ASYMPTOTICALLY ONE-DIMENSIONAL DIFFUSION ON THE SIERPINSKI GASKET AND MULTI-TYPE BRANCHING PROCESSES WITH VARYING ENVIRONMENT

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Abstract

Asymptotically one-dimensional diffusions on the Sierpinski gasket constitute a one parameter family of processes with significantly different behaviour to the Brownian motion. Due to homogenization effects they behave globally like the Brownian motion, yet locally they have a preferred direction. We calculate the spectral dimension for these processes and obtain short time heat kernel estimates in the Euclidean metric. The results are derived using branching process techniques, and we give estimates for the left tail of the limiting distribution for a supercritical multi-type branching process with varying environment.

Running head

Diffusion on a fractal and branching processes

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1 INTRODUCTION

Asymptotically one dimensional processes on fractals were introduced in [HHW94]. For the Sierpinski gasket it was shown that there was a natural class of diffusions on this fractal, with no equivalent in Euclidean space, which have rather different properties than the Brownian motion. One obvious feature was that locally they prefer to move horizontally yet globally appear to diffuse like the Brownian motion. There was little precise information obtained in [HHW94] and, despite some results in [HK98A], the analysis of these processes is still far from complete. In particular it was not previously known if there existed a spectral dimension, the key analytic exponent for fractals, or to what extent there are good short time heat kernel bounds. Our aim in this paper is to investigate some of these problems using branching process techniques.

The Brownian motion on the Sierpinski gasket is the unique continuous diffusion which is invariant under the natural isometries of the fractal. It has the property of decimation invariance in that if the process is stopped on the vertices of level 1 triangles, then its exit probabilities on level 0 are the same. Thus it corresponds to a fixed point for a map of transition probabilities for a random walk on one level to the next. It is known that this map also has degenerate fixed points corresponding to the process moving purely along the horizontal components of the fractal. The idea of [HHW94] was to invert this map and iterate away from the nondegenerate toward the degenerate fixed point. Thus we expect very different local behaviour as we move along the edge or away from it. It was shown that the associated random walks can be scaled to give a diffusion, but there appeared to be two time scales. For the Brownian motion the time scaling factor is 5, but for the asymptotically one dimensional case it appears to be 6 for horizontal and $9/2$ for diagonal movements.

The Sierpinski gasket is a fractal with Hausdorff dimension $d_f = \log 3/\log 2$. Analysis on fractals requires two other dimensional exponents, which first arose in the physics literature where they have a heuristic definition. The spectral dimension d_s can be defined via the asymptotic scaling in the eigenvalue counting function. If $N(\lambda)$ denotes the eigenvalue counting function, then $N(\lambda) \sim \lambda^{d_s/2}$ as $\lambda \to \infty$. The walk dimension d_w is defined via the time to distance scaling for the diffusion X, in that $\mathrm{E}^0 d(0, X_t)^2 \sim t^{2/d_w}$ as $t \to 0$. It has been observed, in all cases where these definitions can be made precise, that these exponents are related through $d_s/2 = d_f/d_w$, where d_f is the Hausdorff dimension of the fractal. For the Brownian motion $d_w = \log 5/\log 2$ and $d_s = 2\log 3/\log 5$. We will establish that the spectral dimension for the asymptotically one dimensional processes is given by $d_s = 2\log 3/\log(9/2)$, and give several versions of the walk dimension.

There are a number of results for diffusion on fractals which deduce transition density estimates for the diffusion of Aronson type. That is bounds above and below of the functional form,

$$p_t(x,y) \sim c_1 t^{-d_s/2} \exp(-c_2 (d(x,y)^{d_w}/t)^{\alpha}),$$

where c_1, c_2 are constants which differ in the upper and lower bounds, α is a function of the shortest path and walk dimension, and $d(x,y)$ is a suitable metric on the fractal. We will find a branching process embedded in the paths of the diffusion and use results about this process to obtain bounds on the transition density — or heat kernel — of an asymptotically one-dimensional diffusion on the Sierpinski gasket, extending the on-diagonal bounds obtained in [HK98A] off the diagonal. We will find appropriate dimensional exponents and work in the Euclidean metric, though we do not obtain bounds of Aronson type.

A multi-type branching process with varying environment (MTBPVE) generalises the classical multi-type branching process. For a finite number d of types, we allow the number of type j offspring of a type i parent at time n to depend on i, j and n. In [Jon95, Jon97], conditions are obtained for the $\mathcal{L}^2$ and a.s. convergence of an MTBPVE normed by its mean. In what follows, the left tail of the normed limit is considered — under so called Böttcher type conditions, that is, with exponential minimal growth — and upper and lower bounds obtained. Bounds of the same form have previously been obtained in [Jon95] for a more restricted class of MTBPVE. Related results for fixed environment multi-type branching processes appear in [Kum93] and [BJ98]. These results are independent of their application to diffusions on fractals.

