THE AUSTRALIAN NATIONAL UNIVERSITY

Au-Delà des Racines Carrées

Alan McIntosh
Centre for Mathematics and its Applications
Australian National University
Canberra, Australia

Dédié à R.Coifman et Y. Meyer

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The lecture is based on joint research with Andreas Axelsson and Stephen Keith at the Centre for Mathematics and its Applications, Australian National University. Full details will appear in our paper "Functional Calculus of Non Self-Adjoint Dirac Operators", which builds on earlier work of Auscher, Hofmann, Lacey, Tchamitchian and myself on the Kato square root problem for elliptic operators.

These results are a continuation of the Calderón Program to which Coifman and Meyer made such a large contribution. I am very happy to have had the opportunity to collaborate with them over 20 years ago, and gratefully acknowledge the stimulus which they provided to my research. I extend my best wishes to both Yves and Raphy.

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This printed version has been prepared by Maren Schmalmack, whom I wholeheartedly thank.

Aim. To prove estimates

$$\|f(\Pi_B)u\|_2 \lesssim \|f\|_\infty \|u\|_2$$

and perturbation results

$$\|f(\Pi_{B+A})u - f(\Pi_B)u\|_2 \lesssim \|f\|_\infty \|A\|_\infty \|u\|_2$$

$$\text{for} \qquad \begin{cases} u & \in & L^2(\mathbb{R}^n, \mathcal{W}) \\ f & \in & H^\infty(S_\mu, \mathbb{C}) \end{cases}$$

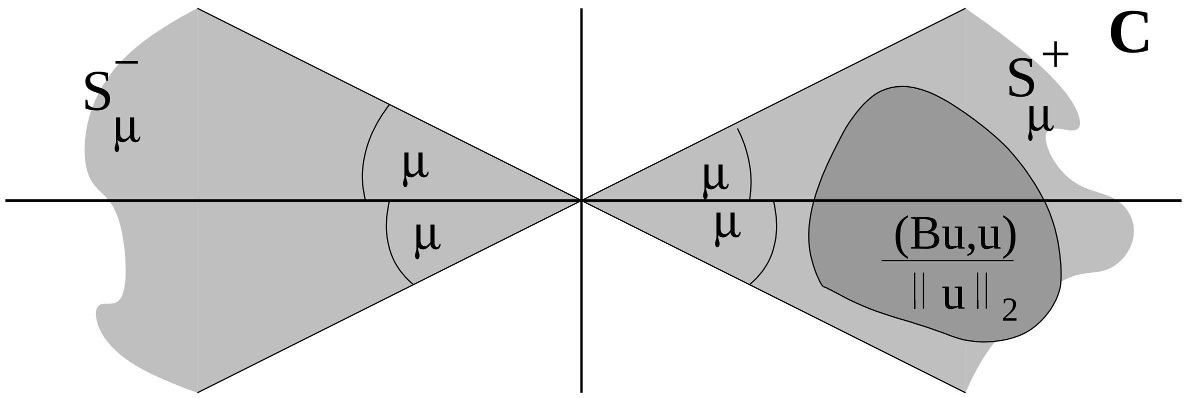
$$\text{where} \qquad \Pi_B = \Gamma + B^{-1}\Gamma^* B$$

is a first order elliptic system, defined as follows.

1) $\mathcal{W} = \mathbb{C}^N, \quad N \geq 2$

2) $0 \leq \omega < \mu < \frac{\pi}{2}$

3) $S_\mu = S_\mu^+ \cup S_\mu^-$ (open)



4) $B,\ B^{-1} \in L^\infty\Big(\mathcal{L}(\mathcal{W})\Big)$,

$$|\arg(Bu, u)| \leq \omega \quad \forall u \in L^2(\mathbb{R}^n, \mathcal{W})$$

i.e. B is a bounded invertible ω-accretive multiplication operator.

5) Γ is a first order p.d. operator with exact nilpotent symbol Γ_ξ, i.e.

$$\mathcal{R}(\Gamma_\xi) = \mathcal{N}(\Gamma_\xi) \quad \forall \xi \in \mathbb{R}^n$$

and thus

*) $\Gamma^2 = 0$; $\overline{\mathcal{R}(\Gamma)} = \mathcal{N}(\Gamma)$

$(\Gamma^*)^2 = 0$; $\overline{\mathcal{R}(\Gamma^*)} = \mathcal{N}(\Gamma^*)$

*) $\mathcal{H} = \mathcal{N}(\Gamma) \oplus \mathcal{N}(\Gamma^*)$

*) the "swapping operator"

$$\Pi = \Gamma + \Gamma^*$$

is first order, self-adjoint, elliptic, i.e. $\forall u \in \dot{H}^1(\mathbb{R}^n)$

$$\|\Gamma u\|_2 + \|\Gamma^* u\|_2 \simeq \|\Pi u\|_2 \simeq \|\nabla \otimes u\|_2$$

*) the "Laplacian"

$$\Delta = \Pi^2 = \Gamma\Gamma^* + \Gamma^*\Gamma$$

is a second order, positive self-adjoint elliptic operator

Definitions

$$\Gamma_B^* = B^{-1}\Gamma^* B$$
$$\Pi_B = \Gamma + \Gamma_B^*$$
$$\Delta_B = (\Pi_B)^2 = \Gamma\Gamma_B^* + \Gamma_B^*\Gamma$$

Properties

*) $(\Gamma_B^*)^2 = 0$, $\overline{\mathcal{R}(\Gamma_B^*)} = \mathcal{N}(\Gamma_B^*)$

*) $\mathcal{H} = \mathcal{N}(\Gamma) \oplus \mathcal{N}(\Gamma_B^*)$ (non-orthogonal decomposition)

*) The swapping operator Π_B is a one-one, closed operator in $L^2(\mathbb{R}, \mathcal{W})$ with spectrum

$$\sigma(\Pi_B) \subset \overline{S_\omega} \quad \text{and}$$

resolvent bounds:

