Kato's Square Root Problem
Background and Recent Results

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This survey is based on lectures presented by the first author at the Blaubeuren Workshop on Functional Calculus, 17–21 June, 2002. We both greatly enjoyed being involved in this activity of the TULKA Internet Seminar.

Further details on holomorphic functional calculi in a Hilbert space, together with references, can be found in the survey paper by Albrecht, Duong and M$^{\mathbf{c}}$Intosh [ADM$^{\mathbf{c}}$]. For the theory of accretive sesquilinear forms and operators, see Chapter 5 of Kato's classic [K]. There is an earlier survey of the Kato square root problem for elliptic operators by M$^{\mathbf{c}}$Intosh in [M$^{\mathbf{c}}$]$_{90}$, and a great deal of information in the book by Auscher and Tchamitchian [AT]. The reader can find a complete proof of its solution in the paper by Auscher, Hofmann, Lacey, M$^{\mathbf{c}}$Intosh and Tchamitchian [AHLM$^{\mathbf{c}}$T].

We shall not attempt to give complete references for the basic material, but refer the reader to the books and papers mentioned above and to those listed at the end of this manuscript. There we have included works suitable for further reading, as well as background material and references quoted.

1 Introduction

The theory of linear partial differential operators was formulated in the 1950^s and 1960^s. In this context accretive operators, semigroups, fractional powers, interpolation and evolution equations were introduced and treated by Yosida, Phillips, Kato, Lions and many others. In the Hilbert space $L^2(\Omega)$ this theory was fairly complete. But one question remained open, now called the *Kato square root problem*.

The square root of a linear operator $L : \mathcal{X} \to \mathcal{X}$ (where $\mathcal{X}$ denotes a Banach space) is a linear operator $\sqrt{L} : \mathcal{X} \to \mathcal{X}$, which satisfies $\sqrt{L}\sqrt{L} = L$. For example, if $\mathcal{X} = \mathbb{C}^n$ and L is represented by the matrix $L = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$, then $\sqrt{L} = \begin{pmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_n} \end{pmatrix}$.

More generally, if the spectrum $\sigma(L)$ of a matrix L satisfies $\sigma(L)\cap(-\infty,0]=\emptyset$, then there exists a unique matrix $\sqrt{L}$ such that

$$(i)\ \mathfrak{Re}\,\sigma(\sqrt{L})>0 \qquad \text{and} \qquad (ii)\ \sqrt{L}\sqrt{L}=L\ ;$$

it is given by

$$\sqrt{L}=\frac{1}{2\pi i}\int_\delta (L-\zeta I)^{-1}\sqrt{\zeta}d\zeta$$

where δ is a simple closed, smooth curve surrounding $\sigma(L)$ clockwise.

However, limit conclusions have to be drawn carefully: If for example $L_n := \begin{pmatrix} \frac{1}{n} & 1 \\ 0 & \frac{1}{n} \end{pmatrix}$, then $L_n \to L := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ as $n\to\infty$, but $\sqrt{L_n} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{\sqrt{n}}{2} \\ 0 & \frac{1}{\sqrt{n}} \end{pmatrix}$ blows up, and $\sqrt{L}$ does not exist.

More generally, if a bounded operator $L:\mathcal{X}\to\mathcal{X}$ satisfies $\sigma(L)\cap(-\infty,0]=\emptyset$, then the square root is still defined by $\sqrt{L}=\frac{1}{2\pi i}\int_\delta(L-\zeta I)^{-1}\sqrt{\zeta}d\zeta:\mathcal{X}\to\mathcal{X}$. Indeed, by the Dunford functional calculus, $f(L):=\frac{1}{2\pi i}\int_\delta(L-\zeta I)^{-1}f(\zeta)d\zeta$ is defined for all f holomorphic in a neighbourhood of $\sigma(L)$, which contains such a curve δ having $\sigma(L)$ in its interior.

In the following, $\mathcal{H}$ is a Hilbert space with inner product $(.,.)$, and we consider operators $T:\mathcal{D}(T)\to\mathcal{H}$ that may not be bounded; the domain $\mathcal{D}(T)$ is always assumed to be a dense subspace of $\mathcal{H}$.

1.1 Accretive operators in a Hilbert space:

Definition: *A linear operator $T:\mathcal{D}(T)\to\mathcal{H}$ is* maximal accretive *if*

$$(i)\ \mathfrak{Re}\,(Tu,u)\geq 0 \text{ for all } u\in\mathcal{D}(T) \qquad \text{and} \qquad (ii)\ \sigma(T)\subset\{z\in\mathbb{C}:\mathfrak{Re}\,z\geq 0\}\ .$$

If so, there is no proper extension $\tilde{T}$ of T that satisfies (i). By definition, an operator T is maximal accretive if and only is its negative $-T$ is "dissipative".

Let $S_{\omega+}$ denote the closed sector $S_{\omega+}:=\{z\in\mathbb{C}:|\arg z|\leq\omega\}\cup\{0\}$.

Definition: *Given $0\leq\omega\leq\frac{\pi}{2}$, T is called* ω-accretive, *provided*

$$(i)\ (Tu,u)\in S_{\omega+} \text{ for all } u\in\mathcal{D}(T) \qquad \text{and} \qquad (ii)\ \sigma(T)\subset S_{\omega+}\ .$$

Therefore $\frac{\pi}{2}-$accretivity means maximal accretivity. T is 0-accretive if and only if T is self-adjoint and non-negative (i.e. $\{(Tu,u):u\in\mathcal{D}(t)\}\subset\mathbb{R}_{\geq 0}$).

Properties 1.1 *Let T be ω-accretive. Then*

(i) $-T$ generates a bounded (holomorphic in case $\omega<\frac{\pi}{2}$) C_0-semigroup $(e^{-tT})_{t\geq 0}$;

(ii) corresponding to each $\alpha\in(0,1]$ there exists a unique $\alpha\omega-$accretive power T^α such that the family of operators satisfies $T^{\alpha+\beta}=T^\alpha T^\beta$ and $T^1=T$;

(iii) the adjoint operator T^* is $\omega-$accretive, and the fractional powers according to (ii) satisfy $(T^*)^\alpha=(T^\alpha)^*$; in case $\alpha<\frac{1}{2}$ we have $\mathcal{D}((T^*)^\alpha)=\mathcal{D}(T^\alpha)$.

If T is ω-accretive and $\zeta \notin S_{\omega+}$, then the estimate

$$\operatorname{dist}(\zeta, S_{\omega+}) \leq \Big| \underbrace{(T\frac{u}{\|u\|}, \frac{u}{\|u\|})}_{\in S_{\omega+}} - \zeta \Big| \leq \Big\|(T-\zeta I)\frac{u}{\|u\|}\Big\|$$

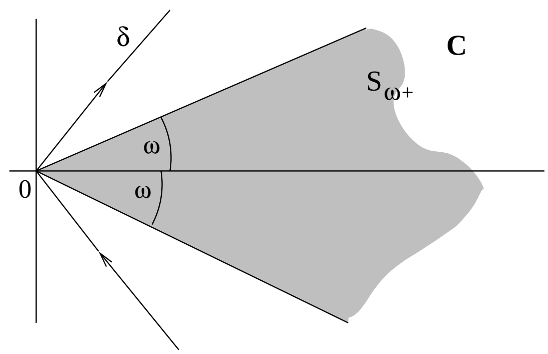
holds for all $u \in \mathcal{D}(T)$, whence

$$\|(T-\zeta I)^{-1}\| \leq \frac{1}{\operatorname{dist}(\zeta, S_{\omega+})} \ .$$

Therefore, if $0 \leq \omega < \mu < \frac{\pi}{2}$ and $\delta : \mathbb{R} \to \mathbb{C}$ with $\delta(t) = -te^{-i\mu}$ for $t < 0$ and $\delta(t) = te^{i\mu}$ for $t \geq 0$, then

$$\int_\delta \|(T-\zeta I)^{-1}\| \left|\frac{\sqrt{\zeta}}{1+\zeta}\right| |d\zeta| \leq \int_\delta \frac{1}{|\zeta| \sin(\mu-\omega)} \left|\frac{\sqrt{\zeta}}{1+\zeta}\right| |d\zeta| < \infty \ ,$$

so that $\int_\delta (T-\zeta I)^{-1} \frac{\sqrt{\zeta}}{1+\zeta} d\zeta$ is absolutely convergent in $\mathcal{L}(\mathcal{H})$.



On the space of all $u \in \mathcal{H}$ for which the value of the integral is in $\mathcal{D}(T)$ thus define

$$\sqrt{T} := (I+T)\frac{\sqrt{T}}{I+T} := (I+T)\frac{1}{2\pi i}\int_\delta (T-\zeta I)^{-1}\frac{\sqrt{\zeta}}{1+\zeta} d\zeta \ .$$

Theorem 1.2 *$T^{\frac{1}{2}} := \sqrt{T}$ is the unique $\frac{\omega}{2}-$accretive operator such that $T^{\frac{1}{2}}T^{\frac{1}{2}} = T$. (In particular $\mathcal{D}(T) = \{x \in \mathcal{D}(\sqrt{T}) : \sqrt{T}x \in \mathcal{D}(\sqrt{T})\}$.)*

1.2 Sectorial forms

Let $\mathcal{V}$ be a dense subspace of $\mathcal{H}$ and $J : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ sesquilinear, i.e. $J[u,v]$ is linear in u and conjugate linear in v. Suppose $0 \leq \omega < \frac{\pi}{2}$.