In previous work asymptotically lower dimensional diffusions have been constructed on some simple fractals [HHW94, Hat97, HW97, HK98A], and estimates have been obtained for the on diagonal heat kernel and a homogenization property proved for the large scale behaviour [HK98A]. For the case in hand, we improve previous estimates for the on-diagonal heat kernel lower bound, which then allow us to determine the spectral dimension for the generator of the diffusion. This follows from a more precise estimate on the tail of the exit probability distribution provided by the branching process results. With this more precise estimate we will be able to establish some off-diagonal heat kernel bounds. However, as there is substantially different local behaviour depending on the starting point of the process, we cannot obtain exact bounds in the Euclidean metric. Previous results for diffusion on fractals suggest that, working in the effective resistance metric, it may be possible to obtain upper and lower estimates which agree up to constants. However, it should be noted from [HK98B], that for general p.c.f. fractals with no symmetry there is not necessarily an Aronson type estimate for the transition density, even in the effective resistance metric.

The structure of the paper is as follows. In the first sections we discuss multi-type varying environment branching processes. By considering functional equations associated with the generating functions for the branching process, we can get general estimates on the left tail of the distribution for the limiting random variable L. In Section 5 we introduce the diffusion process X on the Sierpinski gasket via its Dirichlet form, and describe it as the limit of a sequence of Markov chains. It is convenient to extend this to an infinite fractal in order to avoid boundary effects. In Section 6 we investigate crossing times for X, which can be obtained as the normed limit of an embedded MTBPVE. Then Section 7 establishes the spectral dimension result. Finally, in Theorems14 and 19, we obtain off diagonal transition density estimates for the Euclidean metric, which are not best possible for short time due to the difference in the local geometry of the fractal. The homogenization property proved in [HK98A], shows that the long time estimates will be those of the Brownian motion on the Sierpinski gasket.

Throughout the paper, c_i will be used to denote a positive finite constant whose value remains fixed within each proof, while $c_{n.i}$ will denote a fixed constant appearing in Section n.

2 THE BRANCHING PROCESS

2.1 NOTATION

For a matrix $A \in \mathbb{R}^{d\times d}$, write $A(i,j)$ for its $(i,j)^{\text{th}}$ element, $A(i,\cdot)$ for the row vector given by its i^{th} row and $A(\cdot,j)$ for the column vector given by its j^{th} column. Similarly, for a vector $a \in \mathbb{R}^d$ write $a(i)$ for its i^{th} component. The vector of 1s will be written 1 and the unit vector with a 1 in position i will be written e_i. Write $A \geq 0$ or $a \geq 0$ if every element of A or a is ≥ 0, and write $A > 0$ or $a > 0$ if every element is > 0.

For two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, write $a_n \equiv b_n$ if $\lim_{n\to\infty} a_n/b_n$ exists $\in (0,\infty)$. This implies we can find constants c_1 and c_2 such that $c_1 a_n \leq b_n \leq c_2 a_n$ for all n. For doubly indexed sequences $\{a_{m,n}\}_{n=m}^{\infty}$ and $\{b_{m,n}\}_{n=m}^{\infty}$, write $a_{m,n} \equiv b_{m,n}$ if $\lim_{n\to\infty} a_{m,n}/b_{m,n} = r_m$ exists $\in (0,\infty)$ for all m, and the $\{r_m\}_{m=0}^{\infty}$ are bounded in $(0,\infty)$. This implies we can find constants c_1 and c_2 such that $c_1 a_{m,n} \leq b_{m,n} \leq c_2 a_{m,n}$ for all m and n.

Let the distribution of the number of type j children born to a single type i parent at time n be given by $X_n(i,j)$, for some $\{X_n\}_{n=0}^{\infty}$. For fixed $m \geq 0$, let $Z_m = \{Z_{m,n}\}_{n=m}^{\infty}$ be our branching process, where $Z_{m,n}(i,j)$ is the number of type j descendants at time n of a single type i parent at time m. Note that Z_m takes on values in $\mathbb{Z}_+^{d \times d}$, where each row is an independent process. Write $Z_m(i,\cdot) = \{Z_{m,n}(i,\cdot)\}_{n=m}^{\infty}$ for the process given by row i. Define $M_n = \mathrm{E} X_n$, $V_n[i] = \mathrm{Cov}\, X_n(i,\cdot)$ and $\sigma_n^2(i,j) = \mathrm{Var}\, X_n(i,j) = V_n[i](j,j)$.