$\exists C = C(\|B\|_\infty, \|B^{-1}\|_\infty,\ \omega)$ s.th.

$$\|(\Pi_B - \lambda I)^{-1}u\|_2 \leq \frac{C}{\text{dist}(\lambda, S_\omega)}\|u\|_2$$

[Auscher, M^c Intosh, Nahmod 1997]^2

Theorem [Axelsson, Keith, McIntosh]
Let $\mu > \omega$. There exists $c_\mu(\|B\|_\infty, \|B^{-1}\|_\infty,\ \omega)$ s.th.

$$\boxed{\|f(\Pi_B)\| \le c_\mu \|f\|_\infty, \qquad f \in H^\infty(S_\mu)}$$

Corollaries

(A) $\|F(\Delta_B)\| \le c_\mu \|F\|_\infty;\ F \in H^\infty(S^+_{2\mu})$

(B) $\|\Gamma u\|_2 + \|\Gamma^*_B u\|_2 \simeq \|\Pi_B u\|_2 \simeq \|\sqrt{\Delta_B} u\|_2$

(C) $f(\Pi_B)$ depends analytically on B

(D) Given M, $0 \le \omega < \mu < \frac{\pi}{2}$, then

$$\|f(\Pi_{B+A}) - f(\Pi_B)\| \le c\|f\|_\infty \|A\|_\infty$$

for all $f \in H^\infty(S_\mu)$, A, B, s.th.

$$\begin{cases} \|B\|_\infty, \|B+A\|_\infty,\ \|B^{-1}\|_\infty\ , \\ \qquad\qquad \|(B+A)^{-1}\|_\infty \le\ M \\ \text{arg}(Bu,u),\ \text{arg}((B+A)u,u) \le\ \omega \end{cases}$$

Proof of Corollaries

(A) Take $f(z) = F(z^2)$

(B) Take $f(z) = \frac{z}{\sqrt{z^2}} = \frac{\sqrt{z^2}}{z} = \begin{cases} +1 & , z \in S^+_\mu \\ -1 & , z \in S^-_\mu \end{cases}$

(C) Prove analyticity for "nice" f.
Use uniform bounds to treat all f.

(D) Integrate $\frac{df}{dt}(\Pi_{B+tA})$ from 0 to 1.

Special Cases

(I) B self-adjoint: $B = B^*$, i.e. $\omega = 0$.

Then Π_B is self-adjoint with respect to inner product $(u, v)_B := (Bu, v)$.

Proof.
$$\begin{aligned}(\Gamma u, v)_B &= (B\Gamma u, v)\\ &= (Bu, B^{-1}\Gamma^* Bv)\\ &= (u, \Gamma^*_B)_B\end{aligned}$$

i.e. Γ, Γ^*_B adjoint w.r.t. $(.,.)_B$

i.e. $\Pi_B = \Gamma + \Gamma^*_B$ is self-adjoint w.r.t. $(.,.)_B$.

So Theorem and Corollaries (A), (B) hold by operator theory.

BUT (C), (D) do not follow.

They need bounds for non-self-adjoint perturbations.

<u>Case (II) a)</u> $n = 1,\ \mathcal{W} = \mathbb{C}^2 = \mathbb{C}^{(1)} \oplus \mathbb{C}^{(2)}$

$$\begin{array}{lcccccc}
\Gamma: & & & L^2(\mathbb{R},\mathbb{C}^{(1)}) & \xrightarrow{\frac{d}{dx}} & L^2(\mathbb{R},\mathbb{C}^{(2)}) & \to \{0\} \\
\big\downarrow B & & & \big\downarrow \frac{1}{b} & & \big\downarrow b & \\
\Gamma^*: & \{0\} & \leftarrow & L^2(\mathbb{R},\mathbb{C}^{(1)}) & \xleftarrow{-\frac{d}{dx}} & L^2(\mathbb{R},\mathbb{C}^{(2)}) &
\end{array}$$

$$(*) \quad \begin{cases} b,\ \frac{1}{b} \in L^\infty(\mathbb{R},\mathbb{C}) \\ |\arg b(x)| \le \omega \text{ a.a. } x \in \mathbb{R} \end{cases}$$

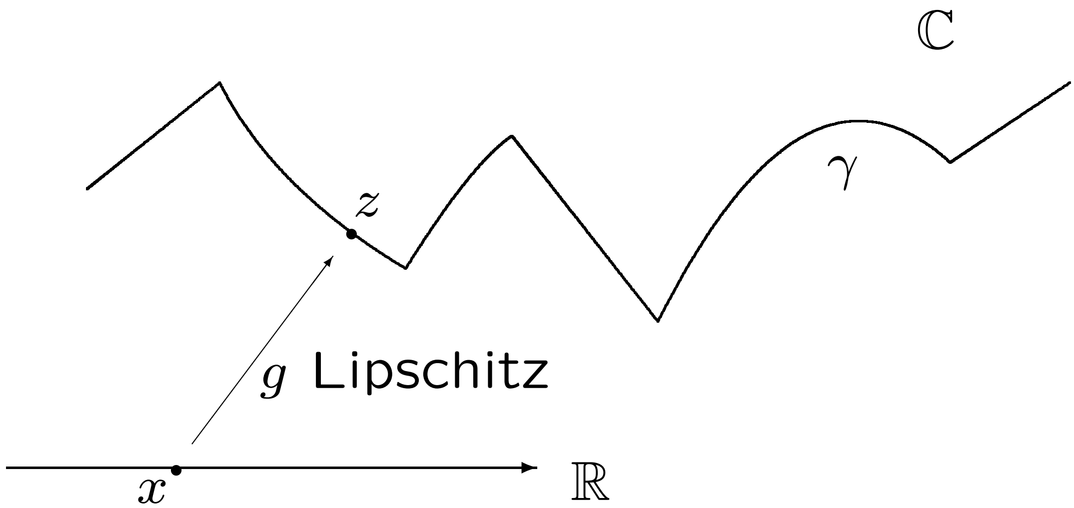
$$\begin{aligned}
\Gamma_B^* u &= -b\,\frac{d}{dx}\,b\,u \qquad \underline{\text{if}}\ u \in L^2(\mathbb{R},\mathbb{C}^{(2)}) \\
\Delta_B u &= -b\,\frac{d}{dx}\,b\,\frac{d}{dx}\,u \quad \underline{\text{if}}\ \ u \in L^2(\mathbb{R},\mathbb{C}^{(1)})
\end{aligned}$$

