 \in \mathsf{H}(\Omega_1)$ *and* $f_2 \in \mathsf{H}(\Omega_2)$ *and* $f_1 = f_2$ *on* Ω, *then* $f_1(T) = f_2(T)$.

(iv) $\sigma(f(T)) = f(\sigma(T))$.

(v) There exists a $C > 0$ *such that*

$$\| f(T) \| \leq C \| f \|_\infty$$

whenever $f \in \mathsf{H}^\infty(\Omega)$.

(vi) If $g \in \mathsf{H}(\Omega)$ *and* $f \in \mathsf{H}(\tilde{\Omega})$ *where* $\sigma(g(T)) = g(\sigma(T)) \subset \tilde{\Omega}$, *then*

$$(f \circ g)(T) = f(g(T)).$$

(vii) If $f_1, f_2 \in \mathsf{H}(\Omega)$, *then*

$$f_1(T)f_2(T) = (f_1 f_2)(T) = f_2(T)f_1(T).$$

(viii) We still have uniqueness if we replace axiom (iii) in the definition of the $\mathsf{H}(\Omega)$ *calculus with:*

(iii') If $f_n \to f$ *uniformly on compact sets of* Ω, *then* $f_n(T)u \to f(T)u$ *for all* $u \in \mathcal{X}$.

The spectral projections for the bounded operators also apply to closed operators. However, in this situation, we also need to give consideration to the point ∞.

Theorem 3.2.8 (Spectral decomposition of a closed operator)**.** *Let* $T \in \mathcal{C}(\mathcal{X})$, $\alpha \in \rho(T)$, $\alpha \neq \lambda \in \mathbf{C}_\infty$ *and there exists a* $C > 0$ *and a sequence* $\zeta_n \in \rho(T)$ *such that* $\zeta_n \to \lambda$. *Then,* *(i) Suppose $\lambda \neq \infty$ and $|\zeta_n - \lambda| \, \| \, R_T(\zeta_n) \, \| \leq C$. Then, defining*

$$\tilde{\mathcal{X}}^\lambda = \{u \in \mathcal{X} : \zeta_n - \lambda R_T(\zeta_n)u \text{ converges in } \mathcal{X}\}$$

we have the decomposition

$$\mathcal{X} \supset \tilde{\mathcal{X}}^\lambda = \mathsf{N}(T - \lambda I) \oplus \overline{\mathsf{R}(T - \lambda I)}.$$

(ii) Suppose $\lambda = \infty$ and $|\zeta_n| \, \| \, R_T(\zeta_n) \, \| \leq C$. Let $w_n = -R_\alpha(\zeta_n)$. Then, defining

$$\tilde{\mathcal{X}}^\infty = \{u \in \mathcal{X} : (I - \zeta_n R_T(\alpha)(\zeta_n))u \text{ converges in } \mathcal{X}\}$$

we have

$$\mathcal{X} \supset \tilde{\mathcal{X}}^\infty = \overline{\mathsf{D}(T)}.$$

Also, we have the characterisation

$$\overline{\mathsf{D}(T)} = \overline{\mathsf{R}(R_T(\alpha))} = \{u \in \mathcal{X} : \zeta_n R_T(\zeta_n)u \to u\} = \{u \in \mathcal{X} : T R_T(\zeta_n)u \to 0\}$$

Furthermore, if $\mathcal{X}, \mathcal{X}'$ have the Hahn-Banach property with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$, then $\tilde{\mathcal{X}}^\lambda = \mathcal{X}$.

Proof. (i) Either we can check that the proof for bounded operators works for closed operators, or we can simply transfer the results by applying the result for bounded operators to $\mathsf{R}_T(\alpha)$.

(ii) Again, we can redo the proof as for bounded operators or transfer the results to $\mathsf{R}_T(\alpha)$. We take the latter approach.

We show that we have resolvent bounds at $0 = -\mathsf{R}_\alpha(\infty)$ for the operator $\mathsf{R}_T(\alpha)$. We note that

$$w_n \mathsf{R}_{\mathsf{R}_T(\alpha)}(w_n) = (\alpha - \zeta_n)\mathsf{R}_T(\zeta_n) + I = I - \zeta_n \mathsf{R}_T(\zeta_n) + \alpha \mathsf{R}_T(\zeta_n)$$

By the resolvent bound for T at ∞, $\alpha \mathsf{R}_T(\zeta_n) \to 0$ and we have the resolvent bound

$$|w_n| \left\| \, \mathsf{R}_{\mathsf{R}_T(\alpha)}(w_n) \, \right\| \leq C + 2.$$

Also,

$$\tilde{X}^\infty = \left\{u \in \mathcal{X} : w_n \mathsf{R}_{\mathsf{R}_T(\alpha)}(w_n)u \text{ converges in } \mathcal{X}\right\}.$$

It follows then that $\mathcal{X} \supset \tilde{\mathcal{X}}^\infty = \mathsf{N}(\mathsf{R}_T(\alpha)) \oplus \overline{\mathsf{R}(\mathsf{R}_T(\alpha))}$ but since $\mathsf{R}_T(\alpha)$ is bijective, $\mathsf{N}(\mathsf{R}_T(\alpha)) = 0$. Also,

$$\mathsf{R}(\mathsf{R}_T(\alpha)) = \mathsf{D}(\mathsf{R}_T(\alpha)^{-1}) = \mathsf{D}(\alpha I - T) = \mathsf{D}(T)$$

which proves the result.

□

We have the following important consequence.

Corollary 3.2.9 (Densely defined closed operator)**.** *If $\mathcal{X}, \mathcal{X}'$ have the Hahn-Banach property with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$, then T is a densely defined operator.*

Typically, from now on, we consider operators with spectrum in a sector or a bisector. Our special points are $\{0, \infty\}$. The Ω are larger sectors that do not contain $\{0, \infty\}$.

Chapter 4

Sectorial Operators

So far, we have considered a functional calculus when $\sigma(T) \subset \Omega \subset \mathbf{C}_\infty$.

Now, suppose $T = \Delta$, the Laplacian on $\mathcal{X} = L^2(\mathbf{R}^n)$, which has spectrum, $\sigma(\Delta) = [0, \infty]$. We want to look at $e^{-t\Delta}$ but $f(\zeta) = e^{-t\zeta}$ is not holomorphic at ∞. Also, we want $e^{-t\sqrt{\Delta}}$ but $f(\zeta) = e^{-t\sqrt{\zeta}}$ is not holomorphic at 0.

So we look at operators T such that:

(i) $\sigma(T)$ is contained in a "sector." Then we consider Ω to be a slightly larger sector without the points $\{0, \infty\}$.

(ii) $\sigma(T)$ is contained in a double sector and we take Ω to be a slightly larger double sector not containing $\{0, \infty\}$.

The first situation (i) generalises the theory of Δ, and (ii) generalises $\begin{bmatrix} 0 & \mathrm{div} \\ -\nabla & 0 \end{bmatrix}$.

4.1 The Ψ functional calculus

Definition 4.1.1 (Sector)**.** Let $0 \leq \omega < \pi$. Define the *closed ω sector*:

$$S_{\omega+} = \{\zeta \in \mathbf{C}_\infty : |\arg \zeta| \leq \omega \text{ or } \zeta = 0, \infty\}.$$

Let $0 < \mu < \pi$. Define the *open μ sector*:

$$S^o_{\mu+} = \{\zeta \in \mathbf{C} : |\arg \zeta| < \mu \text{ or } \zeta \neq 0\}.$$

Definition 4.1.2 (ω-Sectorial operator)**.** Let $0 \leq \omega < \pi$. We say that $T : \mathsf{D}(T) \to \mathcal{X}$ is an *ω-sectorial operator* if:

(i) $T \in \mathcal{C}(\mathcal{X})$.

(ii) $\sigma(T) \subset S_{\omega+}$.

(iii) For all $\mu > \omega$, there exists a $C_\mu \geq 0$ such that

$$|\zeta| \, \| \mathsf{R}_T(\zeta) \| \leq C_\mu$$

for all $\zeta \in \mathbf{C} \setminus \{0\}$ such that $|\arg \zeta| \geq \mu$.

Since our open sector may not contain the entire spectrum (ie., points 0 and ∞), we define a class of functions that gives control by having rapid decay at these points.

Definition 4.1.3 ($\Psi(S^o_{\mu+})$ class functions)**.** Define:

$$\Psi(S^o_{\mu+}) = \left\{ \psi \in \mathsf{H}^\infty(S^o_{\mu+}) : \exists \alpha > 0, C > 0, \psi(\zeta) \leq \frac{C|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}} \right\}.$$

Remark 4.1.4. Note that the decay condition for a function $\psi \in \Psi(S^o_{\mu+})$ is equivalent to:

$$\psi(\zeta) \leq \begin{cases} C|\zeta|^\alpha & |\zeta| \text{ small} \\ C|\zeta|^{-\alpha} & |\zeta| \text{ large} \end{cases}.$$

Definition 4.1.5 ($\Psi(S^o_{\mu+})$ functional calculus)**.** Let T be ω-sectorial and $\psi \in \Psi(S^o_{\mu+})$ where $0 \leq \omega < \mu < \pi$. Define:

$$\psi(T) = \frac{1}{2\pi\imath} \oint_\gamma \psi(\zeta) \mathsf{R}_T(\zeta) \, d\zeta$$

where $\gamma = \{re^{\imath\nu} : \infty > r > 0\} + \{re^{-\imath\nu} : 0 < r < \infty\}$, $\omega < \nu < \mu$ a curve consisting of two rays parametrised anti-clockwise.

Proposition 4.1.6. *The integral*

$$\frac{1}{2\pi\imath} \oint_\gamma \psi(\zeta) R_T(\zeta) \, d\zeta$$

converges absolutely and $\psi(T) \in \mathcal{L}(\mathcal{X})$.

Proof. We compute:

$$\| \psi(T) \| \leq \frac{1}{2\pi} \oint_\gamma \frac{C|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}} C_\nu \frac{1}{|\zeta|} \, |d\zeta| = \frac{1}{2\pi} 2CC_\nu \int_0^\infty \frac{r^\alpha}{1 + r^{2\alpha}} \frac{dr}{r} \leq \tilde{C} < \infty.$$

□

Remark 4.1.7. Note that $\tilde{C}$ depends on ψ and on the resolvent bounds of T.

Example 4.1.8. Some examples of Ψ class functions are

$$\zeta e^{-t\zeta}, \ \frac{\zeta}{1 + \zeta^2}, \ \sqrt{\zeta} e^{-t\sqrt{\zeta}}.$$

By the previous proposition,

$$Te^{-tT}, \ \frac{T}{1 + T^2}, \ \sqrt{T} e^{-t\sqrt{T}} \in \mathcal{L}(\mathcal{X}).$$

Theorem 4.1.9 (Properties of the $\Psi(S^o_{\mu+})$ functional calculus). *(i) The definition of $\psi(T)$ is independent of $\nu \in (\omega, \mu)$.*

(ii) Let $\psi \in \Psi(S^o_{\mu+})$ and fix $\tilde{\mu} \in (\omega, \mu]$. Let $\tilde{\psi} = \psi|_{S^o_{\tilde{\mu}+}}$. Then,

$$\tilde{\psi}(T) = \psi(T).$$

(iii) $\Psi(S^o_{\mu+}) \to \mathcal{L}(\mathcal{X})$ is an algebra homomorphism.

(iv) If $\sigma(T)$ is compact in $S^o_{\mu+}$ (ie., if $0 \in \rho(T)$ and $T \in \mathcal{L}(\mathcal{X})$), then $\psi(T)$ agrees with the functional calculus in §2.3.

(v) If $\psi \in H^\infty(\Omega)$ where Ω is an open set satisfying $S^o_{\mu+} \cup \{0, \infty\} \subset \Omega$, then $\psi(T)$ agrees with the functional calculus in §3.2.

(vi) Let $\psi_n, \psi \in \Psi(S^o_{\mu+})$ and suppose there exists constants $\alpha > 0$ and $C > 0$ such that for all n,

$$|\psi_n(\zeta)| \leq \frac{C|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}}$$

and that $\psi_n \to \psi$ uniformly on compact subsets of $S^o_{\mu+}$. Then, $\psi_n(T) \to \psi(T)$ in $\mathcal{L}(\mathcal{X})$.

Proof. (i) Fix $\omega < \mu < \tilde{\mu}$, and let $\gamma, \tilde{\gamma}$ be corresponding contours. We introduce "cuts"

$$\delta^+_\varepsilon(t) = \varepsilon e^{t\tilde{\mu} + (1-t)\mu},\ 0 \leq t \leq 1$$

and

$$\delta^+_N(t) = N e^{t\tilde{\mu} + (1-t)\mu},\ 0 \leq t \leq 1$$

Now,

$$\left\| \int_{\delta^+_\varepsilon} \psi(\zeta) \mathsf{R}_T(\zeta)\ d\zeta \right\| \leq \tilde{C} l(\delta^+_\varepsilon) \frac{\varepsilon^\alpha}{1 + \varepsilon^{2\alpha}} \frac{1}{\varepsilon} \leq \tilde{C} \varepsilon^\alpha$$

which tends to 0 as $\varepsilon \to 0$. Similarly,

$$\left\| \int_{\delta^+_N} \psi(\zeta) \mathsf{R}_T(\zeta)\ d\zeta \right\| \leq \tilde{C} \frac{1}{N^\alpha}$$

which tends to 0 as $N \to \infty$. We can similarly define δ^-_ε and δ^-_N and show that

$$\left\| \int_{\delta^-_\varepsilon} \psi(\zeta) \mathsf{R}_T(\zeta)\ d\zeta \right\|, \left\| \int_{\delta^-_N} \psi(\zeta) \mathsf{R}_T(\zeta)\ d\zeta \right\| \to 0$$

as $\varepsilon \to 0$, and $N \to \infty$. We leave it as an exercise to check that this proves the claim.