For a sequence of matrices $\{A_n\}_{n=0}^{\infty}$, write $A_{m,n}$ for the forward product from m to $n-1$. That is, $A_{m,n} = A_m A_{m+1} \cdots A_{n-1}$. It follows from the branching property of Z_m that for all $m \leq n \leq p$

$$\mathrm{E}(Z_{m,p} | Z_{m,n}) = Z_{m,n} M_{n,p}.$$

2.2 Convergence assumptions and integral transforms

Let ${}^m R_n = \mathrm{diag}({}^m R_n(1), \ldots, {}^m R_n(d))$ be given by ${}^m R_n(i) = 1^T M_{m,n}(i,\cdot)$. We will assume that

A

$$Z_{m,n}\, {}^m R_n^{-1} \xrightarrow{\mathcal{D}} L_m 1^T \text{ as } n \to \infty,$$

where $L_m \geq 0$ and $w_m = \mathrm{E} L_m$ is a strictly positive probability vector.

B There exist non-zero, finite scalars $\{\alpha_n^m\}_{0 \leq m \leq n}$ such that for all m and n

$${}^m R_p\, {}^n R_p^{-1} \to \alpha_n^m I \text{ as } p \to \infty.$$

Conditions sufficient to imply A and B are given by [Jon97] Theorem 1.

Let $f_{m,n}^i$ be the joint p.g.f. of $Z_{m,n}(i,\cdot)$ and for $x \in [0,1]^d$ put $f_{m,n}(x) = (f_{m,n}^1(x), \ldots, f_{m,n}^d(x))$. Let $\phi_{m,n}^i$ be the Laplace transform of $Z_{m,n}(i,1)/\, {}^m R_n(1)$ and for $s \in \mathbb{R}_+$ put $\phi_{m,n}(s) = (\phi_{m,n}^1(s), \ldots, \phi_{m,n}^d(s))$. Let ϕ_m^i be the Laplace transform of $L_m(i)$ and for $s \in \mathbb{R}_+$ put $\phi_m(s) = (\phi_m^1(s), \ldots, \phi_m^d(s))$. Then, since $Z_{m,n}(\cdot,1)/\, {}^m R_n(1) \xrightarrow{\mathcal{D}} L_m$, $\phi_{m,n}(s) \xrightarrow{n \to \infty} \phi_m(s)$ for all $m \geq 0$ and $s \in \mathbb{R}_+$.

Conditioning on $Z_{m,n}$, we have for all $m \leq n \leq p$, $\phi_{m,p}(s) = f_{m,n}(\phi_{n,p}(s\, {}^n R_p(1)/\, {}^m R_p(1)))$. Sending $p \to \infty$ this gives

$$\phi_m(s) = f_{m,n}(\phi_n(s \alpha_m^n)). \tag{1}$$

This system of functional equations is the source of our upper and lower bounds. Note, there is nothing to be gained by defining $\phi_{m,n}(s)$ for $s \in \mathbb{R}_+^d$ by $\phi_{m,n}(s) = (\phi_{m,n}^1(s(1)), \ldots, \phi_{m,n}^d(s(d)))$, since conditioning on $Z_{m,n}$ just gives, for each i, $\phi_{m,p}^i(s) = f_{m,n}(\phi_{n,p}(s(i)\, {}^n R_p(1)/\, {}^m R_p(1) \cdot 1))$.

2.3 Minimal growth of an MTBPVE; Böttcher condition

Let $J_{m,n}^i \subset \mathbb{Z}_+^d$ be the set of possible population vectors for $Z_{m,n}(i,\cdot)$. That is, $J_{m,n}^i = \{z \in \mathbb{Z}_+^d : \mathrm{P}(Z_{m,n}(i,\cdot) = z^T) > 0\}$. Let $K_{m,n}^i$ be the lower boundary of $J_{m,n}^i$. That is, for all $z \in J_{m,n}^i$, there exists $x \in K_{m,n}^i$ such that $x \leq z$, and for all $x, y \in K_{m,n}^i$, $x \not\leq y$ and $y \not\leq x$. $K_{m,n}^i$ is the set of minimal population vectors for $Z_{m,n}(i,\cdot)$. Note that $K_{m,m}^i = \{e_i\}$, and write K_n^i for $K_{n,n+1}^i$.

Let $K_{m,n} \subset \mathbb{Z}_+^{d \times d}$ be the set of matrices made up by taking the ith row of each matrix from $K_{m,n}^i$. That is, $K_{m,n} = \{A \in \mathbb{Z}_+^{d \times d} : A(i,\cdot) \in K_{m,n}^i \text{ for each } i\}$. Now define operators

$\mathcal{K}_{m,n} : \mathbb{R}_+^d \mapsto \mathbb{R}_+^d$ by $\mathcal{K}_{m,n}(x) = \min_{A \in K_{m,n}} Ax$. $\mathcal{K}_{m,n}$ is well defined due to the structure of $K_{m,n}$, which allows us to take the minimum over each co-ordinate. Let $\mathcal{K}_{m,n}^i(x) = \min_{a \in K_{m,n}^i} a^T x$, then $(K_{m,n}(x))(i) = \mathcal{K}_{m,n}^i(x)$.