(B): <u>If</u> $u \in L^2(\mathbb{R},\mathbb{C}^{(1)})$, then

$$\left\|\frac{du}{dx}\right\|_2 \simeq \left\|\sqrt{-b\,\frac{d}{dx}\,b\,\frac{d}{dx}}\ u\right\|_2$$

i.e. $$\|D_\gamma u\|_2 \simeq \left\|\sqrt{-{D_\gamma}^2}\ u\right\|_2$$

where γ is the Lipschitz curve in $\mathbb{C}$ parametrized by $g : \mathbb{R} \to \mathbb{C}$ where $b = \frac{1}{g'}$ satisfies $(*)$. Here $D_\gamma = \frac{d}{dz}\Big|_\gamma \equiv b\ \frac{d}{dx}$.



Thus (B) gives the boundedness of the Cauchy integral $C_\gamma = \frac{D_\gamma}{\sqrt{-D_\gamma{}^2}}$.

If $\|b-1\|_\infty < \varepsilon$ due to [Calderón 1977].
General case: [Coifman, M^c^Intosh, Meyer 1982]

Case (II) a) $n = 1, \ \mathcal{W} = \mathbb{C}^2 = \mathbb{C}^{(1)} \oplus \mathbb{C}^{(2)}$

$$\begin{array}{lccc}
\Gamma: & L^2(\mathbb{R},\mathbb{C}^{(1)}) & \xrightarrow{\frac{d}{dx}} & L^2(\mathbb{R},\mathbb{C}^{(2)}) \rightarrow \{0\} \\
\Big\downarrow B & \Big\downarrow \frac{1}{b} & & \Big\downarrow b \\
\Gamma^*: \quad \{0\} \leftarrow & L^2(\mathbb{R},\mathbb{C}^{(1)}) & \xleftarrow{-\frac{d}{dx}} & L^2(\mathbb{R},\mathbb{C}^{(2)})
\end{array}$$

$$(*) \quad \begin{cases} b, \ \frac{1}{b} \in L^\infty(\mathbb{R},\mathbb{C}) \\ |\arg b(x)| \leq \omega \ \text{a.a. } x \in \mathbb{R} \end{cases}$$

(A): $\quad \|F(\Delta_B)\| \lesssim \|F\|_\infty \quad \forall F \in H^\infty(S^+_{2\mu})$

$$\Rightarrow \ \|F(-D_\gamma^2)\| \lesssim \|F\|_\infty \quad \forall F \in H^\infty(S^+_{2\mu})$$

$$\Rightarrow \ \|f(-iD_\gamma)\| \lesssim \|f\|_\infty \quad \forall f \in H^\infty(S_\mu)$$

in agreement with results on functional calculi of D_γ by [Coifman and Meyer, 1980] and [M^cIntosh and Qian, 1991]. For the final implication see [Albrecht, Duong, M^cIntosh 1996].

(C): The analytic dependence of $C_\gamma = \dfrac{D_\gamma}{\sqrt{-D_\gamma{}^2}}$ on γ has its roots in the "commutator theorem" [Calderón, 1965] and the "higher commutator theorems" [Coifman and Meyer, 1975, 1978].

Case II b) $n = 1,\ \mathcal{W} = \mathbb{C}^2 = \mathbb{C}^{(1)} \oplus \mathbb{C}^{(2)}$

$$\begin{array}{cccccc} \Gamma: & & L^2(\mathbb{R},\mathbb{C}^{(1)}) & \xrightarrow{\frac{d}{dx}} & L^2(\mathbb{R},\mathbb{C}^{(2)}) \to \{0\} \\ \downarrow B & & \downarrow 1 & & \downarrow b \\ \Gamma^*: & \{0\} \leftarrow & L^2(\mathbb{R},\mathbb{C}^{(1)}) & \xleftarrow{-\frac{d}{dx}} & L^2(\mathbb{R},\mathbb{C}^{(2)}) \end{array}$$

$$(*) \quad \begin{cases} b,\ \frac{1}{b} \in L^\infty(\mathbb{R},\mathbb{C}) \\ |\arg b(x)| \le \omega \text{ a.a. } x \in \mathbb{R} \end{cases}$$

$$\Gamma_B^* u = -\frac{d}{dx}\, b\, u \quad \underline{\text{if}} \quad u \in L^2(\mathbb{R},\mathbb{C}^{(2)})$$

$$\Delta_B u = -\frac{d}{dx}\, b\, \frac{d}{dx}\, u \quad \underline{\text{if}} \quad u \in L^2(\mathbb{R},\mathbb{C}^{(1)})$$

(B): If $u \in L^2(\mathbb{R},\mathbb{C}^{(1)})$, then

$$\left\|\frac{du}{dx}\right\|_2 \simeq \left\|\sqrt{-b\, \frac{d}{dx}\, b\, \frac{d}{dx}}\ u\right\|_2$$

This is "one-dimensional Kato problem", originally proved in [Coifman, M^cIntosh, Meyer, 1982].