Definition: *J is called an $\omega-$*sectorial *sesquilinear form if*

(i) $J[u,u] \in S_{\omega+}$ for all $u \in \mathcal{V}$ and (ii) $\mathcal{V}$ is complete under $\|u\|_{\mathcal{V}}^2 := \|u\|^2 + \mathfrak{Re}\, J[u,u]$.

Let T be the operator with largest domain $\mathcal{D}(T) \subset \mathcal{V}$ such that

$$J[u,v] = (Tu,v) \text{ holds for all } u \in \mathcal{D}(T),\ v \in \mathcal{V} \ .$$

Theorem 1.3 (Lax-Milgram) *If J is ω-sectorial, then the associated operator T is ω-accretive.*

Theorem 1.4 *If J is 0-sectorial, then T is non-negative self-adjoint and*

$$J[u,v] = (\sqrt{T}u, \sqrt{T}v) \text{ holds for all } u,v \in \mathcal{D}(\sqrt{T}) = \mathcal{V} \ .$$

Obviously this means that $\|\sqrt{T}u\|^2 = J[u,u]$ when $\omega = 0$. Kato's (so-called) first question concerned the possibility of a generalization to positive ω:

> "REMARK 1. We do not know whether or not $\mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(A^{*\frac{1}{2}})$ *(where A is a maximal accretive operator)*. This is perhaps not true in general. But the question is open even when A is regularly accretive ($\omega < \frac{\pi}{2}$). In this case it appears reasonable to suppose that both $\mathcal{D}(A^{\frac{1}{2}})$ and $\mathcal{D}(A^{*\frac{1}{2}})$ coincide with $\mathcal{D}(H^{\frac{1}{2}}) = V_J$, where H is the real part of A and J is the regular sesquilinear form which defines A. But all that we know are $V_J \supset \mathcal{D}(A) \subset \mathcal{D}(A^{\frac{1}{2}}) \supset \mathcal{D}(P)$ *(where P is the real part of $A^{\frac{1}{2}}$)* and a similar chain of inclusions with A replaced by A^*."
>
> Tosio Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan, **13** (1961), 246-274.

For maximal accretive operators, a counterexample was given by Lions shortly after [L], so interest turned to ω-accretive operators associated with a sesquilinear form J ($0 < \omega < \frac{\pi}{2}$). In the above notation, this means: Is it always true that

$$\textbf{(K1)} \qquad \mathcal{D}(\sqrt{T}) = \mathcal{V} \ \text{ with } \ \|\sqrt{T}u\|^2 + \|u\|^2 \simeq \mathfrak{Re}\, J[u,u] + \|u\|^2 = \|u\|^2_{\mathcal{V}} \ \textbf{?}$$

('$\simeq$' means equivalence of the norms)
When (K1) holds, also the equality $\mathcal{D}(\sqrt{T^*}) = \mathcal{V}$ is true, and $J[u,v] = (\sqrt{T}u, \sqrt{T^*}v)$ for all $u,v \in \mathcal{V}$.

Now let $J_t : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ be a family of positive hermitian forms (i.e. $\omega = 0$), which extends to a family J_z that is holomorphic in $\{z : |z| < \kappa\}$. By T_z denote the associated operators. (Note that although for real t all T_t are non-negative self-adjoint by Theorem 1.4, the T_z do not have the same property.)

Suppose now that (K1) holds with uniform bounds, i.e. that $\|\sqrt{T_z}u\| \le c\|u\|_{\mathcal{V}}$ for some $c > 0$, all $|z| < \kappa$ and all $u \in \mathcal{V}$. Then, for every $|z_1| < \kappa_1 < \kappa_2 < \kappa$,

$$\left\| \frac{d}{dz}\sqrt{T_z}\Big|_{z=z_1} u \right\| = \left\| \frac{1}{2\pi i} \int\limits_{|z|=\kappa_2} \frac{1}{(z-z_1)^2}\sqrt{T_z}u\,dz \right\| \le \frac{c}{(\kappa_2-\kappa_1)^2}\|u\|_{\mathcal{V}} \ .$$

In particular, if $-\kappa < t_1 < \kappa$, then $\frac{d}{dt}\sqrt{T_t}\Big|_{t=t_1}$ is a bounded operator on $\mathcal{V}$. This motivates Kato's "second question":

> "REMARK 2. If $A = H$ is self-adjoint, the question raised above is answered in the affirmative, for we have $V_J = \mathcal{D}(H^{\frac{1}{2}})$. The question is still open, however, whether or not ["J_T is holomorphic in t" implies "A_t^s is holomorphic in t"] is true with $s = \frac{1}{2}$ when A_t are self-adjoint for real t, although it is true that $\mathcal{D}(A_t^{\frac{1}{2}})$ is independent of t as long as t is real. Thus it must be stated that

our knowledge is quite unsatisfactory regarding the case $s = \frac{1}{2}$."

Tosio Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan, **13** (1961), 246-274.

The question is, for J_t and T_t as above: Is it always true that

$$\textbf{(K2)} \qquad \frac{d}{dt}\sqrt{T_t} : \mathcal{V} \to \mathcal{H} \text{ is bounded } \textbf{?}$$

Both problems were solved negatively by M$^{\mathbf{c}}$Intosh: in 1972 by giving an example of a maximal accretive operator T for which $\mathcal{D}(\sqrt{T}) \neq \mathcal{D}(\sqrt{T^*})$, so that (K1) could not hold in general [M$^{\mathbf{c}}$]$_{72}$, and in 1982 [M$^{\mathbf{c}}$]$_{82}$ contradicting also (K2).

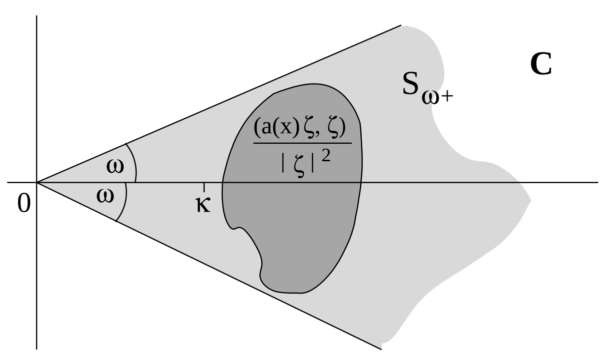
The validity of the above statements for special spaces and operators therefore has to be checked in each case. In particular this has been done extensively for the following forms, which were those with which Kato was most concerned.

1.3 Elliptic forms

Let $\Omega \subset \mathbb{R}^n$, $\mathcal{H} = L^2(\Omega)$ and $\mathcal{V} = \mathcal{W}_0^{1,2}(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_j} \in L^2(\Omega) \text{ with } u|_{b\Omega} \equiv 0\}$ (where $b\Omega$ denotes the boundary of the connected open set Ω).
Let the matrix valued function $a = (a_{j,k}) \in L^\infty(\Omega)^{n\times n}$ satisfy

$$(\diamond) \qquad \mathfrak{Re} \sum_{j,k=1}^{n} a_{j,k}(x)\zeta_k\overline{\zeta_j} \geq \kappa|\zeta|^2$$

for some $\kappa > 0$, all $x \in \Omega$ and all $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$.



Define $J : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ by

$$J[u,v] := \int_\Omega \underbrace{\sum_{j,k=1}^{n} a_{j,k}(x)\frac{\partial u}{\partial x_k}(x)\overline{\frac{\partial v}{\partial x_j}(x)}}_{=\left(a(x)\nabla u(x), \nabla v(x)\right)} dx \quad (+ \text{ lower order terms}) .$$

It can be shown that, in order to solve the Kato problem for such forms, it suffices to treat the case where no lower order terms appear. Then $\mathfrak{Re}\, J[u,u] \leq \|a(.)\|_\infty \|\nabla u\|_2^2 \leq \frac{\|a(.)\|_\infty}{\kappa} \mathfrak{Re}\, J[u,u]$ and thus

$$\|u\|_2^2 + \mathfrak{Re}\, J[u,u] \simeq \|u\|_2^2 + \|\nabla u\|_2^2 \simeq \|u\|_{\mathcal{W}^{1,2}}^2 .$$

As by assumption, $\big(a(x)\nabla u(x), \nabla u(x)\big) \in S_{\omega+}$ for some $\omega \in (0, \frac{\pi}{2})$ and all $u \in \mathcal{V},\ x \in \Omega$, it follows that J is $\omega-$sectorial. Its associated operator L is

$$(*) \qquad Lu(x) = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{j,k}(x)\frac{\partial u}{\partial x_k}(x)\Big)$$

with $\mathcal{D}(L) = \{u \in \mathcal{W}_0^{1,2}(\Omega) : Lu \in L^2(\Omega)\}$. As a consequence of condition $(\diamond)$, L is one-one.

By Theorem 1.3, L is $\omega-$accretive. Thus there exists a unique $\frac{\omega}{2}$-accretive operator $\sqrt{L}$ such that $\sqrt{L}\sqrt{L} = L$. In fact, $\mathcal{D}(\sqrt{L}) = [\mathcal{D}(L), \mathcal{H}]_{\frac{1}{2}}$.