Clearly, $A \in K_{m,n}$ can be written as $\prod_{k=m}^{n-1} A_k$ for $A_k \in K_k$, though this representation will not be unique in general.

Lemma 1 *For all* $0 \le m \le n$,

$$\begin{aligned}\mathcal{K}_{m,n}(x) &= \mathcal{K}_m(\mathcal{K}_{m+1}(\cdots \mathcal{K}_{n-1}(x)\cdots)) \\ &= \min_{A_m \in K_m, \ldots, A_{n-1} \in K_{n-1}} A_m \cdots A_{n-1} x.\end{aligned}$$

Also, for any i *and* $x, y \ge 0$

$$\max\{\mathcal{K}_n^i(x), \mathcal{K}_n^i(y)\} \le \mathcal{K}_n^i(x+y) \le \mathcal{K}_n^i(x) + \mathcal{K}_n^i(y).$$

Proof It suffices to consider $\mathcal{K}_{0,2}$. For any $1 \le i \le d$ and $x \in \mathbb{R}_+^d$ we have

$$\begin{aligned}\mathcal{K}_{0,2}^i(x) = \min_{a \in K_{0,2}^i} a^T x &= \min_{a \in K_0^i, b_{j,k} \in K_1^j} \sum_{j=1}^d \sum_{k=1}^{a(j)} b_{j,k}^T x \\ &= \min_{a \in K_0^i} \sum_{j=1}^d a(j) \mathcal{K}_1^j(x) = \mathcal{K}_0^i(\mathcal{K}_1(x)).\end{aligned}$$

In addition

$$\begin{aligned}\min_{A_0 \in K_0, A_1 \in K_1} (A_0 A_1 x)(i) &= \min_{A_0 \in K_0, A_1 \in K_1} \sum_{j=1}^d A_0(i,j) A_1(j,\cdot) x \\ &= \min_{A_0 \in K_0} \sum_{j=1}^d A_0(i,j) \mathcal{K}_1^j(x) = \mathcal{K}_0^i(\mathcal{K}_1(x)).\end{aligned}$$

The final inequalities are trivial. ∎

Although not stated explicitly in the results that follow, it can be seen that if they are to be non-trivial, then it is necessary for the 'minimal' family size of the process to grow exponentially. This is what we mean by a Böttcher type process; the terminology is taken from the functional equations literature.

3 Branching process upper bound

These results assume we know something of the behaviour of $\mathcal{K}_{m,n}$. In general, it is difficult to say anything exact about $\mathcal{K}_{m,n}$ without placing conditions on the process Z_m. For example, if you assume $\mathcal{K}_m = \mathcal{K}_n$ for all m and n, then these results can be made more specific, though in practice one can generally get better results by considering $\mathcal{K}_{m,n}$ on a case-by-case basis.

Consider the following. From (1), for any i,

$$\phi_m^i(s) = \sum_{z \in J_{m,n}^i} \mathrm{P}(Z_{m,n}(i,\cdot) = z^T) \phi_n(s\alpha_m^n)^z$$

$$
\begin{aligned}
&= \sum_{z \in J^i_{m,n}} \mathrm{P}(Z_{m,n}(i,\cdot) = z^T) \exp\{z^T \log \phi_n(s\alpha^n_m)\} \\
&\le \sum_{z \in J^i_{m,n}} \mathrm{P}(Z_{m,n}(i,\cdot) = z^T) \exp\{-\mathcal{K}^i_{m,n}(-\log \phi_n(s\alpha^n_m))\} \\
&= \exp\{-\mathcal{K}^i_{m,n}(-\log \phi_n(s\alpha^n_m))\}.
\end{aligned}
$$

That is, for all $n \ge m$,

$$
-\log \phi_m(s) \ge \mathcal{K}_{m,n}(-\log \phi_n(s\alpha^n_m)). \tag{2}
$$

Lemma 2 *Fix m, and suppose that we have t_n, y_n and z_n such that for all $n \ge m$*

$$
-\log \phi_n(t_n) \ge y_n \text{ and } \mathcal{K}_{m,n}(y_n) \ge z_n.
$$

Suppose that $t_n \alpha^m_n \to \infty$, and define

$$
\gamma_{m,n} = \frac{\log z_n}{\log t_{n+1}\alpha^m_{n+1}} \text{ and } \gamma^-_m = \liminf_{n\to\infty} \gamma_{m,n}.
$$

Then, for any $\epsilon \ge 0$ such that $\epsilon_n := \gamma^-_m - \epsilon - \gamma_{m,n} \le b_m / \log t_{n+1}\alpha^m_{n+1}$ for some $b_m \ge 0$ and all n, we have for $s \ge t_m$,

$$
-\log \phi_m(s) \ge e^{-b_m} s^\gamma \text{ where } \gamma = \gamma^-_m - \epsilon.
$$

(Here we take e^{-b_m} and s^γ componentwise.)