Case II c) $n = 1,\ \mathcal{W} = \mathbb{C}^2 = \mathbb{C}^{(1)} \oplus \mathbb{C}^{(2)}$

$$\begin{array}{llcl}
\Gamma : & & L^2(\mathbb{R}, \mathbb{C}^{(1)}) \xrightarrow{\frac{d}{dx}} & L^2(\mathbb{R}, \mathbb{C}^{(2)}) \to \{0\} \\
\Big\downarrow B & & \Big\downarrow \frac{1}{a} & \Big\downarrow b \\
\Gamma^* : \{0\} \leftarrow & & L^2(\mathbb{R}, \mathbb{C}^{(1)}) \xleftarrow{-\frac{d}{dx}} & L^2(\mathbb{R}, \mathbb{C}^{(2)})
\end{array}$$

$$b,\ \frac{1}{b} \in L^\infty(\mathbb{R}, \mathbb{C}); \quad a,\ \frac{1}{a} \in L^\infty(\mathbb{R}, \mathbb{C})$$

$$\left\{ \begin{array}{ll} |\arg b(x)| & \leq \omega \\ |\arg a(x)| & \leq \omega \end{array} \right\} \quad \text{a.a. } x \in \mathbb{R}$$

$$\Gamma_B^* u = -a \frac{d}{dx} b\ u \quad \underline{\text{if}} \quad u \in L^2(\mathbb{R}, \mathbb{C}^{(2)})$$

$$\Delta_B u = -a \frac{d}{dx} b \frac{d}{dx}\ u \quad \underline{\text{if}} \quad u \in L^2(\mathbb{R}, \mathbb{C}^{(1)})$$

(B): If $u \in L^2(\mathbb{R}, \mathbb{C}^{(1)})$, then

$$\left\| \frac{du}{dx} \right\|_2 \simeq \left\| \sqrt{-a \frac{d}{dx} b \frac{d}{dx}}\ u \right\|_2$$

[Kenig and Meyer, 1985]
[Auscher, McIntosh, Nahmod, 1997]² Also (A). In this paper we used a similar representation of operators as here. We also treated boundary value problems on an interval.

Case (III) $n \geq 1,\ \mathcal{W} = {\mathbb{C}_{(n)}}^2$ or $\mathcal{V}^2 = \mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)}$
$\mathbb{C}_{(n)} =$ Clifford algebra with n generators
$\mathcal{V} =$ representation space of $\mathbb{C}_{(n)}$.

$$D = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} e_j = \text{ Dirac operator}$$

$$\begin{array}{cccccc}
\Gamma\ : & & L^2(\mathbb{R}^n, \mathcal{V}^{(1)}) & \xrightarrow{D} & L^2(\mathbb{R}^n, \mathcal{V}^{(2)}) \rightarrow \{0\} \\
\big\downarrow B & & \big\downarrow A^{-1} & & \big\downarrow A \\
\Gamma^*: & \{0\} \leftarrow & L^2(\mathbb{R}^n, \mathcal{V}^{(1)}) & \xleftarrow{D} & L^2(\mathbb{R}^n, \mathcal{V}^{(2)})
\end{array}$$

(B) on $\mathcal{V}^{(1)}$:

$$\|Du\|_2 \simeq \|\sqrt{ADAD}\ u\|_2$$

This implies the boundedness of the Clifford-Cauchy integral on a Lipschitz graph.
If $\|A - I\|_\infty < \varepsilon$, due to [Murray 1985].
General case: [McIntosh 1989], [Li, McIntosh, Semmes 1992] and others.

(A) on $\mathcal{V}^{(1)}$ where $\mathcal{V} = \mathbb{C}^n$: [Li, McIntosh, Qian, 1994]

Case (IV) $n \geq 1, \quad \mathcal{W} = \Lambda = \bigoplus_{k=1}^{n} \Lambda^k$

$= \text{multivectors on } \mathbb{R}^n.$

$$\begin{array}{ccccccc}
\Gamma : L^2(\mathbb{R}^n, \Lambda^0) & \xrightarrow{d=\nabla} & L^2(\mathbb{R}^n, \Lambda^1) & \xrightarrow{d} & L^2(\mathbb{R}^n, \Lambda^2) & \xrightarrow{d} \\
B \quad \downarrow B_0 & & \downarrow B_1 & & \downarrow B_2 & \\
\Gamma^* : L^2(\mathbb{R}^n, \Lambda^0) & \xleftarrow{-\delta=-\nabla\cdot} & L^2(\mathbb{R}^n, \Lambda^1) & \xleftarrow{-\delta} & L^2(\mathbb{R}^n, \Lambda^2) & \xleftarrow{-\delta}
\end{array}$$

Theorem: $\|f(d - B^{-1}\delta B)\| \lesssim \|f\|_\infty$

Recall: $\Delta_B = -dB^{-1}\delta B - B^{-1}\delta B d.$

Cor. (A): $\|F(\Delta_B)\| \lesssim \|F\|_\infty$

Cor. (B): $\|du\|_2 + \|\delta B u\|_2 \simeq \|\sqrt{\Delta_B} u\|_2$

(A) on Λ^0 when $B_1 = I, \; B_0 = \frac{1}{b}$:
$\|F(b\Delta)\| \lesssim \|F\|_\infty$ [McIntosh, Nahmod 2000]
(A) on Λ^0 when $B_1 = A, \; B_0 = \frac{1}{b}$:
$\|F(-b\nabla.A\nabla)u\| \lesssim \|F\|_\infty$: previously proved by [Duong, Ouhabaz 1997] under some smoothness assumptions on A.

(B) on Λ^0 when $B_1 = A, \; B_0 = I$:

$$\|\nabla u\|_2 \simeq \|\sqrt{-\nabla.A\nabla} u\|_2$$

This is the Kato square root problem.
If $\|A - I\|_\infty < \varepsilon$: [Coifman, Deng, Meyer 1983] [Fabes, Jerison, Kenig 1982]
Full result:[Auscher, Hofmann, Lacey, McIntosh, Tchamitchian 2002].