Kato's question in this context therefore becomes

$$\text{(K1)} \qquad \text{Is } \ \mathcal{D}(\sqrt{L}) = \mathcal{W}_0^{1,2}(\Omega) \ \text{ with } \ \|\sqrt{L}u\|_2 + \|u\|_2 \simeq \|\nabla u\|_2 + \|u\|_2 \ ?$$

In the case $\Omega = \mathbb{R}^n$ there is a homogeneous version: "Is $\|\sqrt{L}u\|_2 \simeq \|\nabla u\|_2$?"

If the coefficients $a_{j,k}(x)$ and the boundary $b\Omega$ are smooth, then $\mathcal{D}(L) = (\mathcal{W}^{2,2} \cap \mathcal{W}_0^{1,2})(\Omega)$, so in fact $\mathcal{D}(\sqrt{L}) = \mathcal{W}_0^{1,2}(\Omega)$ by complex interpolation (J.L.Lions [L]).

If $a_{j,k} = \overline{a_{k,j}}$, then $\omega = 0$ and L is positive self-adjoint. Theorem 1.4 then yields $\mathcal{D}(\sqrt{L}) = \mathcal{W}_0^{1,2}(\Omega)$ with $\|\sqrt{L}u\|_2^2 = J[u,u]$, i.e. (K1) holds.

Several affirmative **results** under different assumptions have been proved:

1.) $n = 1,\ \Omega = \mathbb{R}$ (and no lower order terms), i.e. $L = -\frac{d}{dx}b(x)\frac{d}{dx}$, where $b \in L^\infty(\mathbb{R})$. Kato's question (K1) then is: "Is $\|\sqrt{L}u\|_2 \simeq \|u'\|_2$?" This is connected with the Calderón question on Cauchy integrals; see Section 3.1 (Coifman, M$^{\mathbf{c}}$Intosh and Meyer [CM$^{\mathbf{c}}$M]).

2.) $n = 1,\ \Omega \subset \mathbb{R}$ (Auscher and Tchamitchian [AT]$_{92}$)

3.) $n \geq 2,\ \Omega = \mathbb{R}^n$, $\|a_{j,k} - \delta_{j,k}\|_\infty \leq \varepsilon$ for $\delta_{j,k} :\equiv \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$ (Coifman, Deng and Meyer [CDM], Fabes, Jerison and Kenig [FJK]$_{84}$, Journé [J])

4.) $n \geq 2,\ \Omega \subset \mathbb{R}^n, \exists s > 0$ s.th. $\|a_{j,k}u\|_{H^s} \leq c\|u\|_{H^s}$ (e.g. $a_{j,k} \in \mathcal{C}^\alpha(\overline{\Omega}),\ \alpha > 0$) (M$^{\mathbf{c}}$Intosh [M$^{\mathbf{c}}$]$_{85}$)

The main achievement for a long time after was the writing of a book by Auscher and Tchamitchian [AT]$_{98}$, consilidating and extending prior results and linking them with $T(b)$-type results. This led to a number of developments in 2000:

1.) (K1) was solved in 2 dimensions (Hofmann and M$^{\mathbf{c}}$Intosh [HM$^{\mathbf{c}}$]);

2.) (K1) was solved in all dimensions for small perturbations of real symmetric operators (Auscher, Hofmann, Lewis and Tchamitchian [AHLT]), thus implying (K2).

Later in 2000, Hofmann, Lacey and M$^{\mathbf{c}}$Intosh [HLM$^{\mathbf{c}}$] solved (K1) for all operators $(*)$ which satisfy Gaussian heat kernel bounds. (This includes the 2-dimensional result, for then the heat kernel bounds always hold [AM$^{\mathbf{c}}$T]). Auscher, Hofmann, Lacey, M$^{\mathbf{c}}$Intosh and Tchamitchian [AHLM$^{\mathbf{c}}$T] then solved (K1) for all operators $(*)$ in all dimensions. The following year saw the solution of (K1) for higher order elliptic operators and systems [AHM$^{\mathbf{c}}$T].

1.4 Motivations and applications

Kato's initial motivation was the study of partial differential equations:

1. In 1960^s he studied **parabolic evolution equations** of the type

$$\begin{cases} \frac{\partial}{\partial t}u(t) + L_t u(t) &= 0 \\ u(0) &= u_0 \in \mathcal{H} \end{cases}$$

where L_t is the operator associated with the $\omega-$sectorial form $J_t : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$.

If all J_t coincide, i.e. if $J = J_t$ is independent of t, then the above equation has the solution $u(t) = e^{-tL}u(0)$ where $L = L_t$.
But if J_t depends on t, then difficulties arise, because the domains $\mathcal{D}(L_t)$ vary with t. Kato used the knowledge that, for $\alpha < \frac{1}{2}$, the domains $\mathcal{D}({L_t}^{\alpha})$ are independent of t. But the results would have been clearer and stronger if he had $\mathcal{D}({L_t}^{\frac{1}{2}}) = \mathcal{V}$ for all $t \geq 0$.
In the case of elliptic operators $L_t = -\sum\limits_{j,k=1}^{n} \frac{\partial}{\partial x_j}(a_{j,k,t}\frac{\partial}{\partial x_k})$ in $\mathcal{H} = L^2(\mathbb{R}^n)$ this is now known.

2. The square root of a positive self-adjoint elliptic operator $L = -\sum\limits_{j,k=1}^{n} \frac{\partial}{\partial x_j}(a_{j,k}\frac{\partial}{\partial x_k})$ plays an important role also in the **hyperbolic wave equation**

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(.,t) + Lu(.,t) &= 0\ , \quad t > 0 \\ u(.,0) &\equiv 0 \\ \frac{\partial}{\partial t}u(.,0) &= g(.) \in L^2(\mathbb{R}^n)\ , \end{cases}$$

for its solution is given by

$$u(.,t) = \frac{1}{2}\Big(e^{it\sqrt{L}}f(.) - e^{-it\sqrt{L}}f(.)\Big)$$

where $i\sqrt{L}f = g \quad \big($or $f = -iL^{-\frac{1}{2}}g \in \dot{\mathcal{W}}^{1,2}(\mathbb{R}^n)$, the completion of $\mathcal{W}^{1,2}(\mathbb{R}^n)$ under the norm $\|\nabla u\|$; note that $\sqrt{L}$ extends to an isomorphism from $\dot{\mathcal{W}}^{1,2}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, see the remark at the beginning of Section 2.2.$\big)$.
As was seen above, (K1) holds in this case. Moreover $\sqrt{L}$ is 0-accretive, thus $\pm i\sqrt{L}$ generates a contraction semigroup, i.e. $\|e^{\pm it\sqrt{L}}v\|_2 \leq \|v\|_2$ for all $v \in L^2(\mathbb{R}^n)$ and $t \geq 0$. Hence for all $t \geq 0$

$$\|\nabla_x u(.,t)\|_2 \simeq \|\sqrt{L}u(.,t)\|_2 \leq \frac{1}{2}\Big(\|e^{it\sqrt{L}}\sqrt{L}f\|_2 + \|e^{-it\sqrt{L}}\sqrt{L}f\|_2\Big) \leq \|\sqrt{L}f\|_2 = \|g\|_2\ .$$

This is called an "energy estimate".

More generally, in the **hyperbolic evolution equation**, the 0-sectorial elliptic forms $J_t : \mathcal{W}^{1,2}(\mathbb{R}^n) \times \mathcal{W}^{1,2}(\mathbb{R}^n) \to \mathbb{C}$ depend on t: Let L_t denote the associated positive self-adjoint operators and let the mapping $t \mapsto J_t$ be smooth. In the 1970^s Kato considered the problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(.,t) + L_t u(.,t) = g(.,t)\ , \qquad t > 0 \\ \qquad + \text{ initial conditions} \end{cases}$$

by reducing it to a first order evolution equation: Let

$$v(.,t) := \frac{\partial}{\partial t}u(.,t) \qquad \text{and} \qquad w(.,t) := \sqrt{L_t}u(.,t)\ .$$

The above problem then is to solve

$$\begin{cases} \frac{\partial}{\partial t}v(.,t) &=& -\sqrt{L_t}w(.,t) + g(.,t) \qquad \text{and} \\ \frac{\partial}{\partial t}w(.,t) &=& \left(\frac{d}{dt}\sqrt{L_t}\right)\frac{1}{\sqrt{L_t}}w(.,t) + \sqrt{L_t}v(.,t) \end{cases}$$

which means

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ w \end{bmatrix} = \Bigg(\underbrace{\begin{bmatrix} 0 & -\sqrt{L_t} \\ \sqrt{L_t} & 0 \end{bmatrix}}_{=:S_t} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{d}{dt}\sqrt{L_t}\right)\frac{1}{\sqrt{L_t}} \end{bmatrix}}_{=:B_t}\Bigg)\begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} g \\ 0 \end{bmatrix}$$

under initial conditions.
For t fixed, the operator S_t is skew-adjoint, and if B_t is bounded on $L^2(\mathbb{R}^n)$ then $S_t + B_t + \|B_t\|$ is maximal accretive (and hence the generator of a C_0-semigroup). It follows (see [K]$_{53}$) that the above problem in this case is uniquely solvable. It therefore remains to show that B_t is bounded, i.e. that for $|t| \leq \kappa$,

$$\|\left(\frac{d}{dt}\sqrt{L_t}\right)\frac{1}{\sqrt{L_t}}f\|_2 \ \leq\ c\|f\|_2 \quad \text{for } f \in L^2(\mathbb{R}^n) \text{ or equivalently,}$$

that
$$\|\frac{d}{dt}\sqrt{L_t}w\|_2 \ \leq\ c\|\sqrt{L_t}w\|_2 \simeq \|w\|_{\dot{\mathcal{W}}^{1,2}(\mathbb{R}^n)} \quad \text{for } w \in \dot{\mathcal{W}}^{1,2}(\mathbb{R}^n);$$

this is a homogeneous version of (K2). See [M$^{\mathbf{c}}$]$_{84}$ for details.
Today it is known that the desired estimate in fact holds true, by which Kato's proof can be completed.