Proof For $s \ge t_m$ let n be such that $t_n\alpha^m_n \le s \le t_{n+1}\alpha^m_{n+1}$, then

$$
\begin{aligned}
-\log \phi_m(s) &\ge \mathcal{K}_{m,n}(-\log \phi_n(s\alpha^n_m)) \\
&\ge \mathcal{K}_{m,n}(-\log \phi_n(t_n)) \\
&\ge z_n \\
&= (t_{n+1}\alpha^m_{n+1})^{\log z_n / \log t_{n+1}\alpha^m_{n+1}} \\
&\ge (t_{n+1}\alpha^m_{n+1})^{\gamma - b_m / \log t_{n+1}\alpha^m_{n+1}} \\
&\ge s^\gamma e^{-b_m}.
\end{aligned}
$$

∎

α^m_n captures the mean growth of Z_m (though note that the columns of $M_{m,n}$ will in general have different growth rates, and α^m_n can have any of these), while z_n captures the minimal growth of Z_m. Thus, depending upon the effect of t_n, γ^-_m will be between 0 and 1. The t_n are required to allow for the varying environment of the process.

Note that if α^k_{k+1}, t_{k+1}/t_k and $z_{k+1}(i)/z_k(i)$ and are all bounded in $(0,\infty)$, then $\gamma^-_m(i)$ is independent of m. If $\alpha^m_n \equiv \alpha^n$, $t_n \equiv \tau^n$ and $z_n(i) \equiv z(i)^n$ for some α, τ and $z(i)$, then $\gamma^-_m(i) = \log z(i)/(\log \tau + \log \alpha)$.

Proposition 3 (Upper bound for left tail.) *Suppose that Conditions A and B hold, and that there exist $B_m \in \mathbb{R}^d_+$, $\gamma \in (0,1)^d$ and $s_m \in \mathbb{R}_+$ such that $-\log \phi^i_m(s) \ge B_m(i) s^{\gamma(i)}$ for all i and $s \ge s_m$. Then, for all i and $l \le l_m(i) := B_m(i)\gamma(i) s_m^{-(1-\gamma(i))}$,*

$$
\mathrm{P}(L_m(i) < l) \le \exp\{-C_m(i) l^{-\gamma(i)/(1-\gamma(i))}\},
$$

where $C_m = B_m^{1/(1-\gamma)}(\gamma^{\gamma/(1-\gamma)} - \gamma^{1/(1-\gamma)}) \ge 0$ componentwise.

Proof For any i, $\mathrm{P}(L_m(i) < l) \le e^{sl}\phi_m^i(s)$ for all $s \in \mathbb{R}_+$. Applying our bound on ϕ_m^i and then minimising over $s \ge s_m$ gives the required result. For more detail of this procedure see, for example, [Jon96] Proposition 6. ▮

3.1 INITIAL BOUND FOR ϕ_n

Here we show how to obtain bounds of the form $-\log \phi_n^i(t_n) \ge y_n(i)$ using second moment conditions. This approach will be used in Section 6, when we discuss the specific application of the branching process to the asymptotically one dimensional diffusion on the Sierpinski gasket.

If we assume that

A2

$$Z_{m,n}\, {}^mR_n^{-1} \xrightarrow{\mathcal{L}^2} L_m 1^T \text{ as } n \to \infty,$$

where $L_m \ge 0$ and $w_m = \mathrm{E}L_m$ is a strictly positive probability vector,

then it can be shown that, for $i = 1, \ldots, d$,

$$\mathcal{V}_m(i) := \mathrm{Var}\, L_m(i) = \sum_{k=m}^{\infty} (\alpha_m^{k+1})^2 w_{k+1}^T \left(\sum_{j=1}^{d} V_k[j] \cdot M_{m,k}(i,j) \right) w_{k+1} < \infty.$$

By Taylor's Theorem we have $\phi_n^i(s) \le 1 - s w_n(i) + s^2(\mathcal{V}_n(i) + w_n(i)^2)/2$. This bound has a minimum of $1 - w_n(i)^2/2(\mathcal{V}_n(i) + w_n(i)^2)$ at $s = w_n(i)/(\mathcal{V}_n(i) + w_n(i)^2)$. Thus, for $s \ge w_n(i)/(\mathcal{V}_n(i) + w_n(i)^2)$, $-\log \phi_n^i(s) \ge w_n(i)^2/2(\mathcal{V}_n(i) + w_n(i)^2)$. That is, we can take

$$t_n = \max_i \{ w_n(i)/(\mathcal{V}_n(i) + w_n(i)^2) \} \text{ and } y_n(i) = w_n(i)^2/2(\mathcal{V}_n(i) + w_n(i)^2).$$

For these to be useful, we need upper and lower bounds for w_n and $\mathcal{V}_n$.