Case IV (b): As above, but with

$$\mathcal{W} = \Lambda \otimes \mathbb{C}^M$$

(B) on $L^2(\mathbb{R}^n, \Lambda^0 \otimes \mathbb{C}^M)\Big(\simeq L^2(\mathbb{R}^n, \mathbb{C}^M)\Big))$ gives Kato problem for certain 2nd order elliptic systems. In [Auscher, Hofmann, McIntosh, Tchamitchian 2001], such a result is obtained under a more general coercivity condition.

Remark:

$$(\mathrm{B}) \Leftrightarrow L^2(\mathbb{R}^n, \Lambda) = L^2_+(\mathbb{R}^n, \Lambda) \oplus L^2_-(\mathbb{R}^n, \Lambda)$$

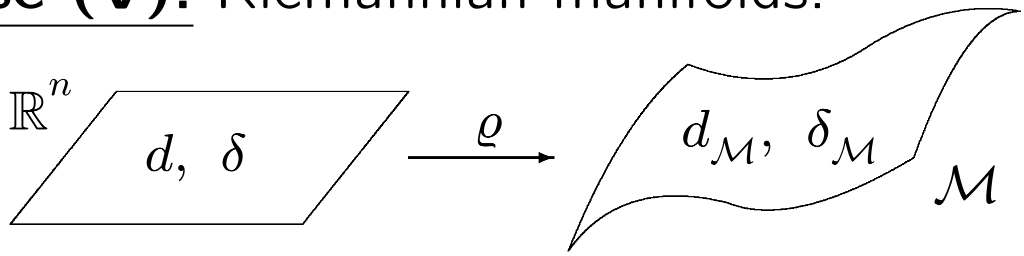
where $\begin{cases} L^2_+ &= \{u \in L^2 : \sqrt{\Delta_B}u = \Pi_B u\} \\ L^2_- &= \{u \in L^2 : \sqrt{\Delta_B}u = -\Pi_B u\} \end{cases}$

If $u_0 \in L^2_\pm$, then $u(x,t) = e^{-t\sqrt{\Delta_B}}u_0(x)$ is a solution of $\begin{cases} \frac{\partial u}{\partial t} \pm \Pi_B u &= 0, \quad t > 0 \\ u\Big|_{t=0} &= u_0 \end{cases}$.

Theorem $\Rightarrow \|u_0\|_2^2 \simeq \int\limits_0^\infty \Big\| \frac{\partial}{\partial t} u(.,t)\|^2 t dt, \quad u_0 \in L^2_\pm.$

Indeed the reverse implication holds too, as seen by applying the procedure in [Coifman, Jones, Semmes 1989], [McIntosh, Qian 1991].

Case (V): Riemannian manifolds.



$$
\begin{aligned}
G &= \varrho^*\varrho_* = \begin{cases} 1 & \text{on } \Lambda^0 \\ (g_{ij}) & \text{on } \Lambda^1 \\ (g_{IJ}) = & \text{on } \Lambda^n \end{cases} \\
D_G &= \varrho^*(d_M - \delta_M)\varrho^{*-1} \\
&= d - \varrho^*\delta_M\varrho^{*-1} \\
&= d - B^{-1}\delta B,\ B^{-1} = \frac{1}{\sqrt{g}}G \\
&= d - \delta_B \\
\Delta_G &= -d\delta_B - \delta_B d \qquad \text{(Hodge Laplacian)}
\end{aligned}
$$

Theorem + (A) + (B) are easy because $B = B^*$. But (C) analyticity of $f(D_G)$ and $F(\Delta_G)$ as functions of $G \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$ is a new result.

Also new is

(D): Let $\mu > 0$.

*) $\|f(D_{G+H}) - f(D_G)\| \lesssim \|f\|_\infty \|H\|_\infty,$
$f \in H^\infty(S_\mu)$

*) $\|F(\Delta_{G+H}) - F(\Delta_G)\| \lesssim \|F\|_\infty \|H\|_\infty,$
$F \in H^\infty(S^+_{2\mu})$

*) $\left\| \frac{D_{G+H}}{\sqrt{\Delta_{G+H}}} - \frac{D_G}{\sqrt{\Delta_G}} \right\| \lesssim \|H\|_\infty.$

Case (VI): Maxwell's Equations

$$
(M)\begin{cases}
dB &= 0 \\
\partial_t B + dE &= 0 \\
\partial_t(\varepsilon E) + \delta\mu^{-1}B &= -J \\
\Big(\Leftrightarrow \partial_t E + \varepsilon^{-1}\delta\mu^{-1}B &= -\varepsilon^{-1}J\Big) \\
\delta(\varepsilon E) &= \rho
\end{cases}
$$

$$
\begin{array}{ccccc}
 & & E & B & \\
 & \Lambda^0 & \Lambda^1 & \Lambda^2 & \Lambda^3 \\
A: & \downarrow I & \downarrow \varepsilon & \downarrow \mu^{-1} & \downarrow I \\
 & \Lambda^0 & \Lambda^1 & \Lambda^2 & \Lambda^3 \\
 & \varrho & J & &
\end{array}
$$

(M) implies:

$$(i\partial_t + d - A^{-1}\delta A)(E - iB) = -\varrho - i\varepsilon^{-1}J$$

i.e.

$$(i\partial_t + \Pi_A)(E - iB) = -\varrho - i\varepsilon^{-1}J$$

Proof of Theorem (Outline):

Need Square Function Estimate

$$\text{(S)} \quad \int_0^\infty \left\| \frac{1}{1+t^2\Pi_B^2} t\Pi_B u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

and dual estimate.

Recall $\mathcal{H} = \mathcal{N}(\Gamma) \oplus \mathcal{N}(\Gamma_B^*)$

$$\begin{aligned} \Pi_B &= \Gamma + \Gamma_B^* = \Gamma + B^{-1}\Gamma^* B \\ \Delta_B &= \Pi_B^2 \end{aligned}$$

So Need

$$(\text{S}_1) \quad \int_0^\infty \left\| \frac{1}{1+t^2\Delta_B} t\Gamma_B^* u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 \quad \forall u \in \mathcal{N}(\Gamma)$$

and similar estimate on $\mathcal{N}(\Gamma_B^*)$.