3. On $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$ consider the **elliptic equation**

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(.,t) - Lu(.,t) &=& 0 \\ u(.,0) &=& u_0(.) \in \mathcal{D}(\sqrt{L}) \end{cases}$$

where still $L = -\sum\limits_{j,k=1}^{n}\frac{\partial}{\partial x_j}(a_{j,k}\frac{\partial}{\partial x_k})$. Then $u(t,x) := e^{-t\sqrt{L}}u_0(x)$ is the solution of the equation satisfying the Neumann boundary condition

$$\frac{\partial u}{\partial t}\Big|_{t=0} = -\sqrt{L}u_0$$

Therefore the homogeneous version of (K1), i.e. the equality $\mathcal{D}(\sqrt{L}) = \mathcal{W}^{1,2}_0(\mathbb{R}^n)$ with $\|\sqrt{L}u(.,0)\|_2 \simeq \|\nabla_x u(.,0)\|_2$, is equivalent to

$$\|\frac{\partial u}{\partial t}\big|_{t=0}\|_2 \simeq \|\nabla_x u_0\|_2\ .$$

This alternative form of Kato's question is also known as a "Rellich inequality" or a "Dirichlet-Neumann inequality".

2 Operators that have a bounded holomorphic functional calculus

As a tool for solving (K1) for elliptic forms in the case $n = 1$, a holomorphic functional calculus can be applied; see [ADM$^{\mathbf{c}}$] and [AM$^{\mathbf{c}}$N] for a more detailed presentation of the material in this section, and for further references.

2.1 Operators of type ω

Let $\mathcal{X}$ be a Banach space, let $0 \leq \omega < \mu < \pi$ and assume $T : \mathcal{D}(T) \to \mathcal{X}$ is a one-one operator with dense domain $\mathcal{D}(T) \subset \mathcal{X}$ and dense range.

Definition: *T is said to be "of type ω" if*

(i) $\sigma(T) \subset S_{\omega+}$ *and*

(ii) $\forall \mu > \omega \; \exists c_\mu > 0$ *s.th.* $\|(T - \zeta I)^{-1}\| \leq \frac{c_\mu}{|\zeta|}$ *holds for each* $\zeta \notin S_{\mu+}$.

Let $S^\circ_{\mu+}$ denote the topological interior of $S_{\mu+}$ and define

$$\Psi(S^\circ_{\mu+}) := \left\{\psi \in H^\infty(S^\circ_{\mu+}) : \; \exists c, s > 0 \textit{ s.th. } |\psi(\zeta)| \leq \min\{c|\zeta|^s, \; c|\zeta|^{-s}\} \text{ for all } \zeta \in S^\circ_{\mu+}\right\} .$$

For T of type ω and $\psi \in \Psi(S^\circ_{\mu+})$ then define

$$\psi(T) := \frac{1}{2\pi i} \int\limits_\gamma \psi(\zeta)(\zeta I - T)^{-1} d\zeta \in \mathcal{L}(\mathcal{X}) ,$$

where the contour γ is defined for some $\nu \in (\omega, \mu)$ by $\gamma(t) = -te^{i\nu}$ if $t \leq 0$ and $\gamma(t) = te^{-i\nu}$ if $t \geq 0$.

Definition: *T* "has a bounded $H^\infty(S^\circ_{\mu+})$ functional calculus" *means that there exists $c_\mu > 0$ such that*

$$\|\psi(T)\| \leq c_\mu \|\psi\|_\infty \textit{ holds for all } \psi \in \Psi(S^\circ_{\mu+}) .$$

For all $f \in H^\infty(S^\circ_{\mu+})$ one can then define $f(T) \in \mathcal{L}(\mathcal{X})$ such that $\|f(T)\| \leq c_\mu \|f\|_\infty$, see [ADM$^{\mathbf{c}}$].

2.2 Quadratic estimates

Now let $\mathcal{X} = \mathcal{H}$ be a Hilbert space and suppose that T is an injective operator of type $\omega < \mu$ in $\mathcal{H}$ (note that in the Hilbert space case every such operator necessarily has dense domain and dense range [CDM$^{\mathbf{c}}$Y]). Given $\psi \in \Psi(S^\circ_{\mu+}) \backslash \{0\}$, let $\psi_t(\zeta) := \psi(t\zeta)$ and define

$$\|u\|_{T,\psi} := \Big(\int\limits_0^\infty \big\|\psi_t(T)u\big\|^2 \frac{dt}{t} \Big)^{\frac{1}{2}}$$

on the space $\mathcal{H}^0_{T,\psi}$ of all $u \in \mathcal{H}$ for which the integral is finite. $\|u\|_{T,\psi}$ is called the quadratic norm associated with T and ψ. This construction is a useful tool in relation to bounded H^∞ functional calculi.

Theorem 2.1 *a) The spaces $\mathcal{H}^0_{T,\psi}$ are independent of ψ and $\mu > \omega$, and $\|u\|_{T,\psi} \simeq \|u\|_{T,\tilde{\psi}}$ holds for all $\psi, \tilde{\psi} \in \Psi(S^\circ_{\mu+})\backslash\{0\}$. Henceforth we write $\mathcal{H}^0_T$ and $\|.\|_T$ in place of any one of these equivalent norms.*

b) There exists a constant $c > 0$ such that $\|f(T)u\|_T \leq c\|f\|_\infty \|u\|_T$ holds for all $f \in H^\infty(S^\circ_{\mu+})$ and every $u \in \mathcal{H}^0_T \cap \mathcal{D}(f(T))$.

Define the Hilbert space $\mathcal{H}_T$ to be the completion of $\mathcal{H}^0_T$ under the norm $\|.\|_T$.

Theorem 2.2 *Let $\mu > \nu > \omega \geq 0$. Let T be a one-one operator operator of type ω in $\mathcal{H}$. Then the following assertions are equivalent:*

(i) T has a bounded $H^\infty(S^\circ_{\mu+})$–functional calculus.

(ii) $\mathcal{H} \subset (\mathcal{H}_T \cap \mathcal{H}_{T^*})$ with $\|u\|_T \leq c\|u\|$ (in this case we say that "T satisfies a quadratic estimate") as well as $\|u\|_{T^*} \leq c\|u\|$ for some $c > 0$ and each $u \in \mathcal{H}$.

(iii) $\mathcal{H} = \mathcal{H}_T$ with $\|u\| \simeq \|u\|_T$.

(iv) T has a bounded $H^\infty(S^\circ_{\nu+})$–functional calculus.

Note that it is not true in all Banach spaces that one can reduce the angle; in $L^p(\mathbb{R})$ ($1 < p < \infty$, $p \neq 2$) this is still an open question.

It has been shown in Section 1.1 that every ω-accretive operator T is of type ω.

In case $\omega = 0$, T is non-negative self-adjoint and has a bounded Borel functional calculus, so it has a bounded $H^\infty(S^\circ_{\varepsilon+})$–functional calculus for each $\varepsilon > 0$, and $\|f(T)\| \leq 1 \cdot \|f\|_\infty$. If $\omega = \frac{\pi}{2}$, there is a similar result:

Theorem 2.3 *Let T be maximal accretive. Then T has a bounded $H^\infty(S_{(\frac{\pi}{2}+\varepsilon)+})$ functional calculus for all $\varepsilon > 0$, and $\|f(T)\| \leq \|f\|_\infty$ for all $f \in H^\infty(S^\circ_{(\frac{\pi}{2}+\varepsilon)+})$.*

This can be proved with von Neumann's Theorem and the transform $V := \frac{I-T}{I+T} \in \mathcal{L}(\mathcal{H})$, as

$$\|u\|^2 - \|Vu\|^2 = \mathfrak{Re}\left((I-V)u, (I+V)u\right) = \mathfrak{Re}\left(T(I+V)u, (I+V)u\right) \geq 0$$

and thus $\|V\| \leq 1$; see [ADM$^{\mathbf{c}}$] for an alternative (direct) proof.

Theorems 2.2 and 2.3 now yield

Corollary 2.4 *If T is ω-accretive and $0 \leq \omega < \nu \leq \frac{\pi}{2}$, then T has a bounded $H^\infty(S^\circ_{\nu+})$ functional calculus.*

A new result has been obtained by Crouzeix and Delyon [CD]: They showed that for all $\mu \in (\omega, \frac{\pi}{2})$ and all $f \in H^\infty(S^\circ_{\mu+})$ one has the bound

$$\|f(T)\| \leq \min\left\{\frac{\pi - \omega}{\omega},\ 2 + \frac{2}{\sqrt{3}}\right\}\|f\|_\infty\ .$$

This generalizes the result $\|f(T)\| \leq \|f\|_\infty$ for maximal accretive operators ($\omega = \frac{\pi}{2}$).