From Conditions A2 and B we have that

$$w_m 1^T = \lim_{p \to \infty} M_{m,p}\, {}^mR_p^{-1} = M_{m,n} \lim_{p \to \infty} M_{n,p}\, {}^nR_p^{-1} ({}^nR_p\, {}^mR_p^{-1}) = M_{m,n} w_n 1^T \alpha_m^n.$$

(In fact, this only needs $\mathcal{L}^1$ convergence.) So, in the terminology of [CN90], $\{\alpha_m^n w_n\}_{n=m}^{\infty}$ is a space-time harmonic sequence for the matrices $\{M_n\}_{n=m}^{\infty}$. In practice, when dealing with diffusion on fractals, the matrices $\{M_n\}$ satisfy the stronger condition of weak-ergodicity, which allows a number of tools to be applied to the estimation of the $\{w_n\}$. See for example [Jon97] Section 2.

To bound $\mathcal{V}_m(i)$, note that by Cauchy-Schwarz and the symmetry of $V_k[j]$,

$$w_{k+1}^T V_k[j] w_{k+1} \le \sum_{x,y} V_k[j](x,y) w_{k+1}^2(x) \le d \sum_x \sigma_k^2(j,x) w_{k+1}^2(x).$$

Whence

$$\begin{aligned} &\sum_{k=m}^{\infty} (\alpha_m^{k+1})^2 \sup_{j,l} M_{m,k}(i,j)\sigma_k^2(j,l) w_{k+1}^2(l) \\ &\quad \le \mathcal{V}_m(i) \le d^3 \sum_{k=m}^{\infty} (\alpha_m^{k+1})^2 \sup_{j,l} M_{m,k}(i,j)\sigma_k^2(j,l) w_{k+1}^2(l). \end{aligned} \qquad (3)$$

To apply this we need, in addition to an estimate for w_{k+1}, an estimate for $M_{m,k}(i,j)$. This is provided by [Jon97] Lemma 15, which gives conditions for the uniform convergence of $M_{m,n}(i,j)/w_m(i)\, {}^mR_n(j) \to 1$.

4 Branching process lower bound

For our upper bound, we used the minimal growth of the branching process to bound its behaviour. For a lower bound, we choose a trajectory of minimal growth, and hope that its probability has the right asymptotics. In all other respects however, the method is very similar to that of the upper bound.

Lemma 4 *Fix m, and suppose that we have t_n, y_n and z_n such that for all $n \geq m$*

$$-\log \phi_n(t_n) \leq y_n \text{ and } \mathcal{K}_{m,n}(y_n) \leq z_n.$$

Let $\mathcal{K}_{m,n}(y_n) = {}^nA_{m,n}y_n$ and put

$$p_n(i) = \mathrm{P}(Z_{m,n}(i,\cdot) = {}^nA_{m,n}(i,\cdot)).$$

Also, suppose that $t_n\alpha_n^m \to \infty$, and define

$$\gamma_{m,n} = \frac{\log(-\log p_n + z_n)}{\log t_{n-1}\alpha_{n-1}^m} \text{ and } \gamma_m^+ = \limsup_{n\to\infty} \gamma_{m,n}.$$

Then, for any $\epsilon \geq 0$ such that $\epsilon_n := \gamma_{m,n} - \gamma_m^+ - \epsilon \leq b_m / \log t_{n-1}\alpha_{n-1}^m$ for some $b_m \geq 0$ and all n, we have for $s \geq t_m$,

$$-\log \phi_n(s) \leq e^{b_m} s^{\gamma} \text{ where } \gamma = \gamma_m^+ + \epsilon.$$

(Here we take e^{b_m} and s^γ componentwise.)