Write: $u = P_t u + (I - P_t)u$ where P_t is convolution with kernel supported in $B(0,t)$ with vanishing moment, $t > 0$. Note that

$$(I-P_t)u = (I-P_t)\Pi^{-1}\Gamma^* u = \Gamma\Pi^{-1}(I-P_t)$$

where $u \in \mathcal{N}(\Gamma)$.

So

$$(\mathsf{S}_h) \qquad \int_0^\infty \left\| \frac{1}{1+t^2\Delta_B} t\Gamma_B^*(I-P_t)u \right\|^2 \frac{dt}{t}$$

$$= \int_0^\infty \Big\| \underbrace{\frac{t\Pi_B}{1+t^2\Delta_B} t\Pi_B}_{\text{bounded}} \underbrace{\Pi_B^{-1}\Gamma}_{\text{bdd.}} \underbrace{(t\Pi)^{-1}(I-P_t)}_{Q_t} u \Big\|^2 \frac{dt}{t}$$

$$\lesssim \|u\|^2 \qquad\qquad (u \in \mathcal{N}(\Gamma))$$

We are adapting the proof of the Kato problem in [Auscher, Hofmann, Lacey, McIntosh and Tchamitchian 2002] and [AHMcT 2001]. These proofs used ideas of [HLMc 2002] and [HMc 2002] which in turn developed [Auscher, Tchamitchian 1998], incorporating $T(b)$ theorems along the lines of [Semmes 1990] and [Christ 1990]. Of course these works owe many ideas to prior results of Calderón, Coifman, Meyer, David, Journé, Semmes and many others.

So <u>Need</u>:

$$(\mathrm{S}_l) \qquad \int_0^\infty \Big\| \underbrace{\frac{1}{1+t^2\Delta_B} t\Gamma_B^*}_{\Theta_t} P_t u \Big\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

<u>Local behaviour of operators</u>

Let $N > 0$; $\quad t > 0, \quad F, E \subset \mathbb{R}^n$ and $u \in L^2(\mathbb{R}^n, \mathcal{W})$ with *sppt(u)*$\subset F$.

<u>Then</u>

$$\Big\{ \int_E |\tfrac{1}{1+t^2\Delta_B} u|^2 \Big\}^{\frac{1}{2}} \lesssim \Big(1 + \tfrac{\mathrm{dist}(E,F)}{t}\Big)^{-N} \|u\|_2$$

$$\Big\{ \int_E |\Theta_t u|^2 \Big\}^{\frac{1}{2}} \lesssim \Big(1 + \tfrac{\mathrm{dist}(E,F)}{t}\Big)^{-N} \|u\|_2$$

<u>**Corollary:**</u> $\Theta_t : L^\infty(\mathbb{R}^n, \mathcal{W}) \to L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{W})$

<u>**Definition**</u> For a.a. $x \in \mathbb{R}^n$, define $\gamma_t(x) \in \mathcal{L}(\mathcal{W})$ by

$$\gamma_t(x).w = \Theta_t(w)(x), \quad w \in \mathcal{W} \subset L^\infty(\mathbb{R}^n, \mathcal{W})$$

(roughly: $\gamma_t(x) = \int \Theta_t(x,y)dy$). Then

$$(*) \qquad \frac{1}{|B(x,t)|} \int_{B(x,t)} \|\gamma_t(x)\|^2_{\mathcal{L}(\mathcal{W})} \lesssim const.$$

Recall: $\Theta_t = \frac{1}{1+t^2\Delta_B} t\Gamma_B^*$

$P_t = \varphi_t *$

Need: $\int_0^\infty \|\Theta_t P_t u\|^2 \frac{dt}{t} \lesssim \|u\|^2$

Let $A_t =$ dyadic averaging operator:

$$A_t f(x) = \frac{1}{|Q(x,t)|} \int_{Q(x,t)} f(y) dy \quad \text{where}$$

$Q_t =$ dyadic cube of scale t which contains x.

$$\begin{aligned} \Theta_t P_t u(x) = \ & \Theta_t P_t u(x) - \gamma_t(x).A_t P_t u(x) && \text{(I)} \\ & + \gamma_t(x).A_t(P_t - I)u(x) && \text{(II)} \\ & + \gamma_t(x).A_t u(x) && \text{(III)} \end{aligned}$$

(I) and (II) use "local behaviour" + "weighted Poincaré inequalities" + following inequality:

$$\Big|\frac{1}{|Q|}\int_Q \Gamma u\Big| \lesssim \frac{1}{\sqrt{\ell(Q)}} \Big(\frac{1}{|Q|}\int_Q |u|^2\Big)^{\frac{1}{4}} \Big(\frac{1}{|Q|}\int_Q |\Gamma u|^2\Big)^{\frac{1}{4}}$$

(III) uses Carleson's theorem.