Example 2.5 *Let L be an elliptic operator of the form $L = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\big(a_{j,k}\frac{\partial}{\partial x_k}\big)$, where the matrix-valued function $a = (a_{j,k})$ satisfies condition $(\diamond)$ in Section 1.3. Then*

$$\Big(\int_0^\infty \big\|\sqrt{tL}e^{-tL}u\big\|^2 \frac{dt}{t}\Big)^{1/2} \simeq \|u\|_2$$

holds for all $u \in L^2(\Omega)$: Indeed, as L is $\omega-$accretive for some $\omega \in (0, \frac{\pi}{2})$, it has a bounded $H^\infty(S^\circ_{\mu+})$ functional calculus ($\mu > \omega$), so Theorem 2.2 with the choice $\psi(\zeta) = \sqrt{\zeta}e^{-\zeta}$ yields the equivalence.

2.3 Operators of double type ω

Let $S_{\omega-} := -S_{\omega+}$ and $S^\circ_{\omega-} := -S^\circ_{\omega+}$. Define the open and closed double sectors $S_\omega := S_{\omega+} \cup S_{\omega-}$ and $S^0_\omega := S^\circ_{\omega+} \cup S^\circ_{\omega-}$, and the function spaces $\Psi(S^0_\omega) \subset H^\infty(S^0_\omega)$ on them exactly as before.

Definition: *The operator T is said to be "of double type ω", if*

(i) $\sigma(T) \subset S_\omega$ *and*

(ii) $\forall \mu > \omega\ \exists c_\mu > 0$ *s.th.* $\|(T - \zeta I)^{-1}\| \leq \frac{c_\mu}{|\zeta|}$ *holds for each* $\zeta \notin S_\mu$.

Most of the results for injective operators of (simple) type ω generalize directly to the case when T is a one-one operator of double type ω. The main result is, that T has an $H^\infty(S^0_\mu)-$functional calculus if and only if $\mathcal{H}_T = \mathcal{H}$ with equivalence of norms. See $[\mathrm{AM^cN}]^1_{97}$ for details of this and for what is to follow.

Theorem 2.6 *Let $0 \leq \omega < \mu < \frac{\pi}{2}$, let S be a one-one self-adjoint operator in $\mathcal{H}$ and let B be bounded, invertible and ω-accretive. Define $T := BS$ and $L := SBS$. Then*

a) T is a one-one operator of double type ω with $\mathcal{D}(T) = \mathcal{D}(S)$, and T^2 is a one-one operator of type 2ω with $\mathcal{D}(T^2) = \mathcal{D}(L)$;

b) L is $\omega-$accretive and associated with the form $J[u,v] = (BSu, Sv)$ on $\mathcal{V} \times \mathcal{V}$ where $\mathcal{V} = \mathcal{D}(S)$. Thus $\mathcal{D}(\sqrt{L}) = \big[\mathcal{H}, \mathcal{D}(L)\big]_{\frac{1}{2}} = \big[\mathcal{H}, \mathcal{D}(T^2)\big]_{\frac{1}{2}}$;

c) if T has a bounded $H^\infty(S^0_\mu)$ functional calculus, then

1.) T^2 has a bounded $H^\infty(S^0_{2\mu+})$ functional calculus;

2.) $\operatorname{sgn}(T) = \frac{\sqrt{T^2}}{T} = \frac{T}{\sqrt{T^2}}$ is bounded, where the function $\operatorname{sgn} \in H^\infty(S^0_{\frac{\pi}{2}})$ is defined by

$$\operatorname{sgn}(\zeta) = \left\{\begin{array}{rl} 1 & \text{if } \mathfrak{Re}\,\zeta > 0 \\ -1 & \text{if } \mathfrak{Re}\,\zeta < 0 \end{array}\right\} = \frac{\sqrt{\zeta^2}}{\zeta} = \frac{\zeta}{\sqrt{\zeta^2}},$$ *and the equality of the domains $\mathcal{D}(T) = \mathcal{D}(\sqrt{T^2}) = \big[\mathcal{H}, \mathcal{D}(T^2)\big]_{\frac{1}{2}}$ holds with $\|Tu\| \simeq \|\sqrt{T^2}u\|$, and thus*

3.) $\mathcal{D}(\sqrt{L}) = \big[\mathcal{H}, \mathcal{D}(T^2)\big]_{\frac{1}{2}} = \mathcal{D}(T) = \mathcal{D}(S) = \mathcal{V}$ with

$$\|\sqrt{L}u\| \simeq \|Tu\| \simeq \|Su\| = \|u\|_{\mathcal{V}}$$

so that (K1) holds for L.

Concerning b), we need the equivalence $\|u\| \simeq \sqrt{\mathfrak{Re}\,(Bu,u)}$. Indeed, let $A = \frac{1}{2}(B + B^*)$ denote the self-adjoint part of B. For each $u \in \mathcal{H}$ then $\mathfrak{Re}\,(Bu,u) = (Au,u) = \|\sqrt{A}u\|^2$, and thus, by B's ω-accretivity,

$$|(Bu,u)| \le \frac{1}{\cos\omega}\mathfrak{Re}\,(Bu,u) = c\,\|\sqrt{A}u\|^2 \ .$$

It follows that $|(Bv,u)| \le c\,\|\sqrt{A}v\|\|\sqrt{A}u\|$ holds for all $u, v \in \mathcal{H}$, and the particular choice $u = Bv$ yields

$$\begin{aligned} \|u\|^2 &\le c\,\|\sqrt{A}B^{-1}u\|\|\sqrt{A}u\| \;=\; c\,|(AB^{-1}u, B^{-1}u)|^{\frac{1}{2}}\|\sqrt{A}u\| \\ &\le \tilde{c}\,\|u\|\,\|\sqrt{A}u\| \qquad \text{whence} \\ \frac{1}{\tilde{c}}\|u\| &\le \|\sqrt{A}u\| \;=\; \sqrt{\mathfrak{Re}\,(Bu,u)} \;\le\; \sqrt{\|B\|\|u\|} \ . \end{aligned}$$

We conclude that, if $T = BS$ has a bounded H^∞ functional calculus, then $L = SBS$ satisfies (K1); the converse implication holds true as well, see $[\mathrm{AM^cN}]^1_{97}$.

3 Kato's questions for elliptic forms

3.1 The one-dimensional problem

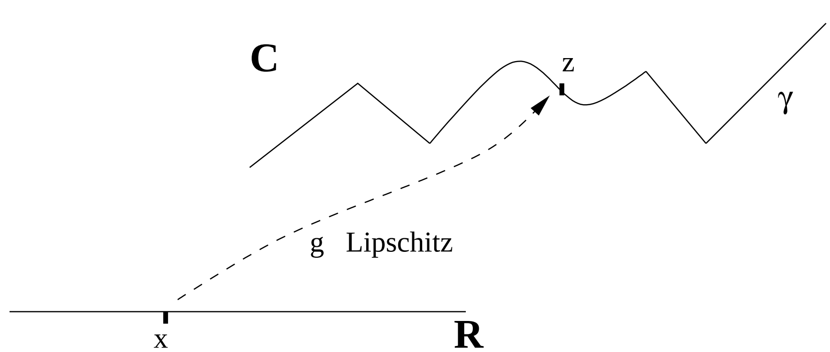
A motivating example for the above results is the Kato problem in case of elliptic forms in dimension $n = 1$:

Let $\mathcal{H} = L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$, $S := D = \frac{1}{i}\frac{d}{dx}$ with $\mathcal{D}(D) = H^1(\mathbb{R}) = \mathcal{W}^{1,2}(\mathbb{R})$. Let $B : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with $Bu = bu$, where $b, \frac{1}{b} \in L^\infty(\mathbb{R})$ and $b(x) \in S_{\omega+}$ for almost all $x \in \mathbb{R}$.
Then $(Tu)(x) = \frac{1}{i}b(x)\frac{d}{dx}u(x)$, and $(Lu)(x) = -\frac{d}{dx}b(x)\frac{d}{dx}u(x)$ is a one-dimensional elliptic operator.
As D is one-one and selfadjoint and as the operator of multiplication by b is a bounded invertible ω-accretive operator in $L^2(\mathbb{R})$, by Theorem 2.6 a) the operator T is one-one of double type ω in $L^2(\mathbb{R})$. So once we show that T and T^* satisfy quadratic estimates, then we have by Theorem 2.6 b) that L satisfies $(K1)$. To show that this in fact is true, much deeper harmonic analysis is required. See $[\mathrm{ADM^c}]$ for an indication of how this can be achieved by applying a $T(b)$-type result of Semmes [S].