Proof Fix m and i. For any s and $n \geq m$, we have from (1) that

$$\phi_m^i(s) \geq p_n(i) \exp\{-{}^nA_{m,n}(i,\cdot)(-\log \phi_n(s\alpha_m^n))\}.$$

Now, for $s \geq t_m$, let n be such that $t_{n-1}\alpha_{n-1}^m \leq s \leq t_n\alpha_n^m$, then

$$\begin{aligned}
-\log \phi_m^i(s) &\leq -\log p_n(i) + {}^nA_{m,n}(i,\cdot)(-\log \phi_n(s\alpha_m^n)) \\
&\leq -\log p_n(i) + z_n(i) \\
&= (t_{n-1}\alpha_{n-1}^m)^{\log(-\log p_n(i) + z_n(i))/\log t_{n-1}\alpha_{n-1}^m} \\
&\leq s^{\gamma}(i) e^{b_m(i)}.
\end{aligned}$$

∎

As before, $\{\alpha_n^m\}$ captures the mean growth of Z_m, and $\{z_n\}$ its minimal growth. Thus, if $-\log p_n$ grows at the same rate as z_n, then we can hope that γ_m^+ will equal γ_m^-. This is not an unreasonable hope, as we will see in Subsection 4.1 below. However, even when $-\log p_n$ behaves well, the $\{t_n\}$ can still interfere.

Proposition 5 (Lower bound for left tail.) *Suppose that Conditions A and B hold, and that there exist $B_m \in \mathbb{R}_+^d$, $\gamma \in (0,1)^d$ and $s_m \in \mathbb{R}_+$ such that $-\log \phi_m^i(s) \leq B_m(i)s^{\gamma(i)}$ for all i and $s \geq s_m$. Then, for all i and $l \leq l_m(i) := 2B_m(i)s_m^{-(1-\gamma(i))}$,*

$$\mathrm{P}(L_m(i) < l) \geq \tfrac{1}{2}\exp\{-C_m(i)l^{-\gamma(i)/(1-\gamma(i))}\},$$

where $C_m = 2^{\gamma/(1-\gamma)}B_m^{1/(1-\gamma)}$ componentwise.

Proof For any i, $\mathrm{P}(L_m(i) < l) \geq (\phi_m^i(s) - e^{-sl})/(1 - e^{-sl})$ for all $s \in \mathbb{R}_+$. Applying our bound on ϕ_m we get for $s \geq s_m$

$$\mathrm{P}(L_m(i) < l) \geq \exp\{-B_m(i)s^{\gamma(i)}\}\frac{1 - \exp\{B_m(i)s^{\gamma(i)} - sl\}}{1 - \exp\{-sl\}}.$$

Putting $s = c_m(i)l^{-1/(1-\gamma(i))}$ with $c_m = (2B_m)^{1/(1-\gamma)}$ now gives the result. For more detail, see the analogous result in [Jon96] Proposition 9. ∎

4.1 INITIAL BOUND FOR ϕ; LIKELIEST PATH

An initial lower bound for ϕ_n^i is readily obtained from Jensen's inequality For all n and i,

$$\phi_n^i(t) = \mathrm{E}e^{-tL_n(i)} \geq e^{-t\mathrm{E}L_n(i)} = e^{-tw_n(i)}.$$

So, for any given t_n, we can take

$$y_n = t_n w_n \text{ and } z_n = t_n \mathcal{K}_{m,n}(w_n).$$

This leaves scope to choose the $\{t_n\}$ optimally. Generally, looking at the numerator of $\gamma_{m,n}$, it is best — if possible — to choose the $\{t_n\}$ so that $-\log p_n(i) \equiv z_n(i)$.

Write ${}^nA_{m,n}$ as $\prod_{k=m}^{n-1} {}^nA_k$, for ${}^nA_k \in K_k$, and let ${}^n\pi_k(i) = \mathrm{P}(X_k(i,\cdot) = {}^nA_k(i,\cdot))$, then

$$p_n(i) \geq \exp\left\{-\sum_{k=m}^{n-1} {}^nA_{m,k}(i,\cdot)(-\log {}^n\pi_k)\right\}.$$

Equality will hold if the representation ${}^nA_{m,n} = \prod_{k=m}^{n-1} {}^nA_k$ is unique. As ${}^nA_{m,n} \in K_{m,n}$, this is not at all unlikely, though it is none-the-less not true in general. It is not difficult to construct an example (even with a fixed environment) where the inequality is strict.