So Need Carleson Estimate:

$$\text{(C)} \qquad \iint\limits_{R_Q} \left\|\gamma_t(x)\right\|^2_{\mathcal{L}(\mathcal{W})} \frac{dx\,dt}{t} \lesssim |Q|$$

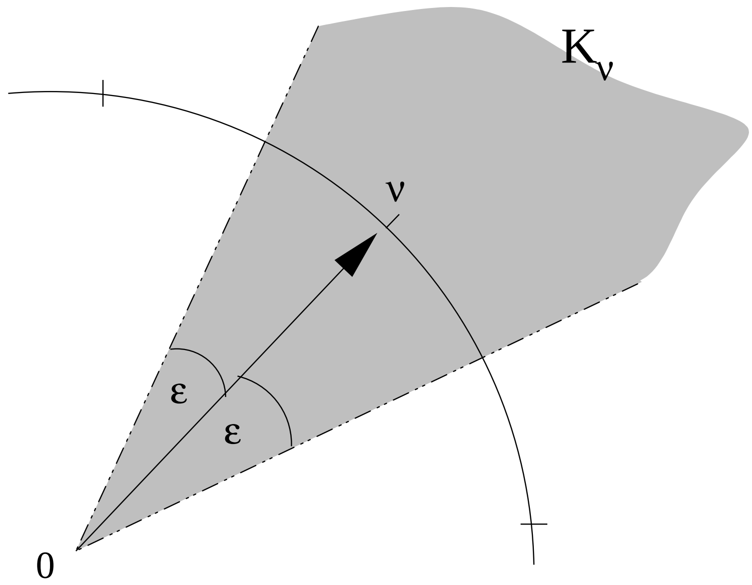
$$\forall \text{ cubes } Q \subset \mathbb{R}^n$$

where $R_Q = Q \times [0, \ell]$, $\ell = \ell(Q)$.

Let $\varepsilon_0 > 0$, to be chosen later. Choose a finite set $\{\nu\} \subset \mathcal{L}(\mathcal{W})$ s.th. $\|\nu\| = 1$ and

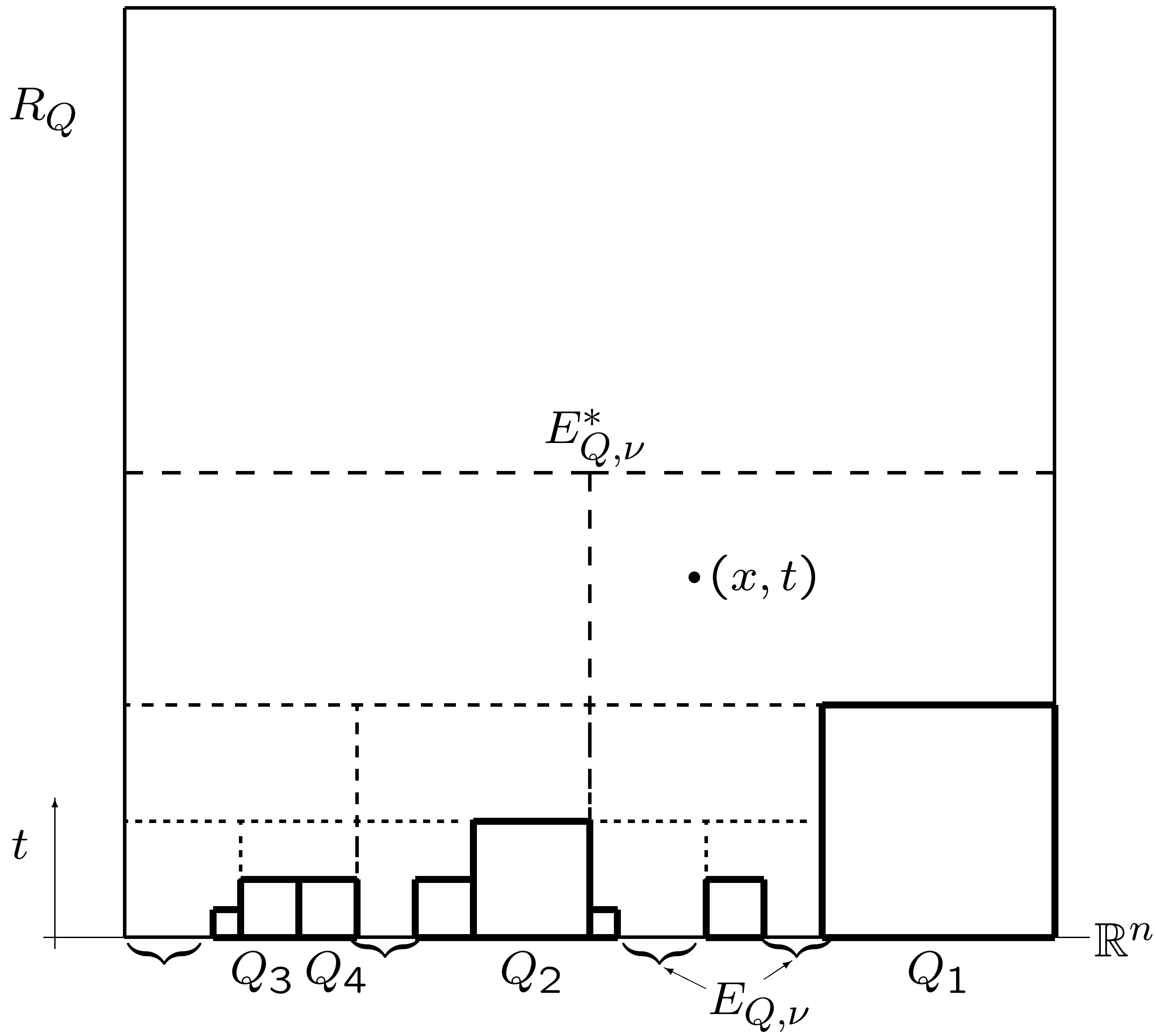
$$\bigcup_\nu K_\nu = \mathcal{L}(\mathcal{W}) \backslash \{0\}, \quad \text{where}$$

$$K_\nu = \{rG \in \mathcal{L}(\mathcal{W}) \mid r > 0,\ \|G\| = 1,\ \|G - \nu\| < \varepsilon_0\}$$



So Need: $\forall \nu \in \{\nu\}$,

$$(\mathrm{C}_{\varepsilon_0}) \qquad \iint\limits_{\substack{(x,t) \in R_Q \\ \gamma_t(x) \in K_\nu}} \|\gamma_t(x)\|^2 \, \frac{dx\,dt}{t} \lesssim |Q|$$



(C_{ε_0}) is a consequence of (C'_{ε_0}):