(vi) Let $\gamma_{a,b} = \{\zeta \in \gamma : a \leq |\zeta| \leq \beta\}$. For $N > \delta > 0$,

$$\gamma = \gamma_{0,\delta} + \gamma_{\delta,N} + \gamma_{N,\infty}.$$

Then,

$$
\begin{aligned}
2\pi \, \| \, \psi_n(T) - \psi(T) \, \| \leq & \left\| \int_{\gamma_{0,\delta}} [\psi_n(\zeta) - \psi(\zeta)] \mathsf{R}_T(\zeta) \; d\zeta \right\| \\
& + \left\| \int_{\gamma_{\delta,N}} [\psi_n(\zeta) - \psi(\zeta)] \mathsf{R}_T(\zeta) \; d\zeta \right\| \\
& + \left\| \int_{\gamma_{N,\infty}} [\psi_n(\zeta) - \psi(\zeta)] \mathsf{R}_T(\zeta) \; d\zeta \right\| .
\end{aligned}
$$

Fix $\varepsilon > 0$. Firstly, we can choose $\delta > 0$ small such that

$$
\left\| \int_{\gamma_{0,\delta}} [\psi_n(\zeta) - \psi(\zeta)] \mathsf{R}_T(\zeta) \; d\zeta \right\| < C \int_0^\delta r^\alpha \frac{1}{r} \; dr = C \int_0^\delta r^{\alpha - 1} = \frac{C}{\alpha} \delta^\alpha < \frac{2\pi\varepsilon}{3}.
$$

Then, we can choose $N > \delta > 0$ large so that

$$
\left\| \int_{\gamma_{\delta,N}} [\psi_n(\zeta) - \psi(\zeta)] \mathsf{R}_T(\zeta) \; d\zeta \right\| < C \int_N^\infty \frac{1}{r^\alpha} \frac{1}{r} < \frac{2\pi\varepsilon}{3}.
$$

Now, there exists an $M > 0$ such that for $n > M$,

$$
\left\| \int_{\gamma_{N,\infty}} [\psi_n(\zeta) - \psi(\zeta)] \mathsf{R}_T(\zeta) \; d\zeta \right\| < \int_\delta^N |\psi_n(\zeta) - \psi(\zeta)| \frac{1}{|\zeta|} \; |d\zeta| \leq C \sup_{\zeta \in \gamma_{\delta,N}} |\psi_n(\zeta) - \psi(\zeta)| < \frac{2\pi\varepsilon}{3}.
$$

Combining these three estimates, we have that whenever $n > M$, $\| \, \psi_n(T) - \psi(T) \, \| < \varepsilon$ which completes the proof.

□

As before, we get a splitting of the space with respect to the operator. However, for sectorial operators, it is also useful to consider the splitting with respect to the two points $\{0, \infty\}$.

Proposition 4.1.10 (Splitting of the space with an ω-sectorial operator)**.** *Let T be an ω-sectorial operator. Then,*

(i) $\mathcal{X} \supset \tilde{\mathcal{X}}^0 = \mathsf{N}(T) \oplus \overline{\mathsf{R}(T)}$.

(ii) $\mathcal{X} \supset \tilde{\mathcal{X}}^\infty = \overline{\mathsf{D}(T)}$.

(iii) $\mathcal{X} \supset \tilde{\mathcal{X}}^{\{0,\infty\}} = \mathsf{N}(T) \oplus (\overline{\mathsf{R}(T)} \cap \overline{\mathsf{D}(T)})$ *where* $\tilde{\mathcal{X}}^{\{0,\infty\}} = \tilde{\mathcal{X}}^0 \cap \tilde{\mathcal{X}}^\infty$.

(iv) $\mathsf{D}(T) \cap \mathsf{R}(T) = \mathsf{R}(T(I+T)^{-1})$.

(v) $\overline{\mathsf{D}(T)} \cap \overline{\mathsf{R}(T)} = \overline{\mathsf{D}(T) \cap \mathsf{R}(T)} = \overline{\mathsf{R}(T(I+T)^{-2})}$.

As before, we have equality in (i)-(iii) if $\mathcal{X}, \mathcal{X}'$ have the Hahn-Banach property with respect to $\langle \mathcal{X}, \mathcal{X}' \rangle$.

Proof. We have already established (i) and (ii). We leave (iii) as an exercise.

(iv) To show that $\mathsf{D}(T)\cap\mathsf{R}(T)=\mathsf{R}(T(I+T)^{-2})$, first note that

$$\begin{aligned}\mathsf{D}((I+T)^2)&=\{u\in\mathsf{D}(I+T):(I+T)u\in\mathsf{D}(I+T)\}\\&=\{u\in\mathsf{D}(T):(I+T)u\in\mathsf{D}(T)\}\\&=\{u\in\mathsf{D}(T):Tu\in\mathsf{D}(T)\}\\&=\mathsf{D}(T^2)\end{aligned}$$

since $\mathsf{D}(I+T)=\mathsf{D}(T)$ which is a linear subspace of $\mathcal{X}$.

Thus,

$$\begin{aligned}\mathsf{R}(T(I+T)^{-2})&=T\mathsf{R}((I+T)^{-2})\\&=T\mathsf{D}(I+T)^2\\&=T\mathsf{D}(T^2)\\&=\{u=Tv:v\in\mathsf{D}(T)\text{ and }u=Tv\in\mathsf{D}(T)\}\\&=\mathsf{D}(T)\cap\mathsf{R}(T).\end{aligned}$$

(v) It follows immediately that

$$\overline{\mathsf{R}(T(I+T)^{-2})}=\overline{\mathsf{D}(T)\cap\mathsf{R}(T)}\subset\overline{\mathsf{D}(T)}\cap\overline{\mathsf{R}(T)}.$$

We leave equality as an exercise.

□

Corollary 4.1.11. *If $\psi\in\Psi(S^o_{\mu+})$, then $\mathsf{N}(T)\subset\mathsf{N}(\psi(T))$.*

Proof. Define:

$$\psi_n(\zeta)=\frac{n\zeta}{1+n\zeta}\psi(\zeta)$$

and note that $\psi_n\to\psi$ uniformly on compact subsets of $S^o_{\mu+}$. Now, Therefore, by (vi) of the previous theorem, $\psi_n(T)\to\psi(T)$ in $\mathcal{L}(\mathcal{X})$. Now, when $u\in\mathsf{N}(T)$,

$$\psi_n(T)u=nT(I+nT)^{-1}\psi(T)u=n\psi(T))(I+nT)^{-1}Tu=0$$

for all n. Therefore $\psi(T)u=\lim_{n\to\infty}\psi_n(T)=0$. □

We now increase the class of functions $\Psi(S^o_{\mu+})$ to functions that approach nonzero values at 0 and ∞.

Definition 4.1.12 ($\Phi(S^o_{\mu+})$ class functions)**.** Define:

$$\begin{aligned}\Phi(S^o_{\mu+})=\big\{&\varphi\in\mathsf{H}^\infty(S^o_{\mu+}):\exists\varphi_0,\varphi_\infty\in\mathbf{C},\alpha>0,C>0,\\&|\varphi(\zeta)-\varphi_0|\le C|\zeta|^\alpha\text{ for small }\zeta,\ |\varphi(\zeta)-\varphi_\infty|\le C|\zeta|^{-\alpha}\text{ for large }\zeta\big\}\end{aligned}$$

Proposition 4.1.13 (Characterisation of $\Phi(S^o_{\mu+})$ functions)**.** *$\varphi \in \Phi(S^o_{\mu+})$ if and only if there exists a $\psi \in \Psi(S^o_{\mu+})$ such that*

$$\varphi(\zeta) = \varphi_0 \left(\frac{1}{1+\zeta}\right) + \varphi_\infty \left(\frac{\zeta}{1+\zeta}\right) + \psi(\zeta) = \varphi_\infty - (\varphi_0 - \varphi_\infty)\left(\frac{1}{1+\zeta}\right) + \psi(\zeta).$$

Proof. Exercise. □

Remark 4.1.14. Let $\Omega \supset S^o_{\mu+} \cup \{0, \infty\}$. Then $\Phi(S^o_{\mu+}) \supset \mathsf{H}(\Omega)$ and $\Phi(S^o_{\mu+}) \supset \Psi(S^o_{\mu+})$.

Proposition 4.1.15. *$\Phi(S^o_{\mu+})$ is an algebra.*

Proof. Exercise. □

Definition 4.1.16 ($\Phi(S^o_{\mu+})$ functional calculus)**.** Define $\Phi(S^o_{\mu+}) \to \mathcal{L}(\mathcal{X})$ functional calculus by:

$$\varphi(T) = \varphi_\infty I - (\varphi_0 - \varphi_\infty)\mathsf{R}_T(-1) + \psi(T)$$

where

$$\psi(T) = \frac{1}{2\pi\imath}\oint_\gamma \psi(\zeta)\mathsf{R}_T(\zeta)\ d\zeta.$$

Proposition 4.1.17. *$\Phi(S^o_{\mu+}) \mapsto \mathcal{L}(\mathcal{X})$ is an algebra homomorphism.*

Proof. The only non-trivial step is to show that whenever $\psi \in \Psi(S^o_{\mu+})$, and defining $f(\zeta) = \left(\frac{1}{1+\zeta}\right)\psi(\zeta) \in \Psi(S^o_{\mu+})$, then

$$f(T) = (I+T)^{-1}\psi(T).$$

We leave it as an exercise to verify that:

$$\frac{1}{1+\zeta}\mathsf{R}_T(\zeta) = -(I+T)^{-1}\left(\frac{1}{1+\zeta}I - (\zeta - T)^{-1}\right).$$

Then,

$$\begin{aligned}
f(T) &= \frac{1}{2\pi\imath}\oint_\gamma \frac{1}{1+\zeta}\psi(\zeta)\mathsf{R}_T(\zeta)\ d\zeta \\
&= \frac{1}{2\pi\imath}\oint_\gamma -(I+T)^{-1}\frac{1}{1+\zeta}\psi(\zeta)\ d\zeta + \frac{1}{2\pi\imath}\oint_\gamma (I+T)^{-1}\psi(\zeta)\mathsf{R}_T(\zeta)\ d\zeta \\
&= \frac{-(I+T)^{-1}}{2\pi\imath}\oint_\gamma \frac{1}{1+\zeta}\psi(\zeta)\ d\zeta + (I+T)^{-1}\frac{1}{2\pi\imath}\oint_\gamma \psi(\zeta)\mathsf{R}_T(\zeta)\ d\zeta \\
&= (I+T)^{-1}\psi(T).
\end{aligned}$$

The first term is zero by Cauchy's Theorem. □

Theorem 4.1.18 (Properties of the $\Phi(S^o_{\mu+})$ functional calculus)**.** *(i) $\Phi(S^o_{\mu+}) \mapsto \mathcal{L}(\mathcal{X})$ is an algebra homomorphism.*

(ii) If $\sigma(T)$ is compact in $S^o_{\mu+}$ (ie., if $0 \in \rho(T)$ and $T \in \mathcal{L}(\mathcal{X})$), then $\varphi(T)$ agrees with the functional calculus in §2.3.

(iii) If $\varphi \in H^\infty(\Omega)$ where Ω is an open set satisfying $S^o_{\mu+} \cup \{0, \infty\} \subset \Omega$, then $\varphi(T)$ agrees with the functional calculus in §3.2.

(iv) If $\varphi(T) \in \Psi(S^o_{\mu+})$ then, $\varphi(T)$ is the $\Psi(S^o_{\mu+})$ functional calculus.

Remark 4.1.19. We have functions like $e^{-t\zeta} \in \Phi(S^o_{\mu+})$, so we have $e^{-tT} \in \mathcal{L}(\mathcal{X})$. Note that we still do not have estimates and we still cannot access a functional calculus of functions $\zeta^{is} \in \mathsf{H}^\infty(S^o_{\mu+})$.

We have the following important theorem.

Theorem 4.1.20 (The Convergence Lemma)**.** *Suppose $f_n \in \Psi(S^o_{\mu+})$ and there exists $C > 0$ such that $\| f_n \|_\infty \leq C$ for all n. Further, suppose $f \in H^\infty(S^o_{\mu+})$ and that $f_n \to f$ uniformly on compact subsets of $S^o_{\mu+}$. Also, suppose there exists $\tilde{C} > 0$ such that $\| f_n(T) \| \leq \tilde{C}$. Then, for all $u \in \tilde{\mathcal{X}}^{\{0,\infty\}} = \mathsf{N}(T) \oplus (\overline{\mathsf{D}(T)} \cap \overline{\mathsf{R}(T)})$, $f_n(T)u$ is a convergent sequence in $\mathcal{X}$.*

Proof. Firstly, suppose $u \in \mathsf{N}(T)$. Then by Corollary 4.1.11, we have that $f_n(T)u \to 0$.