At the same time this gives an affirmative answer to the following "Calderón question":

Let $g : \mathbb{R} \to \mathbb{C}$ be a Lipschitz continuous function such that $b := \frac{1}{g'}$ has the properties as above. Let $\gamma := \{g(x) : x \in \mathbb{R}\}$ denote the corresponding curve and

$$L^2(\gamma) = \Big\{V : \gamma \to \mathbb{C} :\ \|V\|_2 := \Big(\int_\gamma |V(z)|^2|dz|\Big)^{\frac{1}{2}} < \infty\Big\}$$



For $U \in L^2(\gamma)$ define

$$C_\gamma U(z) := \frac{1}{\pi i}\ \text{p.v.}\int_\gamma \frac{1}{z-\zeta}U(\zeta)d\zeta = \frac{1}{\pi i}\lim_{\varepsilon\to 0+}\int_{\{\zeta\in\gamma:|\zeta-z|>\varepsilon\}} \frac{1}{z-\zeta}U(\zeta)d\zeta\ .$$

Calderón's question, also dating back to the 1960's, was

$$\textbf{(Cd)}\qquad \text{Is } C_\gamma : L^2(\gamma)\to L^2(\gamma) \text{ a bounded operator } \textbf{?}$$

(In case of $\|b(x)-1\|_\infty$ sufficiently small this was confirmed by Calderón himself [C].)

On the other hand, for $U \in L^2(\gamma)$ define

$$D_\gamma U(z) := -i \lim_{\substack{h\to 0+\\ z+h\in\gamma}} \frac{U(z+h)-U(z)}{h}$$

whenever the limit exists. Then, if $D_\gamma U \in L^2(\gamma)$, for $u(x) := U(g(x)) \in L^2(\mathbb{R})$ we have $(Du)(x) = \frac{1}{i}\frac{d}{dx}u(x) = \big(D_\gamma U\big)(g(x))g'(x),$ i.e.

$$\big(D_\gamma U\big)(g(x)) = \frac{1}{g'(x)}(Du)(x) = b(x)(Du)(x) = (Tu)(x)\ .$$

Thus D_γ is an injective operator of double type ω in $L^2(\gamma)$, and once we have shown that T has a bounded functional calculus, then D_γ has the same property. Hence in this case, $\text{sgn}(D_\gamma)$ is a bounded operator on $L^2(\gamma)$. But one can show that $\text{sgn}(D_\gamma) = C_\gamma$, so that Calderón's question **(Cd)** is also a consequence of the fact that T has a bounded H^∞ calculus.

We remark that both the one-dimensional Kato question and the Calderón question were initially answered by Coifman, M$^{\mathbf{c}}$Intosh and Meyer [CM$^{\mathbf{c}}$M] using multilinear expansions and quadratic estimates, before the development of the H^∞ functional calculus or of $T(b)$ theorems. Indeed these general methods were in part an outgrowth. However it is worth noting that an understanding of the inter-relationship between the questions of Kato and Calderón was a key stimulus in the solution of both.

3.2 The n-dimensional Kato problem

Let $n \geq 2$ and $\mathcal{H} = L^2(\mathbb{R}^n)$. As in Section 1.3, define the form

$$J[u,v] = \int_{\mathbb{R}^n}\sum_{j,k=1}^n a_{j,k}(x)\frac{\partial u}{\partial x_k}(x)\overline{\frac{\partial v}{\partial x_j}(x)}dx$$

with associated ω-accretive operator $Lu(x) = -\operatorname{div}(a\nabla u)(x) = -\sum\limits_{j,k=1}^n \frac{\partial}{\partial x_j}\big(a_{j,k}\frac{\partial u}{\partial x_k}\big)(x),$ where the matrix-valued function $a(x) = (a_{j,k}(x)) \in L^\infty(\mathbb{R}^n)^{n\times n}$ has the property $(\diamond)$ described in section 1.3.

For this operator L, Auscher, Hofmann, Lacey, M$^{\mathbf{c}}$Intosh and Tchamitchian [AHLM$^{\mathbf{c}}$T] proved (K1):

Theorem 3.1 $\quad \mathcal{D}(\sqrt{L}) = \mathcal{V}$ *with* $\quad \|\sqrt{L}u\|_2 \simeq \|\nabla u\|_2\ .$

In an earlier work Hofmann, Lacey and M^{c}Intosh [HLM^{c}] had derived this result under the additional assumption of heat kernel bounds (G):

(G) $$e^{-t^2L}u(x) = \int_{\mathbb{R}^n} k_{t^2}(x,y)u(y)dy \text{ for all } u \in L^2(\mathbb{R}^n),$$

where the heat kernel $k_{t^2}(x,y)$ satisfies the "Gaussian properties"

1.) $|k_{t^2}(x,y)| \leq \frac{\beta}{t^n} e^{-\frac{|x-y|^2}{\alpha t^2}}$ for some $\alpha, \beta > 0$ and

2.) $\big|k_{t^2}(x+h,y) - k_{t^2}(x,y)\big| + \big|k_{t^2}(x,y+h) - k_{t^2}(x,y)\big| \leq \beta \frac{|h|^\alpha}{t^{\alpha+n}} e^{-\frac{|x-y|^2}{\alpha t^2}}$.

What follows is an outline of the proof under this additional assumption.

As L^* is of the same form, only the direction $\|\sqrt{L}u\|_2 \lesssim \|\nabla u\|_2$ has to be shown. (Note that $\mathcal{V} \subset \mathcal{D}(\sqrt{L}^*)$ with $\frac{\|\sqrt{L}^* u\|_2}{\|\nabla u\|_2} \leq c$ implies that $\|\nabla u\|_2 \leq \frac{|J[u,u]|}{\kappa \|\nabla u\|_2} \leq \frac{c}{\kappa} \frac{|(\sqrt{L}u, \sqrt{L}^* u)|}{\|\sqrt{L}^* u\|_2} \leq \frac{c}{\kappa} \|\sqrt{L}u\|_2$ holds for all u in the domain of L. Since $\mathcal{D}(L)$ is a core for $\mathcal{D}(\sqrt{L})$, the completeness of $\mathcal{V}$ yields the inequality for all $u \in \mathcal{D}(\sqrt{L})$.)

The estimate $\|\sqrt{L}u\|_2 \lesssim \|\nabla u\|_2$ is proved in several steps, developing the implication chain

$$(\mathrm{K1}) \Leftarrow (\mathrm{Q}) \Leftarrow (\mathrm{C}) \Leftarrow (\mathrm{C}_\varepsilon) \Leftarrow (\mathrm{C}'_\varepsilon)$$

and finally showing the validity of $(\mathrm{C}'_\varepsilon)$. The above abbreviations mean:

(Q) $$\int_0^\infty \int_{\mathbb{R}^n} |tLe^{-t^2L}u(x)|^2 \frac{dx\,dt}{t} \lesssim \|\nabla u\|_2^2 \ .$$

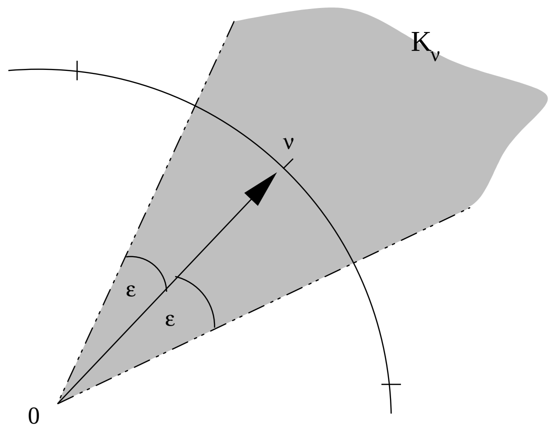
(C) There exists $c > 0$ such that $$\iint_{R_Q} |\gamma_t(x)|^2 \frac{d(x,t)}{t} = \int_0^{\ell(Q)} \int_Q |tLe^{-t^2L}\varphi(x)|^2 \frac{dx\,dt}{t} \leq c\,|Q|$$
holds for all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes, where
$\ell(Q) :=$ side-length of Q,
$R_Q := Q \times [0, \ell(Q)]$,
$|Q| :=$ volume of Q,
$\varphi : \mathbb{R}^n \to \mathbb{R}^n,\ \varphi(x) := (\varphi_1(x), \ldots, \varphi_n(x)) := x$ and $\gamma_t : \mathbb{R}^n \to \mathbb{C}^n$ for $t > 0$ is defined by $\big(\gamma_t(x)\big)_j := \big(tLe^{-t^2L}\varphi(x)\big)_j := tL \int_{\mathbb{R}^n} k_{t^2}(x,y)\varphi_j(y)dy$;
from condition (G) one then can conclude that $M := \sup\{\|\gamma_t\|_\infty : t > 0\} < \infty$.

($\mathbf{C}_\varepsilon$) There exists $c>0$ such that $\displaystyle\iint\limits_{\substack{(x,t)\in R_Q\\ \gamma_t(x)\in K_\nu}} |\gamma_t(x)|^2\,\frac{d(x,t)}{t}\le c|Q|$

holds for a small $\varepsilon>0$ to be determined later, all cubes $Q\subset\mathbb{R}^n$ as above and all unit vectors $\nu\in\mathbb{C}^n$ with corresponding cone $K_\nu:=\{z\in\mathbb{C}^n : |\frac{z}{|z|}-\nu|<\varepsilon\}$.

($\mathbf{C}'_\varepsilon$) There exists $c>0$ and $\eta\in(0,1)$ such that for all cubes $Q\subset\mathbb{R}^n$ and all unit vectors ν there exist some disjoint dyadic subcubes Q_n such that $|\bigcup\limits_n Q_n|<\eta|Q|$ and

$E_Q:=R_Q\backslash\bigcup\limits_n R_{Q_n}$ is a "good" set in the sense that $\displaystyle\iint\limits_{\substack{(x,t)\in E_Q\\ \gamma_t(x)\in K_\nu}} |\gamma_t(x)|^2\,\frac{d(x,t)}{t}\le c\,|Q|$.