(C'_{ε_0}) $\exists \eta > 0$, $C < \infty$ s.th. $\forall Q$, $\forall \nu \in \mathcal{L}(\mathcal{W})$, $\|\nu\| = 1$ there exists a "good" set $E_{Q,\nu} \subset Q$ with

(i) $|E_{Q,\nu}| > \eta |Q|$, (ii) $Q \setminus E_{Q,\nu} = \bigcup_k Q_k$,

Q_k disjoint dyadic subcubes of Q,

$$(iii) \iint_{\substack{(x,t)\in E^*_{Q,\nu} \\ \gamma_t(x) \in K_\nu}} |\gamma_t(x)|^2 \, \frac{dx\,dt}{t} \lesssim |Q|$$

$$(E^*_{Q,\nu} = R_Q \setminus \bigcup_k R_{Q_k}).$$

Have: Cube Q, $\ell(Q)=\ell$, $\nu\in\mathcal{L}(\mathcal{W})$, $\|\nu\|=1$.
Choose $\underline{w}$, $w\in\mathcal{W}$ s.th. $|\underline{w}|=1$, $|w|=1$, $\nu^*(\underline{w})=w$.
Let χ_Q be a smooth cut-off function for Q.
Define $w_Q=\chi_Q w$, and for $\varepsilon>0$,

$$\begin{aligned} f^w_{Q,\varepsilon} &= w_Q - \varepsilon\ell i\Gamma\Big(\frac{1}{1+\varepsilon\ell i\Pi_B}\Big)w_Q \\ &= \Big(1+\varepsilon\ell i\Gamma^*_B\Big)\Big(\frac{1}{1+\varepsilon\ell i\Pi_B}\Big)w_Q \end{aligned}$$

Lemmas. (*) $\|w_Q\|_2,\ \|f^w_{Q,\varepsilon}\|_2 \lesssim |Q|$

(*) $\iint\limits_{R_Q} |\Theta_t f^w_{Q,\varepsilon}|^2\frac{dx\ dt}{t} \lesssim \frac{1}{\varepsilon^2}|Q|$

(*) $|\frac{1}{|Q|}\int\limits_Q f^w_{Q,\varepsilon} - w| \le c\sqrt{\varepsilon}$.

Define $f^w_Q = f^w_{Q,\varepsilon}$ where $\varepsilon=\frac{1}{4c^2}$. Then

$$\mathfrak{Re}\,(w,\frac{1}{|Q|}\int_Q f^w_Q) \ge \frac{1}{2}.$$

Prop. $\exists$ "good part" $E^*_{Q,\nu}$ of R_Q s.th.

$$(i)\qquad \mathfrak{Re}\,(w,\frac{1}{|\underline{Q}|}\int_{\underline{Q}} f^w_Q) \ge c_1 > 0$$

$$(ii)\qquad \frac{1}{|\underline{Q}|}\int_{\underline{Q}} |f^w_Q| \le c_2$$

for all dyadic subcubes $\underline{Q}$ of Q which satisfy $R_{\underline{Q}}\cap E^*_{Q,\nu}\neq\emptyset$.

Choose $\varepsilon_0 \le \frac{c_1}{2c_2}$.

Prop. If $(x,t) \in E^*_{Q,\nu}$ and $\gamma_t \in K_\nu$, then

$$\frac{|\gamma_t(x).A_t f^w_Q(x)|}{\|\gamma_t(x)\|} \ge \frac{1}{2}c_1 > 0$$

Proof. First note

$$\begin{aligned} |\nu.A_t f^w_Q(x)| &\ge \mathfrak{Re}\left(\underline{w}, \nu.A_t f^w_Q(x)\right) \\ &= \mathfrak{Re}\left(w, A_t f^w_Q(x)\right) \\ &\ge c_1 > 0 \end{aligned}$$

Therefore

$$\begin{aligned} &\left|\frac{\gamma_t(x)}{\|\gamma_t(x)\|}.A_t f^w_Q(x)\right| \\ &\quad \ge |\nu.A_t f^w_Q(x)| - \left\|\frac{\gamma_t(x)}{\|\gamma_t(x)\|} - \nu\right\|\left|A_t f^w_Q(x)\right| \\ &\quad \ge c_1 - \varepsilon_0 c_2 \ge \frac{1}{2}c_1. \end{aligned}$$

We used:

$$A_t f^w_Q(x) = \frac{1}{|\underline{Q}|}\int_{\underline{Q}} f^w_Q$$

where $x \in \underline{Q}$, $\underline{Q}$ dyadic of scale t, i.e. $(x,t) \in R_{\underline{Q}}$.

Completion of Proof

$$\iint\limits_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t(x)\in K_\nu}} \|\gamma_t(x)\|^2 \ \frac{dx\,dt}{t}$$

$$\lesssim \iint\limits_{R_Q} |\gamma_t(x).A_t f_Q^w|^2 \frac{dx\,dt}{t}$$

$$\lesssim \iint\limits_{R_Q} |\Theta_t f_Q^w - \gamma_t.A_t f_Q^w|^2 \frac{dx\,dt}{t} + \iint\limits_{R_Q} |\Theta_t f_Q^w|^2 \frac{dx\,dt}{t}$$

$$\lesssim |Q|$$

by estimating first term:

$$\Theta_t f_Q^w - \gamma_t.A_t f_Q^w$$

$$= \Big(\Theta_t - \gamma_t.A_t\Big)\Big(w_Q - \varepsilon\ell i\Gamma\Big(\frac{1}{1+\varepsilon\ell i\Pi_B}\Big)w_Q\Big)$$

$$= -\Big(\Theta_t - \gamma_t.A_t\Big)\Big(\varepsilon\ell i\Gamma\frac{1}{1+\varepsilon\ell i\Pi_B}\Big)w_Q$$

$$+\Theta_t(I-P_t)w_Q$$

$$+(\Theta_t P_t - \gamma_t.A_t)w_Q$$

as before.

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