Now, fix $u \in \mathsf{D}(T) \cap \mathsf{R}(T)$ By Prop 4.1.10 (iv), there exists $v \in \mathcal{X}$ such that $u = T(I+T)^{-2}v$. Define:

$$\psi_n(\zeta) = \frac{\zeta}{(1+\zeta)^2} f_n(\zeta)$$

and so there exists a $C' > 0$ such that

$$|\psi_n(\zeta)| \leq C' \frac{|\zeta|}{1 + |\zeta|^2}.$$

Setting

$$\psi(\zeta) = \frac{\zeta}{(1+\zeta)^2} f(\zeta)$$

we find that $\psi_n \to \psi$ uniformly on compact subsets of $S^o_{\mu+}$. So, by Theorem 4.1.9, $\psi_n(T)v \to \psi(T)v$. So,

$$f_n(T)u = f_n(T)T(I+T)^{-2}v = f_n(T)u \to \varphi(T)v.$$

By Prop 2.6.3, $f_n(T)u$ converges for all $u \in \overline{\mathsf{R}(T(I+T)^{-2})} = \overline{\mathsf{D}(T)} \cap \overline{\mathsf{R}(T)}$. ☐

We leave the following further statements as exercises.

Remark 4.1.21. Suppose that the hypothesis of Theorem 4.1.20 are satisfied. Then, under the following additional hypothesis, we have the stronger consequences.

1. If $f \in \Phi(S^o_{\mu+})$, then whenever $u \in \overline{\mathsf{D}(T)} \cap \overline{\mathsf{R}(T)}$, $f_n(T)u \to f(T)$ and when $u \in \mathsf{N}(T)$, $f_n(T)u \to 0$.

2. If $\alpha > 0$ and $|f_n(\zeta)| \leq |C||\zeta|^{\alpha}$ when ζ is small, then whenever $u \in \overline{\mathsf{D}(T)}$, $f_n(T)u$ converges.

3. If $\alpha > 0$ and $|f_n(\zeta)| \leq |C||\zeta|^{-\alpha}$ when ζ is large, then whenever $u \in \mathsf{N}(T) \oplus \overline{\mathsf{R}(T)}$, $f_n(T)u$ converges.

4. If (i) + (ii) holds, then $f_n(T) \to f(T)$ on $\overline{\mathsf{D}(T)}$.

5. If (i) + (iii) holds, then $f_n(T) \to f(T)$ on $\overline{\mathsf{R}(T)}$.

4.2 Semigroup Theory

In §2.5, we considered the family of operators $(e^{-tT})_{0<t<\infty}$ when T is a bounded operator. We now consider this theory where T is a sectorial operator.

More precisely, in this section, we fix μ and ω such that $0 \leq \omega < \mu < \frac{\pi}{2}$ and we suppose our operator is T ω-sectorial.

Proposition 4.2.1. *Let $f_t(\zeta) = e^{-t\zeta}$ and let*

$$\psi_t(\zeta) = \left(e^{-t\zeta} - \frac{1}{1+t\zeta} \right).$$

Then, $\psi_t \in \Psi(S^o_{\mu+})$ and

$$f_t(\zeta) = \frac{1}{1+t\zeta} + \psi_t(\zeta).$$

Proof. We note that $f \in \Phi(S^o_{\mu+})$ and

$$|\psi_t(\zeta)| \leq \frac{C|t\zeta|}{1+t^2|\zeta|^2}$$

which proves the assertion. □

Definition 4.2.2 (Exponential Semigroup)**.** Define:

$$e^{-tT} = f_t(T) = (I+tT)^{-1} + \psi_t(T).$$

Proposition 4.2.3. *$e^{-tT} \in \mathcal{L}(\mathcal{X})$ for $0 < t < \infty$.*

Proof. Fix $0 < t < \infty$. Then by resolvent bounds at 0, there exists a $C > 0$ such that

$$\| (I+tT)^{-1} \| = \left\| -\frac{1}{t} \mathsf{R}_T \left(-\frac{1}{t} \right) \right\| \leq C.$$

Also,

$$\| \psi_t(T) \| \leq \frac{1}{2\pi} \oint_{\gamma} |\psi(t\zeta)| \, |\frac{d\zeta}{\zeta}| = \frac{1}{2\pi} \oint_{\gamma} |\psi(\zeta)| \, |\frac{d\zeta}{\zeta}| < \infty.$$

The result follows by combining these two estimates. □

Proposition 4.2.4 (Properties of the Semigroup $(e^{-tT})_{0<t<\infty}$). *(i) There exists a constant $C > 0$ such that $\| e^{-tT} \| \leq C$ whenever $0 \leq t \leq \infty$.*

(ii) $e^{-tT}e^{-sT} = e^{-(t+s)T}$.

(iii) The semigroup is continous. That is, the mapping $(0,\infty) \to \mathcal{L}(X)$ defined by $t \mapsto e^{-tT}$ is continuous.

(iv) The semigroup is differentiable in t, and

$$\frac{d}{dt}e^{-tT} = -Te^{-tT} \supset -e^{-tT}T.$$

(v) If $u \in \mathsf{N}(T)$, then $e^{-tT}u = u$.

(vi) If $u \in \overline{\mathsf{R}(T)}$, then $\lim_{t\to\infty} e^{-tT}u = 0$.

(vii) If $u \in \overline{\mathsf{D}(T)}$, then $\lim_{t\to 0} e^{-tT}u = u$.

(viii) $u \in \mathsf{D}(T)$ if and only if

$$\lim_{t\to 0} \frac{1}{t}(e^{-tT}u - u)$$

exists and

$$-Tu = \lim_{t\to 0} \frac{1}{t}(e^{-tT}u - u).$$

Proof. We have already proved (i), and (ii) is easy.

To prove (iii), fix a $t_0 \in (1,\infty)$. We consider $\frac{1}{2}t_0 < t < 2t_0$. We already have that $(I + tT)^{-1}$ is continuous in t. Also, there exists and $\alpha > 0$ such that

$$|\psi_t(\zeta)| \leq \frac{C|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}}$$

for $\zeta \in S^o_{\mu+}$, and in fact, uniformly bounded for our choice of t. Also, $\psi_t \to \psi_t$ as $t \to \infty$ uniformly on compact sets and by applying Theorem 4.1.9, $\psi_t(T) \to \psi_{t_0}(T)$ in $\mathcal{L}(X)$ which establishes continuity.

Now we prove (vi). Fix $u \in \overline{\mathsf{R}(T)}$. Firstly, note that

$$(I + tT)^{-1}u = -\frac{1}{t}\mathsf{R}_T\left(-\frac{1}{t}\right) \to 0$$

since $-\frac{1}{t} \to 0$ as $t \to \infty$. Now, whenever $1 \leq t < \infty$, $\psi_t \to 0$ uniformly on compact subsets. That is, $|\psi_t(\zeta)| \leq \frac{C}{|\zeta|}$ and $\| \psi_t(T) \| \leq C$ and applying Corollary 4.1.11 $\psi_t(T)u \to 0$.

We leave the rest as an exercise. □

Remark 4.2.5. The last case, (viii) illustrates that

$$-T = \frac{d}{dt}|_{t=0}\ e^{-tT}$$

in the strong operator topology where we define $\frac{d}{dt}|_{t=0}$ as the limit in the theorem. We say that $-T$ is the generator of the semigroup.

What we have achieved has some far reaching consequences. We define the following "evolution equation." We suppose that for simplicity that $\mathcal{X} = \overline{\mathsf{D}(T)} = \overline{\mathsf{R}(T)}$.

Definition 4.2.6 (Evolution Equation)**.** Suppose there exists $u : (0, \infty) \to \mathcal{L}(\mathcal{X})$ such that

$$\begin{cases} \frac{d}{dt}u(t) + Tu(t) = 0 \\ \lim_{t \to 0} u(t) = w \end{cases}.$$

We say that u is an evolution equation.

Proposition 4.2.7. *Given T an ω-sectorial operator (with $\omega < \frac{\pi}{2}$), there exists a unique solution u to the evolution equation.*

Proof. The solution is given by $u(t) = e^{-tT}w$. The solution is uniformly bounded in t:

$$\| u(t) \| \leq C \| w \| .$$

Furthermore, $u(t) \to 0$ as $t \to \infty$ and $u(t) \to w$ as $t \to 0$. One can check that u is indeed unique. □

So, this proposition shows that we have in fact solved an entire class of parabolic PDE. In particular, $T = -\Delta$ is the heat equation.

4.3 H^∞ Functional Calculus

We return to the situation where our operator T is ω-sectorial with $0 \leq \omega < \pi$.

Definition 4.3.1 (Bounded H^∞ functional caculus)**.** We say that T has a bounded $\mathsf{H}^\infty(S^o_{\mu+})$ functional calculus if there exists $C > 0$ such that

$$\| \psi(T) \| \leq C \| \psi \|_\infty$$

for all $\psi \in \Psi(S^o_{\mu+})$.

A justification of this definition is necessary, because we have only considered $\Psi(S^o_{\mu+})$ class functions.

Proposition 4.3.2. *Let $f \in H^\infty(S^o_{\mu+})$. Then there exists a sequence $\psi_k \in \Psi(S^o_{\mu+})$ and a constant $C > 0$ such that $\| \psi_k \|_\infty \leq C$ and $\psi_k \to f$ uniformly on compact subsets of $S^o_{\mu+}$.*

Definition 4.3.3 ($\mathsf{H}^\infty(S^o_{\mu+})$ functional calculus)**.** Fix f and let $\psi_k \in \Psi(S^o_{\mu+})$ be a uniformly bounded sequence such that $\psi_k \to f$ uniformly on compact subsets. Define:

$$f(T)u = \lim_{k \to \infty} \psi_k(T)u$$

for all $u \in \overline{\mathsf{D}(T)} \cap \overline{\mathsf{R}(T)}$.

Remark 4.3.4. The limit in the definition exists, since $\psi_k(T)u$ is a cauchy sequence whenever $u \in \overline{\mathsf{D}(T)} \cap \overline{\mathsf{R}(T)}$.

Proposition 4.3.5. *$f(T)$ is well defined and there exists $C > 0$ such that $\| f(T)u \| \leq C \| u \|$ for all $u \in \overline{\mathsf{D}(T)} \cap \overline{\mathsf{R}(T)}$.*

Chapter 5

Operators on Hilbert Spaces

We start with a summary of Hilbert space theory. See for example [Hal98], [Kat76], [DS88].

Definition 5.0.6 (Inner Product)**.** Let $\mathcal{H}$ be a linear space over $\mathbf{C}$ and let $\langle\cdot\,,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbf{C}$ be a map satisfying:

(i) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ for all $u, v, w \in \mathcal{H}$ and $\alpha, \beta \in \mathbf{C}$.

(ii) $\langle u, \alpha v + \beta w\rangle = \overline{\alpha}\langle u, v\rangle + \overline{\beta}\langle u, w\rangle$ for all $u, v, w \in \mathcal{H}$ and $\alpha, \beta \in \mathbf{C}$.

(iii) If $\langle u, v\rangle = 0$ for all v then $u = 0$. *(Positive definiteness)*

Remark 5.0.7. An inner product is a notion of angle.

Definition 5.0.8 (Orthogonal)**.** Let $\mathcal{H}$ be a linear space over $\mathbf{C}$ with an inner product $\langle\cdot\,,\cdot\rangle$. Then, we say that u, v are orthogonal and write $u \perp v$ if $\langle u, v\rangle = 0$.

Remark 5.0.9. An inner product has an associated norm given by:

$$\| u \| = \sqrt{\langle u, u\rangle}.$$

Definition 5.0.10 (Hilbert Space)**.** A *Hilbert space* is a linear space $\mathcal{H}$ over $\mathbf{C}$ with an inner product $\langle\cdot\,,\cdot\rangle$ which is a complete topological space with respect to the associated norm.

Throughout this section, we will take $\mathcal{H}$ to mean a Hilbert space with an inner product $\langle\cdot\,,\cdot\rangle$.

Proposition 5.0.11 (Properties of a Hilbert space)**.**

(i) $\langle u, v\rangle = \frac{1}{4}[\| u+v \| - \| u-v \| + \imath \| u+\imath v \| - \imath \| u - \imath v \|]$. (Polarisation Identity)

(ii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$. (Hermitian)

(iii) $|\langle u, v\rangle| \leq \| u \| \, \| v \|$. (Cauchy-Schwartz)

(iv) $\| u + v \| \leq \| u \| + \| v \|$. (Minkowski)

(v) $\| u+v \|^2 + \| u-v \|^2 = 2\| u \|^2 + 2\| v \|^2$. (Parallelogram)

(vi) Whenever $u \perp v$, $\| u+v \|^2 = \| u \|^2 + \| v \|^2$. (Pythagoras)

Example 5.0.12. (i) $\mathcal{H} = \ell^2$ with $\langle u, v \rangle = \sum_{i=1}^{\infty} u_j \overline{v_j}$.

(ii) $\mathcal{H} = L^2(\Omega)$ with $\langle u, v \rangle = \int_\Omega u\overline{v}$.

The main theorem at the heart of Hilbert Space theory is the following.