(i) The implication "(Q)⇒(K1)" holds because

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^n}\big|tLe^{-t^2L}u(x)\big|^2\frac{dx\,dt}{t} &= \int_0^\infty \|t\sqrt{L}e^{-t^2L}(\sqrt{L}u)\|_2^2\frac{dt}{t}\\ &= \frac12\int_0^\infty \|\sqrt{\tau L}e^{-\tau L}(\sqrt{L}u)\|_2^2\frac{d\tau}{\tau}\\ &\simeq \|\sqrt{L}u\|_2^2 \quad \text{(by Example 2.5).}\end{aligned}$$

(ii) In order to show "(C)⇒(Q)", methods of harmonic analysis are used. Let $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ with $\int\phi=1$ and define $\phi_t(x):=\frac{1}{t^n}\phi(\frac{x}{t})$ for positive t. The operator P_t is then defined by

$$P_tv(x):=\frac{1}{t^n}\int_{\mathbb{R}^n}\phi(\frac{x-y}{t})v(y)dy=(\phi_t*v)(x)\ .$$

As by assumption (C) $d\mu:=|\gamma_t(x)|^2\frac{d(x,t)}{t}$ defines a Carleson measure μ on $\mathbb{R}^n\times\mathbb{R}_+$, Carleson's Theorem gives

$$\begin{aligned}\int\limits_0^\infty\int\limits_{\mathbb{R}^n}\big|\gamma_t(x)\big|^2\big|P_t\nabla u\big|^2\frac{dx\,dt}{t} &= \int\limits_{\mathbb{R}^n\times\mathbb{R}_+}\big|P_t\nabla u\big|^2 d\mu(t,x)\\ &= \int\limits_{\mathbb{R}^n\times\mathbb{R}_+}\big|(\phi_t*\nabla u)(x)\big|^2 d\mu(x,t)\lesssim\|\nabla u\|_2^2\ .\end{aligned}$$

On the other hand it can be shown that the functions b_t, defined by

$b_t(x) := (tLe^{-t^2L}u)(x) - [\gamma_t(x).(P_t\nabla u)(x)]$ (where $v.w$ denotes the dot product in $\mathbb{C}^n$)

satisfy

$$\int_0^\infty \int_{\mathbb{R}^n} |b_t(x)|^2 \frac{dx\,dt}{t} \lesssim \|\nabla u\|_2^2 \ .$$

Hence

$$\begin{aligned}
\Big(\int_0^\infty \int_{\mathbb{R}^n} \big|tLe^{-t^2L}u(x)\big|^2 \frac{dx\,dt}{t}\Big)^{\frac{1}{2}} &= \Big(\int_0^\infty \int_{\mathbb{R}^n} \big|b_t(x) + [\gamma_t(x).(P_t\nabla u)(x)]\big|^2 \frac{dx\,dt}{t}\Big)^{\frac{1}{2}} \\
&\le \Big(\int_0^\infty \int_{\mathbb{R}^n} \big|b_t(x)\big|^2 \frac{dx\,dt}{t}\Big)^{\frac{1}{2}} + \Big(\int_0^\infty \int_{\mathbb{R}^n} \big|\gamma_t(x)\big|^2 \big|P_t\nabla u(x)\big|^2 \frac{dx\,dt}{t}\Big)^{\frac{1}{2}} \\
&\lesssim \|\nabla u\|_2 \ .
\end{aligned}$$

(iii) It is clear that (C) is implied by (C_ε).

(iv) If (C'_ε) holds, then repeating the argument one gets that

$$\begin{aligned}
\iint\limits_{\substack{(x,t)\in R_Q \\ \gamma_t \in K_\nu}} .. &= \iint\limits_{\substack{(x,t)\in E_Q \\ \gamma_t \in K_\nu}} .. + \sum_n \iint\limits_{\substack{(x,t)\in R_{Q_n} \\ \gamma_t \in K_\nu}} .. \\
&\le c|Q| + \sum_n \Big(\iint\limits_{\substack{(x,t)\in E_{Q_n} \\ \gamma_t \in K_\nu}} .. + \sum_j \iint .. \Big) \\
&\le c|Q| + \sum_n \Big(c|Q_n| + \sum_j \Big(c|Q_{n,j}| + \sum_k \iint .. \Big)\Big) \\
&\le c|Q|\big(1 + \eta + \eta^2 + \ldots\big) = \frac{c}{1-\eta}|Q| \ , \qquad \text{i.e. } (\mathrm{C}_\varepsilon) \text{ holds.}
\end{aligned}$$

The aim now is to prove (C'_ε) for some positive ε (that will be determined later). Let $F^{(\varepsilon)} : \mathbb{R}^n \to \mathbb{C}^n$ be

$$F^{(\varepsilon)}(x) := (e^{-\varepsilon^2 L}\varphi)(x) = \big((e^{-\varepsilon^2 L}\varphi_1)(x), \ldots, (e^{-\varepsilon^2 L}\varphi_n)(x)\big)^T \ ,$$

which can be defined by the heat kernel, see (C). Without loss of generality Q can be assumed to have side-length $\ell(Q) = 1$. The following will be used:

Properties 3.2 *There exists a constant c independent of ε such that*

(α) $\int_{5Q} |\nabla F^{(\varepsilon)}|^2 \le c$

(β) $\int_{5Q} |LF^{(\varepsilon)}|^2 \le \frac{c}{\varepsilon^2}$

(γ) $\|F^{(\varepsilon)} - \varphi\|_\infty = \|(e^{-\varepsilon^2 L} - I)\varphi\|_\infty \le c\varepsilon$ *and thus* *(δ)* $|\int\limits_Q(\nabla F^{(\varepsilon)}-I)dx|\leq c\varepsilon$ *(note that* $\nabla F^{(\varepsilon)}=\big(\nabla(e^{-\varepsilon^2L}\varphi_1),\dots,\nabla(e^{-\varepsilon^2L}\varphi_n)\big)^T$ *is a matrix).*

For any fixed $\nu\in\mathbb{C}^n$ with $|\nu|=1$ define $F_\nu^{(\varepsilon)}:\mathbb{R}^n\to\mathbb{C}$ by

$$F_\nu^{(\varepsilon)}=F^{(\varepsilon)}.\,\overline{\nu}=e^{-\varepsilon^2L}\varphi\,.\,\overline{\nu}$$

The above properties then can be transferred:

Properties 3.3 *There exists a constant c not depending on* ε *such that*

(α′) $\int\limits_{5Q}|\nabla F_\nu^{(\varepsilon)}|^2\leq c$

(β′) $\int\limits_{5Q}|LF_\nu^{(\varepsilon)}|^2\leq\frac{c}{\varepsilon^2}$

(δ′) $|\int\limits_Q(\nu\,.\nabla F_\nu^{(\varepsilon)}(x)-1)dx|=|\int\limits_Q(\nu\,.\nabla F_\nu^{(\varepsilon)}(x)dx-1|\leq c\varepsilon$ *and hence*

$$\mathfrak{Re}\int_Q\nu\,.\nabla F_\nu^{(\varepsilon)}(x)dx\;\geq\;1-c\varepsilon\ .$$

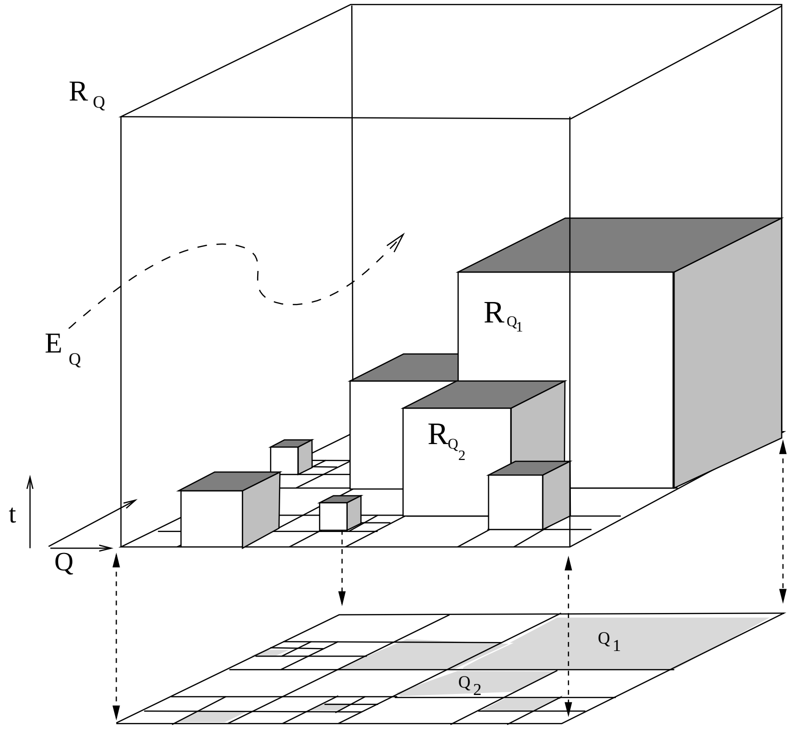
Now “good” subsets of R_Q are going to be distinguished: Let $\{Q_k'\}$ be the set of all pairwise disjoint maximal dyadic (i.e. arised from bisections of the sides) subcubes of Q for which

$$\mathfrak{Re}\int_{Q_k'}\nu\,.\nabla F_\nu^{(\varepsilon)}(x)dx\leq\frac{3}{4}|Q_k'|\ .$$

Moreover, let $\{Q_k''\}$ be the set of all disjoint maximal dyadic subcubes $Q_k''\subset Q$ for which

$$\int_{Q_k''}|\nabla F_\nu^{(\varepsilon)}(x)|dx\geq\frac{1}{8\varepsilon}|Q_k''|\ .$$

Define $B:=B_1\cup B_2=\bigcup_k Q_k$ with $B_1:=\bigcup_k Q_k'$ and $B_2:=\bigcup_k Q_k''$ and $\{Q_k\}:=\{Q_k'\}\cup\{Q_k''\}$. Set $E_Q:=R_Q\backslash\big(\bigcup_k R_{Q_k}\big)$. (Note that the defined sets depend on ε.)