Theorem 5.0.13 (Closest point)**.** *Let $M \subset \mathcal{H}$ be closed linear subspace. Then, for any $u \in \mathcal{H}$, there exists $v \in M$ such that $\| u - v \| = \operatorname{dist}(u, M)$ and $(u - v) \perp M$.*

Corollary 5.0.14. *Let $M^\perp = \{w \in \mathcal{H} : w \perp v \ \forall v \in M\}$. Then, $\mathcal{H} = M \overset{\perp}{\oplus} M^\perp$ and $M^{\perp\perp} = M$.*

Now, for the moment, let $\mathcal{X}$ be a Banach space. In §2.7, we considered the dual space to be $\mathcal{X}' = \mathcal{L}(\mathcal{X}, \mathbf{C})$. That is, if $f \in \mathcal{X}'$, then $f : \mathcal{X} \to \mathbf{C}$, bounded, and linear. In a Hilbert space, it is often useful to replace this linearity with conjugate linearity.

Definition 5.0.15 (Adjoint space)**.** Let $\mathcal{X}$ be a Banach space. Define:

$$\mathcal{X}^* = \{f : \mathcal{X} \to \mathbf{C} \text{ conjugate linear }\}.$$

Then, $\mathcal{X}^*$ is called the *adjoint space* of $\mathcal{X}$.

Remark 5.0.16. It is worth noting that the theory we built for $\mathcal{X}'$ in §2.7 works in much the same way if, instead, we worked with $\mathcal{X}^*$.

Theorem 5.0.17 (Reisz representation)**.** *Let $u \in \mathcal{H}$ and let $f_u \in \mathcal{H}^*$ be defined by $f_u(v) = \langle u, v \rangle$ for all $v \in \mathcal{H}$. Then the map $\mathcal{H} \to \mathcal{H}^*$ given by $u \mapsto f_u$ is a linear isometry.*

Proposition 5.0.18 (Hanh-Banach property of Hilbert spaces)**.** *Every Hilbert space $\mathcal{H}$ has the Hahn-Banach property with respect to $\langle \mathcal{H}, \mathcal{H}^* \rangle$.*

Proof. Let $M \subset \mathcal{H}$ be a closed linear subspace. Let $u \in \mathcal{H} \setminus M$. Then, we can write $u = v + w \in M \oplus M^\perp$. Then, $w \neq 0$ and now consider $f_w \in \mathcal{H}^*$. Then, we have $w \in M^\perp$ and $f_w(w) = \langle w, w \rangle = \| w \|^2 \neq 0$. Also, if $x \in M$, $f_w(x) = \langle w, x \rangle = 0$. □

5.1 The numerical range

From here on, we assume that $T \in \mathcal{L}(\mathcal{H})$.

Definition 5.1.1 (Numerical range, Numerical radius)**.** Define the *numerical range* of T:

$$\operatorname{nr}(T) = \{\langle Tu, u \rangle \in \mathbf{C} : u \in \mathcal{H}, \ \| u \| = 1\}.$$

Define the *numerical radius* of T:

$$\operatorname{nrad}(T) = \sup |\operatorname{nr}(T)| = \sup_{u \neq 0} \frac{|\langle Tu, u \rangle|}{\| u \|^2}.$$

Remark 5.1.2. Note that $\mathrm{nrad}(T) \leq \| T \|$.

Proposition 5.1.3. $\frac{1}{2} \| T \| \leq \mathrm{nrad}(T) \leq \| T \|$.

Corollary 5.1.4. $\mathrm{nr}(T) = \{\lambda\}$ *if and only if* $T = \lambda I$.

Proof. $\mathrm{nr}(T) = \{\lambda\} \iff \mathrm{nr}(T - \lambda I) = \{0\} \iff \| T - \lambda i I \| = 0 \iff T = \lambda I.$ $\square$

Theorem 5.1.5 (Toeplitz-Hausdorff (1906))**.** *The numerical range* $\mathrm{nr}(T)$ *is convex.*

Example 5.1.6. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, $\sigma(T) = \{0\}$, $\| T \| = 1$. We compute the numerical range:

$$\begin{aligned}\mathrm{nr}(T) &= \left\{ \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rangle \in \mathbf{C} : |\alpha|^2 + |\beta|^2 = 1 \right\} \\ &= \left\{ \beta\overline{\alpha} \in \mathbf{C} : |\alpha|^2 + |\beta|^2 = 1 \right\}.\end{aligned}$$

The, the numerical radius is $\mathrm{nrad}(T) = \frac{1}{2}$.

Example 5.1.7. Let $\mathcal{H} = \ell^2$, $\tilde{\lambda} = (\lambda_j)_{j=1}^{\infty}$, $\lambda_j \in \mathbf{C}$ and $||\tilde{\lambda}||_\infty = \sup_j |\lambda_j| < \infty$. Define $T = \mathrm{diag}(\lambda_j)$. Then, for $u \in \ell^2$, $(Tu)_j = \lambda_j u_j$. Now, $\| T \| = ||\tilde{\lambda}||_\infty$ point spectrum (set of eigenvalues) is $\sigma_{\mathrm{pt}}(T) = \{\lambda_j\}$ and the spectrum is $\sigma(T) = \overline{\sigma_{\mathrm{pt}}(T)}$. We compute the numerical range:

$$\mathrm{nr}(T) = \{\langle Tu, u \rangle \in \mathbf{C} : \| u \| = 1\} = \left\{ \sum_j \lambda_j u_j \overline{u_j} : \sum_j |u_j|^2 = 1 \right\}.$$

We denote the convex hull of a set S (set of finite sums) by $\mathrm{co}(S)$. Then,

$$\sigma_{\mathrm{pt}}(T) \subset \mathrm{co}(\sigma_{\mathrm{pt}}(T)) \subset \mathrm{nr}(T) \subset \overline{\mathrm{co}(\sigma_{\mathrm{pt}}(T))}$$

and

$$\sigma(T) = \overline{\sigma_{\mathrm{pt}}(T)} \subset \overline{\mathrm{co}(\sigma_{\mathrm{pt}}(T))} = \overline{\mathrm{nr}(T)}.$$

Also, in this example, $\mathrm{nrad}(T) = \| T \|$.

In particular, the preceding example highlights that $\sigma(T) \subset \overline{\mathrm{nr}(T)}$, and not $\mathrm{nr}(T)$. In fact, this is true in general, and we prove this as a theorem. But first, we need some results about *adjoints* of operators.

Definition 5.1.8 (Adjoint)**.** Let $T \in \mathcal{L}(\mathcal{H})$. We define $T^* : \mathcal{H} \to \mathcal{H}$ by $\langle T^* u, v \rangle = \langle u, Tv \rangle$ for all $u, v \in \mathcal{H}$.

Remark 5.1.9. In the general theory, the adjoint of an operator acts on the adjoint space, just as the dual of an operator acts on the dual space as see in §2.7. But this definition makes sense in the light of the Riesz representation theorem as we can identify the adjoint space $\mathcal{H}^*$ with $\mathcal{H}$.

Proposition 5.1.10 (Properties of the Adjoint)**.** *(i)* $T^* \in \mathcal{L}(\mathcal{H})$ *and* $\| T^* \| = \| T \|$.

(ii) $T^{**} = T$ *and* $(\alpha T + \beta S)^* = \overline{\alpha} T^* + \overline{\beta} T^*$.

(iii) $(ST)^* = T^*S^*$ *and* $I^* = I$.

(iv) *If* S *is invertible, then* S^* *is invertible and* $(S^*)^{-1} = (S^{-1})^*$.

(v) $\zeta \in \rho(T)$ *if and only if* $\overline{\zeta} \in \rho(T^*)$ *and* $R_T(\zeta)^* = R_{T^*}(\overline{\zeta})$.

(vi) $\lambda \in \sigma(T)$ *if and only if* $\overline{\lambda} \in \sigma(T)$.

(vii) $\alpha \in \mathrm{nr}(T)$ *if and only if* $\overline{\alpha} \in \mathrm{nr}(T^*)$.

(viii) $\mathcal{H} = N(T) \overset{\perp}{\oplus} \overline{R(T^*)}$.

Proof. (i) Note that,

$$\| T^* \| = \sup_{u,v \neq 0} \frac{|\langle T^*u, v\rangle|}{\| u \| \| v \|} = \| T \| .$$

(viii) We have

$$\begin{aligned} u \in \mathsf{N}(T) &\iff \langle Tu, v\rangle = 0 \ \forall v \in \mathcal{H} \\ &\iff \langle u, T^*v\rangle = 0 \ \forall v \in \mathcal{H} \\ &\iff u \perp \mathsf{R}(T^*) \\ &\iff u \perp \overline{\mathsf{R}(T^*)} \end{aligned}$$

We leave the rest as an exercise. □

Proposition 5.1.11. *Let* $T \in \mathcal{L}(\mathcal{H})$. *If* $\alpha, \beta \geq 0$ *such that*

(i) $\| u \| \leq \alpha \| Tu \|$ *for all* $u \in \mathcal{H}$

(ii) $\| v \| \leq \beta \| T^*v \|$ *for all* $v \in \mathcal{H}$

then T *is bijective,* $T^{-1} \in \mathcal{L}(\mathcal{H})$ *and*

$$\left\| T^{-1} \right\| \leq \min\{\alpha, \beta\} .$$

Proof. Condition (i) immediately gives that T is one-one with closed range. Condition (ii) gives that T^* is one-one and so $N(T^*) = 0$. It follows by (viii) and (ii) of Proposition 5.1.10 that $\overline{\mathsf{R}(T)} = \mathcal{H}$. Putting these two facts together gives that $\mathsf{R}(T) = \mathcal{H}$ and so $T^{-1} \in \mathcal{L}(\mathcal{H})$. Furthermore, (i) gives that $\left\| T^{-1}w \right\| \leq \alpha \| w \|$. Using the same argument, we can find that $T^{*-1} \in \mathcal{L}(\mathcal{H})$ and $\left\| T^{*-1}w \right\| \leq \beta \| w \|$. Since $\| T \| = \| T^* \|$, it follows that $\left\| T^{-1} \right\| \leq \min\{\alpha, \beta\}$. □

We now return to prove the general statement that $\sigma(T) \subset \overline{\mathrm{nr}(T)}$.

Theorem 5.1.12. *Let* $T \in \mathcal{L}(\mathcal{H})$. *Then,*

(i) $\sigma(T) \subset \overline{\mathrm{nr}(T)}$.

(ii) For all $\zeta \in \mathbf{C} \setminus \overline{\mathrm{nr}(T)}$,

$$\| R_T(\zeta) \| \leq \frac{1}{\operatorname{dist}(\zeta, \overline{\mathrm{nr}(T)})}.$$

Proof. Fix $\zeta \in \mathbf{C} \setminus \overline{\mathrm{nr}(T)}$. let $d = \operatorname{dist}(\zeta, \overline{\mathrm{nr}(T)})$. Then, for all $u \in \mathcal{H} \setminus \{0\}$,

$$\left| \zeta - \frac{\langle Tu, u \rangle}{\| u \|^2} \right| \geq d.$$

By Cauchy-Schwartz and multiplying across by $\| u \|^2$,

$$\| (\zeta I - T)u \| \| u \| \geq |\langle (\zeta I - T)u, u \rangle| \geq d \| u \|^2 .$$

So, $\| u \| \geq \frac{1}{d} \| (\zeta I - T)u \| \| u \|$. Similarly, we apply the same argument with T^* in place of T and $\overline{\zeta}$ in place of ζ to find the estimate $\| u \| \geq \frac{1}{d} \left\| (\overline{\zeta} I - T^*)u \right\| \| u \|$. By Proposition 5.1.11, we get $\mathsf{R}(\zeta I - T) = \mathcal{H}$ and $(\zeta I - T)^{-1} \in \mathcal{L}(\mathcal{H})$ so $\zeta \in \rho(T)$ proving $\sigma(T) \subset \overline{\mathrm{nr}(T)}$. Also, by the same proposition,

$$\| \mathsf{R}_T(\zeta) \| \leq \frac{1}{d} = \frac{1}{\operatorname{dist} \zeta, \overline{\mathrm{nr}(T)}}$$

which completes the proof. □

5.2 Functional calculus and numerical range

Let $p(\zeta) = \sum_{k=0}^N \alpha_k \zeta^k$. We define $\overline{p}(\zeta) = \sum_{k=0}^N \overline{\alpha_k} \zeta^k$. Note that $\overline{p}(\zeta) = \overline{p(\overline{\zeta})}$. Then, note that

$$p(T)^* = (\sum_{k=0}^N \alpha_k T^k)^* = \sum_{k=0}^N \overline{\alpha_k} T^{*k} = \overline{p}(T).$$

We use the case of polynomials as a prototype to consider the "conjugation" of an arbitrary function f.

Definition 5.2.1 (Conjugate function)**.** Let $\Omega \subset \mathbf{C}$, Ω open and let $\Omega^{\mathrm{conj}} = \{\overline{\zeta} : \zeta \in \Omega\}$, the conjugate of Ω. Then, whenever $f \in \mathsf{H}(\Omega)$, we define $\overline{f} \in \mathsf{H}(\Omega^{\mathrm{conj}})$ by $\overline{f}(\zeta) = \overline{f(\overline{\zeta})}$.

Proposition 5.2.2. $f(T)^* = \overline{f}(T)$.

Remark 5.2.3. Suppose Ω is connected and $\Omega \cap \mathbf{R} \neq \varnothing$. Then,

$$f(\zeta) = \overline{f}(\zeta) \ \forall \zeta \in \Omega \iff f(x) = \overline{f}(x) \ \forall x \in \Omega \cap \mathbf{R} \iff f(x) \in \mathbf{R} \ \forall x \in \Omega \cap \mathbf{R}$$

Definition 5.2.4 (Self adjoint)**.** We say that $T \in \mathcal{L}(\mathcal{H})$ is a *self adjoint* operator if $T = T^*$.

Proposition 5.2.5. $T = T^*$ *if and only if* $\mathrm{nr}(T) \subset \mathbf{R}$.

Proof. To prove that when $T = T^*$ then $\mathrm{nr}(T) \subset \mathbf{R}$, observe that

$$\langle Tu, u \rangle = \langle T^* u, u \rangle = \langle u, Tu \rangle = \overline{\langle Tu, u \rangle}.$$

We leave the other direction as an exercise. □

Proposition 5.2.6. *There exists $a, b \in \mathbf{R}$, satisfying $a \leq b$ such that $\overline{\mathrm{nr}(T)} = [a, b]$ or $\mathrm{nr}(T) = \{a\}$. Moreover, $a, b \in \sigma(T)$, so that $\overline{\mathrm{nr}(T)} = \mathrm{co}(\sigma(T))$ and $\| T \| = \mathrm{nrad}(T) = \sup |\sigma(T)|$.*

Theorem 5.2.7. *Let T be self adjoint and suppose Ω is open in $\mathbf{C}$, $\mathrm{nr}(T) \subset \Omega$. Suppose that $f \in H(\Omega)$ such that $f = \overline{f}$. Then, $f(T)$ is self adjoint and*

$$\| f(T) \| \leq \sup_{\zeta \in \mathrm{nr}(T)} |f(\zeta)| \leq \| f \|_\infty .$$

Proof. Firstly, we observe that $f(T)^* = \overline{f}(T^*) = f(T)$. Now,

$$\| f(T) \| = \sup |\sigma(f(T))| = \sup |f(\sigma(T))| = \sup_{\zeta \in \sigma(T)} |f(\zeta)| \leq \sup_{\zeta \in \mathrm{nr}(T)} |f(\zeta)| \leq \| f \|_\infty$$

□

Exercise 5.2.8. Prove that the preceding theorem remains true without the condition $f = \overline{f}$.

Remark 5.2.9. In fact, for self adjoint operators, we have a *Borel functional calculus* via the spectral representation. Furthermore, the estimate $\| f(T) \| \leq \| f \|_\infty$ holds for all Borel measurable functions $f : \sigma(T) \to \mathbf{C}$.

5.3 Accretive operators

Definition 5.3.1 (Accretive)**.** An operator $T \in \mathcal{L}(\mathcal{H})$ is called *accretive* if $\mathrm{Re} \langle Tu, u \rangle \geq 0$ for all $u \in \mathcal{H}$.

Proposition 5.3.2. *$T \in \mathcal{L}(\mathcal{H})$ is accretive if and only if $T + T^*$ is a non negative self adjoint operator*

Proof. Note that

$$\begin{aligned} \mathrm{Re}\langle Tu, u \rangle \geq 0 \ \forall u \in \mathcal{H} &\iff \langle Tu, u \rangle + \langle u, Tu \rangle \geq 0 \ \forall u \in \mathcal{H} \\ &\iff \langle (T + T^*)u, u \rangle \geq 0 \ \forall u \in \mathcal{H} \\ &\iff T + T^* \text{ is non-negative self adjoint.} \end{aligned}$$

which establishes the claim. □

In particular, this theorem tells us that there exists $\sqrt{T + T^*} \in \mathcal{L}(\mathcal{H})$. Also, we find $\sqrt{T} \in \mathcal{L}(\mathcal{H})$ by integrating around the numerical range and using the resolvent bounds at 0.

Also, for a bounded accretive operator T, by considering $f_t(\zeta) = e^{-t\zeta}$ for $t > 0$, we find $\| e^{-tT} \| \leq 1$. That is, we get a *contraction* semigroup. In fact, this works even when T is unbounded.

Definition 5.3.3 (Bounded strictly accretive)**.** An operator $T \in \mathcal{L}(\mathcal{H})$ is called *strictly accretive* if there exists $\kappa > 0$ such that $\mathrm{Re}\langle Tu, u\rangle \geq \kappa \| u \|^2$ for all $u \in \mathcal{H}$.

Theorem 5.3.4. *Suppose that T is a bounded strictly accretive operator, and let $\mu > \frac{\pi}{2}$. If $f \in H^\infty(S^o_{\mu+})$, then*

$$\| f(T) \| \leq 1 \cdot \sup_{\zeta \in \imath\mathbf{R}} |f(\zeta)| = 1 \cdot \sup_{\mathrm{Re}\,\zeta \geq 0} |f(\zeta)|.$$

Proof. Let $\psi \in \Psi(S^o_{\mu+})$. We can take $\gamma = -\imath\mathbf{R}$ since $\mu > \frac{\pi}{2}$ and then

$$\begin{aligned}
\psi(T) &= \frac{1}{2\pi\imath} \int_\gamma \psi(\zeta)(\zeta I - T)^{-1}\, d\zeta \\
&= \frac{1}{2\pi\imath} \int_{\imath\mathbf{R}} \psi(\zeta)[(T - \zeta I)^{-1} + (T^* + \zeta I)^{-1}\, d\zeta \quad \text{(Since } (T^* + \zeta I)^{-1} \text{ is holomorphic on } S^o_{\mu+}) \\
&= \frac{1}{2\pi\imath} \int_{\imath\mathbf{R}} (T^* + \zeta I)^{-1}(T^* + T)(T - \zeta I)^{-1}\, d\zeta
\end{aligned}$$

Now, there exists $\psi_n \in \Psi(S^o_{\mu+})$ such that $\psi_n \to f$ uniformly on compact subsets of $S^o_{\mu+}$. So, $\psi_n(T) \to f(T)$. By the decay condition on $(T^* + \zeta I)^{-1}(T^* + T)(T - \zeta I)^{-1}$,

$$f(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\imath y)(T^* + \imath y I)^{-1}(T^* + T)(T - \imath y I)^{-1}\, dy$$

and

$$\langle f(T)u, v\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\imath y)\langle (T^* + T)^{\frac{1}{2}}(T - \imath y I)^{-1}u, (T^* + T)^{\frac{1}{2}}(T - \imath y I)^{-1}v\rangle\, dy$$

for all $u, v \in \mathcal{H}$.

Putting $f \equiv 1$, $u = v$, we get,

$$\| u \|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| (T^* + T)^{\frac{1}{2}}(T - \imath y I)^{-1}u \right\|^2\, dy.$$

By the application of Cauchy-Schwartz,

$$\begin{aligned}
|\langle f(T)u, v\rangle| &\leq \frac{1}{2\pi} \sup_{y \in \mathbf{R}} |f(iy)| \int_{-\infty}^{\infty} \left\| (T^* + T)^{\frac{1}{2}}(T - \imath y I)^{-1}u \right\| \left\| (T^* + T)^{\frac{1}{2}}(T - \imath y I)^{-1}v \right\|\, dy \\
&\leq \sup_{\zeta \in \imath\mathbf{R}} |f(\zeta)| \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| (T^* + T)^{\frac{1}{2}}(T - \imath y I)^{-1}u \right\|^2 dy \right)^{\frac{1}{2}} \\
&\qquad\qquad \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| (T^* + T)^{\frac{1}{2}}(T - \imath y I)^{-1}v \right\|^2 dy \right)^{\frac{1}{2}} \\
&= \sup_{\zeta \in \imath\mathbf{R}} |f(\zeta)| \, \| u \| \, \| v \|
\end{aligned}$$

□

Definition 5.3.5 (Bounded ω-accretive)**.** Let $0 \leq \omega < \frac{\pi}{2}$. We say an operator $T \in \mathcal{L}(\mathcal{H})$ is bounded ω-accretive if $\mathrm{nr}(T) \subset S_{\omega+}$.

Note that T is ω-accretive means that T is $\frac{\pi}{2}$-accretive, which is the same as saying that T is accretive.

Proposition 5.3.6. *$T \in \mathcal{L}(\mathcal{H})$ is bounded strictly accretive implies there exists $\omega < \frac{\pi}{2}$ such that T is bounded and ω-accretive.*

Proof. Note that $|\operatorname{nr}(T)| \leq \| T \|$. Let $\kappa > 0$ be the constant in the strict accretive hypothesis. Then, take ω such that $\| T \| \cos\ \omega = \kappa$. $\square$

Proposition 5.3.7. *$T \in \mathcal{L}(\mathcal{H})$ bounded strictly accretive implies that T^{-1} is bounded strictly accretive.*

Proof. Firstly, we note that the strict accretive hypothesis implies that $0 \in \rho(T)$ so that $\mathsf{R}_T(0) = T^{-1} \in \mathcal{L}(\mathcal{H})$. Also,

$$\| \mathsf{R}_T(0) \| \leq \frac{1}{\operatorname{dist}(0, \operatorname{nr}(T))} \leq \frac{1}{\kappa}.$$

Since T is also ω-accretive with some $\omega < \frac{\pi}{2}$,

$$\langle T^{-1}u, u \rangle = \langle T^{-1}u, TT^{-1}u \rangle = \overline{\langle T(T^{-1}u), (T^{-1}u) \rangle} \in S_{\omega+}.$$

So,

$$\operatorname{Re} \langle T^{-1}u, u \rangle = \operatorname{Re} \langle T(T^{-1}u), (T^{-1}u) \rangle \geq \kappa \left\| T^{-1}u \right\|^2 \geq \frac{\kappa}{\| T \|} \| u \|^2 .$$

$\square$

Example 5.3.8. (i) Let $\mathcal{H} = \ell^2$, and let $T = \operatorname{diag}(\lambda_j)_{j=1}^{\infty}$. We already know that $\overline{\operatorname{nr}(T)} = \overline{\operatorname{co}\{\lambda_j\}}$. Then, T is bounded strictly accretive if and only if $|\lambda_j| \leq \omega$ and $\operatorname{Re} \lambda_j \geq \kappa > 0$ for all j.

(ii) Let $\mathcal{H} = L^2(\mathbf{R}^n)$. Let $b \in L^\infty(\mathbf{R}^n)$ such that $\operatorname{Re} b(x) \geq \kappa > 0$ for almost every $x \in \mathbf{R}^n$. Let T be the map $u \mapsto bu$. Then, $\| T \| = \| b \|_\infty$ and

$$\operatorname{Re}\langle Tu, u \rangle = \operatorname{Re} \int_{\mathbf{R}^n} b(x)|u(x)|^2 \ dx \geq \kappa \| u \|^2 .$$

(iii) Let $\mathcal{H} = L^2(\mathbf{R}^n, \mathbf{C}^N)$, and $T \in L^\infty(\mathbf{R}^n, M^{N \times N})$ such that ess $\sup|T(x)| = \| T \|_\infty < \infty$, and $\operatorname{Re} T(x) \geq \kappa I$ for almost all $x \in \mathbf{R}^n$. Then, $\mathbb{T} \in \mathcal{L}\left(L^2(\mathbf{R}^n, \mathbf{C}^N)\right)$ defined by $\mathbb{T}u(x) = T(x)u(x)$ is bounded strictly accretive.

5.4 Closed operators

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ and suppose that $\overline{\mathsf{D}(T)} = \mathcal{H}$.

Definition 5.4.1 (Adjoint)**.** The adjoint T^* of T is the operator with the largest domain $\mathsf{D}(T^*)$ such that $\langle Tu, v \rangle = \langle u, T^*v \rangle$ for all $u \in \mathsf{D}(T), v \in \mathsf{D}(T^*)$.

Remark 5.4.2. Equivalently, we could define

$$\mathsf{D}(T^*) = \{v \in \mathcal{K} : \exists f \in \mathcal{H} \text{ s.t. } \langle Tu, v\rangle = \langle u, f\rangle \ \forall u \in \mathsf{D}(T)\}$$

and

$$T^* v = f.$$

This is indeed well defined. Suppose there exists $g \in \mathcal{H}$ such that

$$\langle Tu, v\rangle = \langle u, f\rangle = \langle u, g\rangle$$

for all $u \in \mathsf{D}(T)$. Then, $f = g$ if and only if $\mathsf{D}(T)$ is dense in $\mathcal{H}$. This is why we made the the density assumption on $\mathsf{D}(T)$.

Remark 5.4.3. Note that

$$\mathcal{H} \oplus \mathcal{K} = \mathcal{G}(T) \overset{\perp}{\oplus} \mathcal{G}(-T^*)$$

since

$$\langle (u, Tu), (-T^* v, v)\rangle = -\langle u, T^* v\rangle + \langle Tu, v\rangle = 0.$$

This gives us another approach to define the adjoint. Take $\mathcal{G}(T)^\perp$ in $\mathcal{H} \oplus \mathcal{K}$ and check if it is the graph of an operator. Clearly such is the case if T is densely defined. So, define $-T^*$ to be the operator with the corresponding graph $\mathcal{G}(T)^\perp$. See [Kat76].

Proposition 5.4.4 (Properties of T^*)**.** *(i)* $T^* \in \mathcal{C}(\mathcal{K}, \mathcal{H})$.

(ii) $\overline{\mathsf{D}(T^*)} = \mathcal{K}$.