The following theorem is used both to determine ε and as a tool in the further proof; recall that $|Q| = 1$.

Properties 3.4 (I) *For an appropriate choice of $\varepsilon > 0$ there exists $\eta \in (0,1)$ not depending on Q or ν such that*

$$|B_1 \cup B_2| < 1 - \eta \ .$$

(II) *If $(x,t) \in E_Q$ and $z \in K_\nu$, then*

$$|z| \leq 2|z \ .A_{x,t}(\nabla F_\nu^{(\varepsilon)})|$$

where $\left(A_{x,t}(\nabla F_\nu^{(\varepsilon)})\right)_j := \frac{1}{Q(x,t)} \int_{Q(x,t)} (\nabla F_\nu^{(\varepsilon)})_j(y)dy$ and $Q(t,x) \subset Q$ is the smallest dyadic cube such that $(x,t) \in R_{Q(x,t)}$.

Proof: (I) By Properties 3.3 there is a constant c such that for arbitrary $\varepsilon > 0$

$$\sqrt{c} \geq \Big(\int_{2Q} |\nabla F_\nu^{(\varepsilon)}|^2 \Big)^{\frac{1}{2}} \geq \Big(\int_{Q} |\nabla F_\nu^{(\varepsilon)}|^2 \Big)^{\frac{1}{2}} \geq \int_{Q} |\nabla F_\nu^{(\varepsilon)}| \ .$$

Moreover, by the choice of B_2 we have

$$\int_Q |\nabla F_\nu^{(\varepsilon)}| \geq \int_{B_2} |\nabla F_\nu^{(\varepsilon)}| = \sum_k \int_{Q_k''} |\nabla F_\nu^{(\varepsilon)}| \geq \sum_k \frac{1}{8\varepsilon} |Q_k''| = \frac{1}{8\varepsilon} |B_2| \ .$$

Hence $|B_2| \leq \sqrt{c} 8 \varepsilon$.

This, Properties 3.3, and the specifying conditions of B_1 now yield

$$\begin{aligned}
1-c\varepsilon &\leq \mathfrak{Re}\int\limits_Q \nu.\nabla F_\nu^{(\varepsilon)} = \mathfrak{Re}\int\limits_{Q\setminus B} \nu.\nabla F_\nu^{(\varepsilon)} + \mathfrak{Re}\int\limits_{B_1} \nu.\nabla F_\nu^{(\varepsilon)} + \mathfrak{Re}\int\limits_{B\setminus B_1} \nu.\nabla F_\nu^{(\varepsilon)} \\
&\leq |Q\setminus B|^{\frac{1}{2}}\Big(\int\limits_Q |\nabla F_\nu^{(\varepsilon)}|^2\Big)^{\frac{1}{2}} + \frac{3}{4}|B_1| + |B_2|^{\frac{1}{2}}\Big(\int\limits_Q |\nabla F_\nu^{(\varepsilon)}|^2\Big)^{\frac{1}{2}} \\
&\leq \sqrt{c}|Q\setminus B|^{\frac{1}{2}} + \frac{3}{4} + \sqrt{\sqrt{c}8\varepsilon}\sqrt{c}
\end{aligned}$$

and therefore

$$(\bullet) \qquad |Q\setminus B| \geq \Big(\frac{1}{4\sqrt{c}} - \sqrt{c}\varepsilon - \sqrt{\sqrt{c}8\varepsilon}\Big)^2 \geq \frac{1}{(8\sqrt{c})^2} =: \eta \quad \text{for sufficiently small } \varepsilon > 0\,.$$

(II) Abbreviate $\big(A_{x,t}(\nabla F_\nu^{(\varepsilon)})\big)_j =: V_j(x,t)$ and let $V(x,t) = (V_1(x,t)\ldots, V_n(x,t))$ denote the corresponding vector in $\mathbb{C}^n$.

It follows from $(x,t)\in E_Q$ that $Q(x,t)\notin\{Q_k\}$ and thus that

$$(a) \qquad \frac{1}{|Q(x,t)|}\mathfrak{Re}\int\limits_{Q(x,t)} \nu.\nabla F_\nu^{(\varepsilon)} > \frac{3}{4} \qquad \big(\text{ i.e. } \ \mathfrak{Re}\,\nu.V \geq \frac{3}{4}\big) \quad \text{and}$$

$$(b) \qquad \frac{1}{|Q(x,t)|}\int\limits_{Q(x,t)} |\nabla F_\nu^{(\varepsilon)}| \leq \frac{1}{8\varepsilon} \qquad \big(\text{and hence } \ |V| \leq \frac{1}{8\varepsilon}\,\big).$$

This yields (note that $K_\nu = \{z\in\mathbb{C} : |z-\nu(z.\overline{\nu})| < \varepsilon|z|\}$)

$$\begin{aligned}
|z.V| &\geq \Big|(\nu.V)(z.\overline{\nu})\Big| - \Big|\Big(z-\nu(z.\overline{\nu})\Big).V\Big| \\
&\geq \frac{3}{4}\,|z.\overline{\nu}| - \varepsilon|z|\frac{1}{8\varepsilon} \ \geq \ \Big(\frac{3}{4}-\frac{1}{8}\Big)|z| \\
&\geq \frac{1}{2}|z|.
\end{aligned}$$

Therefore Properties 3.4 is proved.

Now fix an ε so that the lower bound $(\bullet)$ holds.

The **proof of $(\mathbf{C}'_\varepsilon)$** then is the following: With η defined in 3.4 we have

$$\begin{aligned}
\Big(\iint\limits_{\substack{(x,t)\in E_Q\\ \gamma_t(x)\in K_\nu}} |\gamma_t(x)|^2\,\frac{d(x,t)}{t}\Big)^{\frac{1}{2}} &\leq \Big(2\iint\limits_{R_Q} \big|\gamma_t(x)\ .A_{x,t}(\nabla F_\nu^{(\varepsilon)})\big|^2\frac{d(x,t)}{t}\Big)^{\frac{1}{2}} \qquad \text{by 3.4 (II)} \\
&\leq \sqrt{2}\Big(\iint\limits_{R_Q} \big|\gamma_t(x)\ .P_t\nabla F_\nu^{(\varepsilon)}\big|^2\frac{d(x,t)}{t}\Big)^{\frac{1}{2}} \\
&\qquad + \sqrt{2M}\Big(\iint\limits_{R_Q} \big|(P_t - A_{x,t})\nabla F_\nu^{(\varepsilon)}\big|^2\frac{d(x,t)}{t}\Big)^{\frac{1}{2}},
\end{aligned}$$

where the convolution operator P_t was defined in (ii), the proof of "(C)⇒(Q)". Call the right-hand integrals I_1 and I_2, respectively. It can be shown then that

$$I_2 \leq \int\limits_{5Q} |\nabla F_\nu^{(\varepsilon)}(x)|^2 dx \leq c \ ,$$

whereas similarly to the fact mentioned in (ii) one can derive

$$\begin{aligned}
\sqrt{I_1} &= \Big(\iint\limits_{R_Q} \Big|tLe^{-t^2L}\varphi(x).\ P_t\nabla F_\nu^{(\varepsilon)}(x)\Big|^2 \frac{d(x,t)}{t}\Big)^{\frac{1}{2}} \\
&\leq \Big(\iint\limits_{R_Q} \Big|tLe^{-t^2L}\underbrace{F_\nu^{(\varepsilon)}(x)}_{e^{-\varepsilon^2L}\varphi(x).\nu}\Big|^2 \frac{d(x,t)}{t}\Big)^{\frac{1}{2}} + C \\
&\leq \Big(\int\limits_0^1\int\limits_Q \big|tLe^{-(t^2+\varepsilon^2)L}\varphi(x)\big|^2 \frac{dx\,dt}{t}\Big)^{\frac{1}{2}} + C \\
&< \Big(\int\limits_\varepsilon^2\int\limits_Q \underbrace{\big|\tau Le^{-\tau^2L}\varphi(x)\big|}_{=|\gamma_t(x)|\leq M}{}^2 \frac{dx\,d\tau}{\tau}\Big)^{\frac{1}{2}} + C \qquad \text{by substitution } \tau^2 = t^2+\varepsilon^2 \\
&< C(\varepsilon)\ .
\end{aligned}$$

This completes the outline of the proof. For further details, see the papers [HLM$^{\mathbf{c}}$] and [AHLM$^{\mathbf{c}}$T].

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