(iii) $T^{**} = T$.

(iv) $\mathcal{H} = \mathsf{N}(T) \overset{\perp}{\oplus} \overline{\mathsf{R}(T^*)}$ *and* $\mathcal{K} = \mathsf{N}(T^*) \overset{\perp}{\oplus} \overline{\mathsf{R}(T)}$.

(v) If T *is invertible (ie,.* $T^{-1} \in \mathcal{L}(\mathcal{K}, \mathcal{H})$*), then* T^* *is invertible and* $T^{*-1} = T^{-1*}$.

(vi) $\zeta \in \rho(T) \iff \overline{\zeta} \in \rho(T^*)$ *and* $\mathsf{R}_T(\zeta)^* = \mathsf{R}_{T^*}(\overline{\zeta})$.

(vii) $\lambda \in \sigma(T) \iff \overline{\lambda} \in \sigma(T^*)$.

Proof. We prove (ii). Suppose $u \in \mathsf{D}(T^*)^\perp$. That is $\langle u, v\rangle = 0$ for all $v \in \mathsf{D}(T^*)$ which is if and only if $\langle (0, u), (-T^* v, v)\rangle = 0$ for all $v \in \mathsf{D}(T^*)$ in $\mathcal{H} \oplus \mathcal{K}$. So, $(0, u) \in \mathcal{G}(-T^*)^\perp = \mathcal{G}(T)$ which implies $u = 0$. So, $\overline{\mathsf{D}(T^*)} = \mathcal{K}$.

The rest of the proposition is left as an exercise. □

Remark 5.4.5. Unlike in the case for bounded operators, we do not always have that $\mathrm{nr}(T) = \mathrm{nr}(T)^{\mathrm{conj}}$. This is because the domains $\mathsf{D}(T)$ and $\mathsf{D}(T^*)$ may be different, even in the case $\mathcal{H} = \mathcal{K}$.

Definition 5.4.6 (Self adjoint)**.** $T \in \mathcal{C}(\mathcal{H})$ self adjoint means that T is densely defined and $T = T^*$.

Remark 5.4.7. As a consequence, $\mathsf{D}(T) = \mathsf{D}(T^*)$. Also, $\sigma(T) \subset \overline{\mathrm{nr}(T)} \subset \mathbf{R}$ and for all $\zeta \in \mathbf{C} \setminus \mathbf{R}$,

$$\| \mathsf{R}_T(\zeta) \| \leq \frac{1}{\operatorname{Im} \zeta}.$$

Definition 5.4.8 (ω-accretive operator)**.** Let $0 \leq \omega \leq \frac{\pi}{2}$. Then, we say that T is ω-accretive if

(i) $T \in \mathcal{C}(\mathcal{H})$,

(ii) $\mathrm{nr}(T) \subset S_{\omega+}$,

(iii) $\sigma(T) \subset S_{\omega+}$.

Remark 5.4.9. Note that condition (ii) is equivalent to saying that $|\arg\langle Tu, u\rangle| \leq \omega$ for all $u \in \mathsf{D}(T)$. When, $T \in \mathcal{L}(\mathcal{H})$, (ii) implies (iii).

Proposition 5.4.10. *Suppose T satisfies (i) and (ii) in the definition of ω-accretivity. Then whenever $\zeta \in \mathbf{C} \setminus \overline{\mathrm{nr}(T)}$, we have $\|(T - \zeta I)u\| \geq \mathrm{dist}(\zeta, S_{\omega+})\|u\|$. So $T - \zeta I$ is injective with closed range.*

Remark 5.4.11. In this case, (iii) holds if and only if $\overline{\mathsf{R}(T - \zeta I)} = \mathcal{H}$. The proof is by the stability of the semi-Fredholm index.

Proposition 5.4.12. *If T is ω-accretive, then*

$$\| R_T(\zeta) \| \leq \frac{1}{\mathrm{dist}(\zeta, S_{\omega+})}$$

for all $\zeta \in \mathbf{C} \setminus S_{\omega+}$ and T is ω-sectorial.

Remark 5.4.13. An immediate consequence is that $\overline{\mathsf{D}(T)} = \mathcal{H}$ and $\mathcal{H} = \mathsf{N}(T) \oplus \overline{\mathsf{R}(T)}$. (Note that $\oplus$ does not imply orthogonality.)

Theorem 5.4.14. *Suppose that (i) and (ii) holds. Then, (iii) holds (ie., T is ω-accretive) if and only if $\overline{\mathsf{D}(T)} = \mathcal{H}$ and $\mathrm{nr}(T^*) \subset S_{\omega+}$.*

Proof. Suppose that $\overline{\mathsf{D}(T)} = \mathcal{H}$ and $\mathrm{nr}(T^*) \subset S_{\omega+}$. Then,

$$\left\| (T^* - \overline{\zeta} I)^{-1} u \right\| \geq \mathrm{dist}(\zeta, S_{\omega+}) \| u \|$$

whenever $\zeta \notin S_{\omega+}$. So, $(T^* - \overline{\zeta} I)$ is injective which implies that $\mathsf{N}(T^* - \overline{\zeta} I) = 0$. It follows that $\overline{\mathsf{R}(T - \zeta I)} = \mathcal{H}$ which implies $\zeta \in \rho(T)$. This proves $\sigma(T) \subset S_{\omega+}$.

To prove the other direction, suppose T is ω-accretive. By the previous theorem, ω-accretive implies ω-sectorial and we have that $\overline{\mathsf{D}(T)} = \mathcal{H}$. Fix $\varepsilon > 0$. Then, $(T + \varepsilon I)$ is ω-accretive and $(T + \varepsilon I)^{-1}$ is bounded and ω-accretive. So, $(T^* + \varepsilon I)^{-1}$ is bounded and ω-accretive and $\mathrm{nr}(T^* + \varepsilon I) \subset S_{\omega+}$. Then let $\varepsilon \to 0$ to conclude $\mathrm{nr}(T^*) \subset S_{\omega+}$. $\square$

We summarise the properties of ω-accretivity.

Proposition 5.4.15 (Properties of ω-accretivity)**.** *Let T be ω-accretive. Then,*

(i) Whenever $\zeta \notin S_{\omega+}$, we have

$$\| R_T(\zeta) \| \leq \frac{1}{\mathrm{dist}(\zeta, S_{\omega+})}.$$

(ii) T *is* ω*-sectorial.*

(iii) $\mathcal{H} = \mathsf{N}(T) \oplus \overline{\mathsf{R}(T)}$.

(iv) If T *is injective, then* $\overline{\mathsf{R}(T)} = \mathcal{H}$.

(v) $\overline{\mathsf{D}(T)} = \mathcal{H}$.

(vi) T *has a bounded* H^∞ *functional calculus, and whenever* $\psi \in \Psi(S^o_{\mu+})$ *for* $\mu > \frac{\pi}{2}$,

$$\| \psi(T) \| \leq 1 \cdot \| \psi \|_\infty .$$

Remark 5.4.16. We can get bounds for even more general f, for instance, $f_t(\zeta) = e^{t\zeta}$, and we find $\left\| e^{-tT} \right\| \leq 1$. For a general $f \in \mathsf{H}^\infty(S^o_{\mu+})$ with $\mu > \omega$, we can show $\| f(T) \| \leq C \| f \|_\infty$

Example 5.4.17. Let $\mathcal{H} = L^2[0,\infty)$, and let $T = \frac{d}{dx}$ with $\mathsf{D}(T) = \{u \in \mathcal{H} : u' \in \mathcal{H}, u(0) = 0\} = W^{1,2}_0[0,\infty)$.

Now, $T \in \mathcal{C}(\mathcal{H})$, and $\overline{\mathsf{D}(T)} = \mathcal{H}$. Also, $T^* = -\frac{d}{dx}$ with $\mathsf{D}(T^*) = \{u \in \mathcal{H} : u' \in \mathcal{H}\} = W^{1,2}[0,\infty)$. Formally,

$$\langle Tu, v\rangle = \int_0^\infty u'\overline{v} = -\int_0^\infty u\overline{v}' + 0 = \langle u, T^*v\rangle.$$

Then, the numerical range is $\mathrm{nr}(T) = \{\langle Tu, u\rangle : u \in \mathsf{D}(T)\}$ and

$$\operatorname{Re}\langle Tu, u\rangle = \frac{1}{2}\left(\int u'\overline{u} + \overline{\int u'\overline{u}}\right) = \frac{1}{2}\int \left(u'\overline{u} + u\overline{u}'\right) = 0.$$

So, $\mathrm{nr}(T) \subset \imath\mathbf{R}$ (in fact $= \imath\mathbf{R}$) and T satisfies (i) and (ii) in the definition of $\frac{\pi}{2}$-accretive.

Note that

$$(T^* - \overline{\lambda}I)u = 0 \iff -u' - \overline{\lambda}u = 0,\ u \in \mathcal{H}(0,\infty) \iff u(x) = Ce^{\lambda x}.$$

So, $(T^* - \overline{\lambda}I)$ is 1-1 if and only if $\operatorname{Re}\overline{\zeta} = \operatorname{Re}\zeta \leq 0$ and this happens if and only if $\overline{\mathsf{R}(T - \lambda I)} = \mathcal{H}$. Therefore, we conclude that T is $\frac{\pi}{2}$-accretive.

Note that if we instead considered the operator $-T$, then it would satisfy (i), (ii) but not (iii).

The following is the main example.

Example 5.4.18 (Uniformly elliptic divergence form PD operator)**.** Let $\mathcal{H} = L^2(\mathbf{R}^n)$, $\kappa > 0$ and $A \in L^\infty(\mathbf{R}^n, M^{n\times n})$ such that $\operatorname{Re}\langle A(x)w, w\rangle \geq \kappa|w|^2$ for almost all $x \in \mathbf{R}^n$ and for all $w \in \mathbf{C}^n$.

Define an associated sesquilinear form

$$J[u,v] = \int_{\mathbf{R}^n} \langle A(x)\nabla u(x), \nabla v(x)\rangle\ dx$$

for all $u, v \in W^{1,2}(\mathbf{R}^n)$. Now, we have that $J[u,u] \simeq \| \nabla u \|^2$.

We can think of J as being a complex energy. We think of ∇ as the unbounded operator $\nabla : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n, \mathbf{C}^n)$ where

$$\nabla f = (\partial_1 f, \ldots, \partial_n f)$$

with $\mathsf{D}(\nabla) = W^{1,2}(\mathbf{R}^n)$. Similarly, we consider div $: L^2(\mathbf{R}^n, \mathbf{C}^n) \to L^2(\mathbf{R}^n)$ defined by

$$\operatorname{div} \tilde{u} = \sum_{j=1}^{n} \partial_j u_j$$

with $\mathsf{D}(\operatorname{div}) = \{\tilde{u} \in L^2(\mathbf{R}^n, \mathbf{C}^n) : \operatorname{div} \tilde{u} \in L^2(\mathbf{R}^n)\}$. Then, $\operatorname{div} = -\nabla^*$. This is because

$$\langle \nabla f, \tilde{u} \rangle = \int_{\mathbf{R}^n} \sum_{j=1}^{n} \partial_j f u_j = -\int_{\mathbf{R}^n} \sum_{j=1}^{n} f \partial_j u_j = \langle f, \operatorname{div} \tilde{u} \rangle.$$

The operator associated with the form J is the operator L with the largest domain $\mathsf{D}(L)$ such that

$$J[u,v] = \langle Lu, v \rangle$$

for all $u \in \mathsf{D}(L)$ and for all $v \in \mathsf{D}(J) = W^{1,2}(\mathbf{R}^n)$.

We find that $\mathsf{D}(L) \subset W^{1,2}(\mathbf{R}^n)$. In fact, even without this fact, we have

$$\langle Lu, u \rangle = J[u,u] = \int_{\mathbf{R}^n} \langle A(x) \nabla u(x), \nabla u(x) \rangle \, dx \in S_{\omega+}.$$

By Lax-Milgram Theorem, we find $\sigma(L) \subset S_{\omega+}$. In other words, L is ω-accretive.

Furthermore, $L^* = -\operatorname{div} A^* \nabla$.

5.5 Fractional powers

Definition 5.5.1 (Fractional Powers)**.** Let T be a ω-sectorial operator and $0 < \alpha < 1$. Define

$$T^\alpha = (I + T)\psi_\alpha(T)$$

where

$$\psi_\alpha(\zeta) = \frac{\zeta^\alpha}{1+\zeta} \in \Psi(S^0_{\mu+})$$

for $\mu > \omega$. Define $T^0 = I$ and $T^1 = T$.

We list the important properties of fractional powers.

Proposition 5.5.2 (Properties of fractional powers of T)**.** *Let T be an ω-sectorial operator. Then,* *(i)* T^α *is* $\alpha\omega$*-sectorial.*

(ii) $T^\alpha T^\beta = T^{\alpha+\beta}$ *as long as* $\alpha + \beta \leq 1$.

(iii) In particular, if $\alpha = \frac{1}{2}$*, then we write* $\sqrt{T} = T^{\frac{1}{2}}$ *and it is the unique* $\frac{\omega}{2}$ *operator satisfying* $\sqrt{T}\sqrt{T} = T$.

(iv) $\mathsf{D}(T) \subset \mathsf{D}(\sqrt{T}) \subset \mathcal{H}$.

(v) $(T^\alpha)^* = (T^*)^\alpha$.

(vi) If T *is* ω*-accretive, then* T^α *is* $\alpha\omega$*-accretive.*

We leave the proof of this as an exercise.

Example 5.5.3. We return to Example 5.4.18. Recall that $L = -\mathrm{div}\, A\nabla$ and $L^* = -\mathrm{div}\, A^*\nabla$. Now, assume that $\omega = 0$ so that A is self adjoint. Then $L = L^*$ and so $\sqrt{L}^* = \sqrt{L^*} = \sqrt{L}$. Also,

$$\left\| \sqrt{L}u \right\|^2 = \langle \sqrt{L}u, \sqrt{L}u \rangle \stackrel{\dagger}{=} \langle Lu, u \rangle = \langle A\nabla u, \nabla u \rangle_{L^2} \simeq \| \nabla u \|^2$$

for all $u \in \mathsf{D}(L)$ and from this it follows that

$$\| u \|^2 + \left\| \sqrt{L}u \right\|^2 \simeq \| u \|^2 + \| \nabla u \|^2 \simeq \| u \|^2_{W^{1,2}} .$$

By a density argument we can show that $\mathsf{D}(\sqrt{L}) = W^{1,2}(\mathbf{R}^n)$ with equivalent norms.

But this is all for self-adjoint operators and it is well known classical theory. In the non self-adjoint case, the equality † fails.

This was the question that was asked by Kato in 1961. Suppose $\omega \neq 0$. Then is the following true?

$$\begin{cases} \mathsf{D}(\sqrt{L}) = W^{1,2}(\mathbf{R}^n) \quad \text{with} \\ \left\| \sqrt{L}u \right\| \simeq \| \nabla u \| \end{cases}$$

In 1972, McIntosh gave a negative answer in the case that A is not multiplication. It was not until 2002 that the question was answered by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian. See [AHL+02].

5.6 Bisectorial operators

In this section, let $0 \leq \omega < \frac{\pi}{2}$.

Definition 5.6.1 ($S_{\omega-}, S^o_{\mu-}, S_\omega, S^o_\mu$)**.** Define the *left closed sector* $S_{\omega-} = -S_{\omega+}$. and the *left open sector* $S^o_{\mu-} = -S^o_{\mu+}$. Then, the *closed bisector* is defined as $S_\omega = S_{\omega+} \cup S_{\omega-}$ and the *open bisector* $S^o_\mu = S^o_{\mu+} \cup S^o_{\mu-}$.

Definition 5.6.2 (Bisectorial operator). $T \in \mathcal{C}(\mathcal{H})$ is called an ω-bisectorial operator if

(i) $\sigma(T) \subset S_\omega$, and

(ii) For all $\mu > \omega$ ($\mu < \frac{\pi}{2}$), there exists a constant $C_\mu > 0$ such that

$$\| \mathsf{R}_T(\zeta) \| \leq \frac{C_\mu}{|\zeta|}$$

for all $\zeta \in \mathbf{C} \setminus S_\mu$.

Remark 5.6.3. As before, we have $\mathcal{H} = \mathsf{N}(T) \oplus \overline{\mathsf{R}(T)}$ and $\mathcal{H} = \overline{\mathsf{D}(T)}$ because resolvent bounds at 0 and ∞ are unchanged.

Proposition 5.6.4. *If T is ω-bisectorial, then T^2 is 2ω-sectorial. In particular, $\sqrt{T^2}$ is ω-sectorial.*

Proof. Firstly, note that $\zeta \in \mathbf{C} \setminus S_{2\mu}$ if and only if $\pm\sqrt{\zeta} \in \mathbf{C} \setminus S_\mu$. By the ω-bisectoriality of T, there exists a C_μ such that

$$\left\| \mathsf{R}_T(\pm\sqrt{\zeta}) \right\| \leq \frac{C_\mu}{|\sqrt{\zeta}|}.$$

Now, $(\zeta I - T^2) = -(\sqrt{\zeta} I - T)(-\sqrt{\zeta} I - T)$ and so it follows that

$$\| \mathsf{R}_{T^2}(\zeta) \| \leq \frac{C_\mu}{|\sqrt{\zeta}|} \frac{C_\mu}{|\sqrt{\zeta}|} = \frac{C_\mu^2}{|\zeta|}.$$

□

Proposition 5.6.5. *Let $A, A^{-1} \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{C}(\mathcal{H})$. Then $ASA^{-1} \in \mathcal{C}(\mathcal{H})$ and $ASA^{-1} - \zeta I = A^{-1}(S - \zeta I)A$.*

A direct consequence is the following.

Corollary 5.6.6. *If $S \in \mathcal{C}(\mathcal{H})$ and $A \in \mathcal{L}(\mathcal{H})$ with A invertible, then $\sigma\left(ASA^{-1}\right) = \sigma(S)$. If S is ω-(bi)sectorial then ASA^{-1} is ω-(bi)sectorial. Furthermore, $f(A^{-1}SA) = A^{-1}f(S)A$.*

Theorem 5.6.7. *Suppose $S = S^* \in \mathcal{C}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$ strictly ω-accretive. Then SB and BS are ω-bisectorial. Moreover, if $S \geq 0$ then they are ω-sectorial.*

Proof. Let $\zeta \in \mathbf{C} \setminus S_\omega$. Then,

$$\begin{aligned}
\left\| B^{-1} \right\| \| (\zeta I - BS)u \| \| u \| &\geq |\langle B^{-1}(\zeta I - BS)u, u\rangle| \\
&= |\zeta\langle B^{-1}u, u\rangle - \langle Su, u\rangle| \\
&= |\langle B^{-1}u, u\rangle||\zeta - \frac{\langle Su, u\rangle}{\langle B^{-1}u, u\rangle}|
\end{aligned}$$

for all $u \in \mathsf{D}(S)$. Since S is self adjoint, $\langle Su, u\rangle \in \mathbf{R}$, and by the ω-accretivity hypothesis on B,

$$\frac{\langle Su, u\rangle}{\langle B^{-1}u, u\rangle} \in S_\omega.$$

It follows that,

$$|\langle B^{-1}u, u\rangle| \left|\zeta - \frac{\langle Su, u\rangle}{\langle B^{-1}u, u\rangle}\right| \geq \frac{\kappa}{\|B\|^2} \|u\|^2 \operatorname{dist}(\zeta, S_\omega).$$

So,

$$\|(\zeta I - BS)u\| \geq \frac{\kappa^2}{\|B\|^2} \operatorname{dist}(\zeta, S_\omega) \|u\|$$

for all $u \in \mathsf{D}(S)$ which implies that $(\zeta I - BS)$ is one-one with closed range. Also,

$$(\zeta I - BS)^* = (\bar{\zeta} I - (BS)^*) = (\bar{\zeta} I - SB^*) = B^{*-1}(\bar{\zeta} I - B^* S)B^*.$$

By a similar argument as before, $(\bar{\zeta} I - B^* S)$ is one-one and so $\overline{\mathsf{R}(\zeta I - BS)} = \mathcal{H}$. This implies that $\zeta \in \rho(BS)$ and

$$\|\mathsf{R}_{BS}(\zeta)\| \leq \frac{\|B\|^2}{\kappa^2} \frac{1}{\operatorname{dist}(\zeta, S_\omega)}.$$

The result for SB follows from Corollary 5.6.6 on noting that $SB = B^{-1}(BS)B$. We used the fact that $(BS)^* = SB^*$. We leave this as an exercise. □

5.7 Bounded H^∞ functional calculus of bisectorial operators

Definition 5.7.1 ($\Psi(S^o_\mu)$ class and $\Psi(S^o_\mu)$ functional calculus)**.** Let T be an ω-bisectorial operator and let $\mu > \omega$ with $\mu < \frac{\pi}{2}$. Define

$$\Psi(S^o_\mu) = \left\{\psi \in \mathsf{H}^\infty(S^o_\mu) : \exists \alpha > 0, C \geq 0 \text{ s.t. } |\psi(\zeta)| \leq \frac{C|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}}\right\}.$$

For $\psi \in \Psi(S^o_\mu)$, define

$$\psi(T) = \frac{1}{2\pi\imath} \int_\gamma \psi(\zeta) \mathsf{R}_T(\zeta)\, d\zeta$$

where

$$\gamma = \{te^{\imath\nu} : \infty > t > 0\} + \{-te^{-\imath\nu} : 0 < t < \infty\} + \{te^{\imath\nu} : \infty > t > 0\} + \{te^{-\imath\nu} : 0 < t < \infty\}$$

with $\omega < \nu < \mu$.

As for sectorial operators, we have the following result.

Proposition 5.7.2 (Properties of the $\Psi(S^o_\mu)$ functional calculus)**.** *(i) The integral*

$$\psi(T) = \frac{1}{2\pi\imath} \int_\gamma \psi(\zeta) R_T(\zeta)\, d\zeta$$

converges absolutely in $\mathcal{L}(\mathcal{H})$.

(ii) The definition of $\psi(T)$ is independent of γ.

(iii) The map $\Psi(S^o_\mu) \to \mathcal{L}(\mathcal{H})$ defined by $\psi \mapsto \psi(T)$ is an algebra homomorphism.

Further properties are analogous to those of ω-sectorial operators. We just note:

Theorem 5.7.3 (The Convergence Lemma for bisectorial opeators)**.** *Suppose $f_n \in \Psi(S^o_\mu)$, $f \in H^\infty(S^o_\mu)$, there exists $C \geq 0$ such that $\| f_n \|_\infty \leq C$ and that $\| f_n(T) \| \leq C$. If $f_n \to f$ uniformly on compact subsets of S^o_μ, then $f_n(T)u$ converges for all $u \in \mathcal{H}$.*

Remark 5.7.4. Note that if $u \in \mathsf{N}(T)$, then $f_n(T)u = 0 \to 0$.

Definition 5.7.5 (Bounded $\mathsf{H}^\infty(S^o_\mu)$ functional calculus)**.** We say that an ω-bisectorial operator has *bounded $H^\infty(S^o_\mu)$ functional calculus* if there exists a $C \geq 0$ such that for all $\psi \in \Psi(S_\mu)$,

$$\| \psi(T) \| \leq C \| \psi \|_\infty .$$

We define the following important class of functions.

Definition 5.7.6 ($\mathsf{H}^\infty(S^o_\mu, \{0\})$)**.** Define

$$\mathsf{H}^\infty(S^o_\mu, \{0\}) = \left\{ f : S^o_\mu \cup \{0\} \to \mathbf{C};\ f|_{S^o_\mu} \in \mathsf{H}^\infty(S^o_\mu) \right\}.$$

Remark 5.7.7. Notice that a function $f \in \mathsf{H}^\infty(S^o_\mu, \{0\})$ is typically not holomorphic or even continous at 0.

Definition 5.7.8 ($\mathsf{H}^\infty(S^o_\mu, \{0\})$ functional calculus)**.** A bounded *$H^\infty(S^o_\mu, \{0\})$ functional calculus* is a map $\Phi_T : \mathsf{H}^\infty(S^o_\mu, \{0\}) \to \mathcal{L}(\mathcal{H})$ satisfying:

(i) Φ_T is an algebra homomorphism.

(ii) Φ_T is bounded. That is, there exists a $C > 0$ such that $\| \Phi_T(f) \| \leq C \| f \|_\infty$.

(iii) $\Phi_T(1) = 1$ and $\Phi_T(\mathsf{R}_\alpha) = -\mathsf{R}_T(\alpha)$.

(iv) If $f_n, f \in \mathsf{H}^\infty(S^o_\mu, \{0\})$ and there exists a $C > 0$ such that $\| f_n \|_\infty \leq C$ and $f_n(\zeta) \to f(\zeta)$ for all $\zeta \in S^o_\mu \cup \{0\}$, then $f_n(T)u \to f(T)u$.

We write $f(T) = \Phi_T(f)$.

The following equivalence is important.

Proposition 5.7.9. *An ω-bisectorial operator T has a bounded $H^\infty(S^o_\mu)$ functional calculus if and only if T has a bounded $H^\infty(S^o_\mu, \{0\})$ functional calculus.*

Lemma 5.7.10. *Let $f \in H^\infty(S^o_\mu)$. Then, there exists a sequence $f_n \in \Psi(S^o_\mu)$ such that $\| f_n \|_\infty \leq \frac{1}{\cos^2 \mu} \| f \|_\infty$ and $f_n \to f$ uniformly on compact subsets of S^o_μ.*

Proof. Let
$$\varphi_n(\zeta) = \frac{\imath n}{\imath n + \zeta}$$
and
$$\tilde{\varphi_n}(\zeta) = \frac{\zeta}{\frac{\imath}{n} + \zeta}.$$
Then, $\| \tilde{\varphi_n} \|_\infty, \| \varphi_n \|_\infty \leq \frac{1}{\cos \mu}$. Also,$|\varphi_n(\zeta) \leq |\zeta|^{-1}$ for $|\zeta|$ large and $|\tilde{\varphi_n}(\zeta)| \leq |\zeta|$ when $|\zeta|$ small. Define $\psi_n = \varphi_n \tilde{\varphi_n}$ and $\psi_n \in \Psi(S^o_\mu)$ with $\| \psi_n \| \leq \frac{1}{\cos^2 \mu}$. Then, $\psi_n(\zeta) \to 1$ for each $\zeta \in S^o_\mu$. Set $f_n = \psi_n f$ and notice that $f_n(\zeta) \to f(\zeta)$ pointwise and therefore, $f_n \to f$ uniformly on compact subsets. Also, $\| f_n \|_\infty \leq \frac{1}{\cos^2 \mu} \| f \|_\infty$. ☐

Proposition 5.7.11. *Suppose that T has a bounded $\mathsf{H}^\infty(S^o_\mu)$ functional calculus and let $P^T_0 : \mathcal{H} \to \mathsf{N}(T)$ denote the projection onto the null space. If $f \in \mathsf{H}^\infty(S^o_\mu, \{0\})$, then,*
$$f(T)u = f(0)P^T_0 u + \lim_{n\to\infty} f_n(T)u$$
where $f_n \in \Psi(S^o_\mu)$ with $f_n \to f$ uniformly on compact subsets, $\| f_n \|_\infty \leq C \| f \|_\infty$ where C can depend on μ.

Remark 5.7.12. By the Convergence Lemma, $f_n(T)u$ converges for all $u \in \mathcal{H}$.

Corollary 5.7.13. *The functional calculus $\mathsf{H}^\infty(S^o_\mu, \{0\}) \to \mathcal{L}(\mathcal{H})$ defined by $f \mapsto f(T)$ is unique.*

We can now consider decomposing the space according to spectral subspaces somewhat analogous to what we did in §2.4.

Definition 5.7.14 ($\chi_\pm, \chi_0, \mathrm{sgn}$)**.** Define $\chi_\pm, \chi_0, \mathrm{sgn} \in \mathsf{H}^\infty(S^o_\mu, \{0\})$ by
$$\chi_\pm(\zeta) = \begin{cases} 1 & \zeta \in S^o_{\mu\pm} \\ 0 & \text{otherwise} \end{cases}$$
and
$$\chi_0(\zeta) = \begin{cases} 1 & \zeta = 0 \\ 0 & \text{otherwise} \end{cases}.$$
Then, we define $\mathrm{sgn} = \chi_+ - \chi_-$.

Proposition 5.7.15. *If T has a bounded $\mathsf{H}^\infty(S^o_\mu)$ functional calculus, then, $\chi_\pm(T), \chi_0(T), \mathrm{sgn}(T) \in \mathcal{L}(\mathcal{H})$. Also, $\chi_\pm(T), \chi_0(T)$ are projections that satisfy $\chi_\pm(T)\chi_\mp(T) = \chi_\pm(T)\chi_0(T) = 0$. Furthermore,*
$$\chi_0(T)u = P^T_0 u = \lim_{n\to\infty} \frac{\imath}{n} \mathsf{R}_T\left(\frac{\imath}{n}\right) u.$$

Proof. By application of Cor 5.7.13 we have $\chi_\pm(T), \chi_0(T), \mathrm{sgn}(T) \in \mathcal{L}(\mathcal{H})$ by noting that $\chi_\pm, \chi_0 \in \mathsf{H}^\infty(S_\mu, \{0\})$.

Then, note that $\chi_\pm^2 = \chi_\pm$, $\chi_0^2 = \chi_0$, $\chi_\pm\chi_\mp = 0$, $\chi_\pm\chi_0 = 0$ and apply the algebra homomorphism property of the functional calculus.

The formula $\chi_0(T) = \lim_{n\to\infty} \frac{\imath}{n} \mathsf{R}_T\left(\frac{\imath}{n}\right)$ follows since it is the projection into $\mathsf{N}(T)$. ☐

Definition 5.7.16 (Spectral Projections)**.** Define $P_\pm = \chi_\pm(T)$ and $P_0 = \chi_0(T)$ and $\mathcal{H}_\pm = \mathsf{R}(P_\pm)$.

We have the following important splitting of the space into spectral subspaces.

Proposition 5.7.17. *If T has a bounded H^∞ functional calculus, then $\mathcal{H} = \mathsf{N}(T) \oplus \mathcal{H}_+ \oplus \mathcal{H}_-$.*

We leave the proof of the following important proposition as an exercise.

Proposition 5.7.18. *Whenever T has a bounded H^∞ functional calculus,*

$$\operatorname{sgn}(T)T = \sqrt{T^2}$$
$$T = \operatorname{sgn}(T)\sqrt{T^2}$$

and

$$\| Tu \| \simeq \left\| \sqrt{T^2}u \right\|$$

for all $u \in \mathcal{H}$.

So far, we have assumed that T has a bounded H^∞ functional calculus. However, not all ω-bisectorial operators have a H^∞ functional calculus. We give three examples of operators that do have a bounded H^∞ functional calculus.

Example 5.7.19.

$T = \operatorname{diag}(\lambda_j)$ in ℓ^2 where $\lambda_j \in S_\omega$.

$T = M_\varphi$ in $L^2(\Omega)$, $\varphi \in L^\infty(\Omega)$ with $\operatorname{ess\,ran} \varphi \subset S_\omega$.

That these operators have a bounded H^∞ functional calculus is a relatively easy fact. The following is a much deeper result related to current research. See [AMK].

Theorem 5.7.20. *Let $T = BD$ in $\mathcal{H} = L^2(\mathbf{R}^n, \mathbf{C}^N)$, $N \geq 1$, with $B \in L^\infty(\mathbf{R}^n, M^{N\times N}) = L^\infty(\mathbf{R}^n, \mathcal{L}\left(\mathbf{C}^N\right))$ with $\operatorname{Re} B \geq \kappa I$, where $\kappa > 0$. Let D be a homogeneous first order partial differential operator with constant coefficients satisfying $D = D^*$ and $\| Du \| \simeq \| \nabla \otimes u \|$, when $u \in \mathsf{R}(D)$.*

Example 5.7.21. Suppose $N = n + 1$ so $\mathcal{H} = L^2(\mathbf{R}^n) \oplus L^2(\mathbf{R}^n, \mathbf{C}^n)$ and

$$B = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{pmatrix}$$

where A satisfies the condition in Example 5.4.18.

Then, BD has a bounded H^∞ functional calculus and $\| BDu \| \simeq \left\| \sqrt{(BD)^2}u \right\|$. When $u = (f, 0)$, this gives

$$\left\| \sqrt{-\operatorname{div} A\nabla} f \right\| \simeq \| \nabla f \|$$

which is the Kato square root problem.

Consider the following evolution equation.

$$\text{(E)} \qquad \begin{cases} \dfrac{d}{dt}U(t) + DBU(t) = 0 \\ U(0) = u \in \mathcal{H} \end{cases}$$

If $u \in \mathcal{H}_+$, then there exists a unique solution $U \in C^1(\mathbf{R}^+, \mathcal{H})$ such that $U(t) \to 0$ as $t \to \infty$ and $U(t) \to u$ as $t \to 0$. Namely, this solution is

$$U(t) = e^{-tDB}u = e^{-t\sqrt{(DB)^2}}u$$

because $DB = \sqrt{(DB)^2}$ on $\mathcal{H}_+$.

If $u \in \mathcal{H}_-$, then there is a unique solutions $U \in C^1(\mathbf{R}^-, \mathcal{H})$ and it is given by

$$U(t) = e^{-tDB}u = e^{t\sqrt{(DB)^2}}u, \qquad t < 0$$

because $DB = -\sqrt{(DB)^2}$ on $\mathcal{H}_-$.

Now, if $u \in \mathsf{N}(T)$, the $U = u$ for all $t \in \mathbf{R}$.

The following very special case is well known. We present it to acknowledge the roots of the subject.

Example 5.7.22. Let $\mathcal{H} = L^2(\mathbf{R})$, $D = \frac{1}{\imath}\frac{d}{dx}$, and $B = I$. Suppose we set the initial condition $U(0) = 0$. Then,

$$\begin{aligned} \text{(E)} &\iff \partial_t U + \frac{1}{\imath}\partial_x U = 0 \\ &\iff \partial_x U + \frac{1}{\imath}\partial_t U \\ &\iff \text{U is holomorphic} \end{aligned}$$

$u \in \mathcal{H}_+$ if and only if $Du = \sqrt{D^2}u$. This happens if and only if $\xi\hat{u}(\xi) = |\xi|\hat{u}(\xi)$ which is if and only if sppt $\hat{u} \subset [0, \infty)$. Similarly, $u \in \mathcal{H}_-$ if and only if sppt $\hat{u} \subset (-\infty, 0]$. So, $u \in \mathcal{H}_+$ if u extends to a holomorphic function on the upper half plane. Similarly $u \in \mathcal{H}_-$ if u extends to a holomorphic function on the lower half plane.

Also, $\operatorname{sgn}(D)Du = \sqrt{D^2}u$ and $(\xi\hat{u})\check{} = |\xi|\hat{u}(\xi)$. So, $\operatorname{sgn}(D) = H$ is the *Hilbert transform*, which is given by

$$\operatorname{sgn}(D)u = \frac{1}{\pi\imath}\ \text{p.v.} \int_{-\infty}^{\infty} \frac{u(x)}{x-y}dx.$$

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Notation

T^* Adjoint operator of T.

$\mathcal{L}(\mathcal{X}, \mathcal{Y})$ Bounded linear functions from normed space $\mathcal{X}$ to Banach space $\mathcal{Y}$ equipped with the operator norm.

$\mathcal{L}(\mathcal{X})$ The space $\mathcal{L}(\mathcal{X}, \mathcal{X})$.

$\mathbf{C}_\infty$ Extended complex plane or Riemann sphere $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$.

$\mathcal{C}(\mathcal{X}, \mathcal{Y})$ Closed operators from $\mathcal{X}$ to $\mathcal{Y}$.

$\mathrm{co}(S)$ Convex hull of S.

$\overline{f}$ The conjugation of the function f given by $\overline{f(\overline{\zeta})}$.

Δ The Laplacian operator.

$\mathrm{diag}(\lambda_j)$ Matrix with the diagonal entry j set to λ_j.

ℓ^p Infinite sequences with the p norm.

$\mathsf{H}(\Omega)$ The space $\mathsf{H}(\Omega, \mathbf{C})$.

$\mathsf{H}(\Omega, \mathcal{X})$ The space of all holomorphic functions from Ω to $\mathcal{X}$.

$\mathsf{H}^\infty(\Omega)$ Bounded holomorphic functions on open $\Omega \subset \mathbf{C}$.

$\mathsf{H}^\infty(\Omega, \mathcal{X})$ Space of bounded holomorphic functions from Ω to $\mathcal{X}$.

id The identity map $id : z \mapsto z$.

$\langle \cdot , \cdot \rangle$ Inner product.

$\langle \mathcal{X}, \mathcal{X}' \rangle$ Duality of $\mathcal{X}$ and $\mathcal{X}'$ under the bilinear pairing

$\langle v, U \rangle$ Bilinear pairing of v and U

$\mathcal{X}$ Banach Space.

$\mathcal{X}'$ Dual space of $\mathcal{X}$

$\mathcal{X}^*$ Adjoint space of $\mathcal{X}$.

$\mathcal{X}_1 \oplus \mathcal{X}_2$ Direct sum of Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$.

$\mathcal{Y}^{\perp}$ The annihilator of $\mathcal{Y}$

$\|\cdot\|_V$ Norm on a space V.

$\mathrm{nr}(T)$ Numerical range of T.

$\mathrm{nrad}(T)$ Numerical radius of T.

$\Phi(S^o_{\mu+})$ Bounded holomorphic functions on the sector $S^o_{\mu+}$ that are constant at 0 and ∞.

$\mathcal{P}$ Algebra of polynomials.

$\mathcal{P}_R$ Algebra of power series with radius of convergence larger than R.

$\Psi(S^o_{\mu+})$ Holomorphic functions ψ decaying at a power $\alpha > 0$ at $0, \infty$

$\Psi(S^o_{\mu})$ Holomorphic functions on the open bisector S^o_{μ}.

$\mathcal{R}_K$ Algebra of rational functions with no poles in K.

$\rho(T)$ Resolvent set of T.

R_T Resolvent operator of T.

$\sigma_p(T)$ The point spectrum or the set of eigenvalues.

$\text{s-}\lim_{n\to\infty} S_n$ The limit taken in the strong convergence topology.

$\sigma(T)$ The spectrum of T.

$C_0(M)$ Functions over a metric space M that decay at ∞.

$C_b(M)$ Space of continuous complex valued bounded functions on a metric space M.

$K \Subset \Omega$ K compact and $K \subset \Omega$.

$L^p(\Omega)$ Functions $f : \Omega \to \mathbf{R}$ with finite integral p norm.

$M^{N\times N}$ The space of $N \times N$ matrices

P_0^T Projection $\mathcal{H} \mapsto \mathsf{N}(T)$.

$S \subset T$ Operator T is an extension of S.

S^{conj} The complex conjugate of the set S.

$S^o_{\mu+}$ Open μ-sector.

$S^o_{\mu-}$ The left open sector $-S^o_{\mu+}$.

S^o_{μ} The open bisector $S^o_{\mu+} \cup S^o_{\mu-}$.

$S_{\omega+}$ Closed ω-sector.

$S_{\omega-}$ The left closed sector $-S_{\omega+}$.

S_{ω} The closed bisector $S_{\omega+} \cup S_{\omega-}$.

$u \perp v$ u is perpendicular to v, ie., $\langle u, v\rangle = 0$.

$W^{p,k}(\Omega)$ The Sobolev space in L^p with k derivatives.

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