To see why it is reasonable to expect $-\log p_n(i)$ to grow similarly to $z_n(i)$, we consider the fixed environment case. With a fixed environment, $w_n = w$ for all n and some $w > 0$, and (given some reasonable conditions, see [BJ98]) we can take ${}^nA_{m,n} = A^{n-m}$ for all m and n and some A. Thus $\pi := {}^n\pi_k$ is independent of n and k. Now, let λ be the largest (real) eigenvalue of A, and suppose that $\lambda > 1$ and that it has a strictly positive eigenvector x. Then, as $w > 0$ and $\pi < 1$ (for a non-degenerate process),

$$A^n w \equiv A^n(-\log \pi) \equiv \lambda^n x.$$

Taking $t_n = 1$ for all n, this gives for any i,

$$-\log p_n(i) \equiv \sum_{k=m}^{n-1} \lambda^{k-m} \equiv \lambda^{n-m} \equiv z_n(i).$$

5 ASYMPTOTICALLY ONE-DIMENSIONAL PROCESS ON THE SIERPINSKI GASKET

The finite Sierpinski gasket G^0 is a compact triangular subset of $\mathbb{R}^2$ formed as the fixed point, $G^0 = \cup_{i=1}^3 \psi_i(G^0)$, of a set of three similitudes $\{\psi_1, \psi_2, \psi_3\}$, where, for $(x,y) \in \mathbb{R}^2$,

$$\psi_1(x) = (\tfrac{1}{2}x, \tfrac{1}{2}y) + (\tfrac{1}{4}, \tfrac{\sqrt{3}}{4}),\ \psi_2(x,y) = (\tfrac{1}{2}x, \tfrac{1}{2}y) \text{ and } \psi_3(x) = (\tfrac{1}{2}x, \tfrac{1}{2}y) + (\tfrac{1}{2}, 0).$$

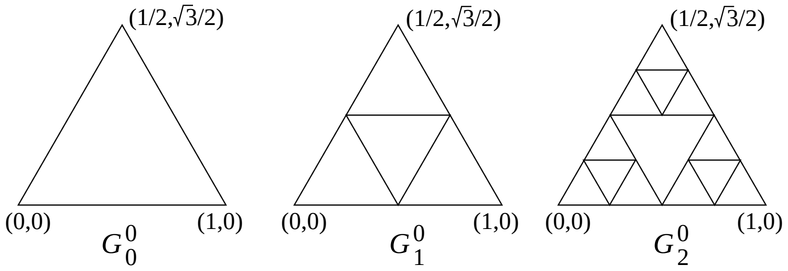

Figure 1: G_0^0, G_1^0 and G_2^0

An alternative description is provided by a shift space. Let $A = \{1,2,3\}$ and let A^n denote the set of sequences of length n formed from elements of A. We write $\psi_a = \psi_{i_1} \circ \ldots \circ \psi_{i_n} : a = (i_1, \ldots, i_n)$. Let G_0^0 be the complete graph on the three vertices V_0^0 of the unit equilateral triangle based at the origin. We denote the set of vertices of the triangles formed by iteration of the maps n times $V_n^0 = \cup_{a \in A^n} \psi_a(V_0^0)$. The fractal can then be recovered as $G^0 = cl(\lim_{n\to\infty} V_n^0)$. The graph $G_n^0 = \cup_{a\in A^n} \psi_a(G_0^0) = (V_n^0, E_n^0)$, where E_n^0 denotes the edge set. See Figure 1. The triangles at level n are labelled by elements of A_n, and the vertices can be described as $a\dot{i}$ where $\dot{i}$ denotes an infinite sequence of i.

In what follows, to avoid boundary effects at $(1/2, \sqrt{3}/2)$, $(0,0)$ and $(1,0)$, we will actually consider diffusion on the infinite Sierpinski gasket $G = \cup_{n=0}^{\infty}(2^n G^0 \cup \rho(2^n G^0))$, where $\rho(x,y) = (-x,y)$ reflects (x,y) about $x = 0$. Let $\phi_1^n(x,y) = (x,y) + (2^{n-1}, \sqrt{3}2^{n-1})$, $\phi_2^n(x,y) = (x,y)$ and $\phi_3^n(x,y) = (x,y) + (2^n, 0)$, then recursively define

$$G_n^{m+1} = \cup_{i=1}^{3}(\phi_i^m(G_n^m) \cup \rho \circ \phi_i^m(G_n^m)),$$

and put

$$G_n = (V_n, E_n) = \lim_{m\to\infty} G_n^m.$$

Then G_n is an infinite graph, made up of triangles of size 2^{-n}, with $G_n = 2^{-n}G_0$. Also, we see that $G = cl(\lim_{n\to\infty} V_n)$, and that G^0 is just the subset of G bounded by $(1/2, \sqrt{3}/2)$, $(0,0)$ and $(1,0)$. Note that where we write $x \in G_n$ below, we mean $x \in V_n$.

We construct a process X on G via a sequence of compatible electrical networks, and hence build the Dirichlet form for the process. We briefly summarise this approach, for more detail see [HK98A] Sections 4, 8.

For $(x,y) \in E_n$, we will write $\tau(x,y) = h$ if (x,y) is a horizontal edge, and $\tau(x,y) = d$ if (x,y) is a diagonal edge. For $\omega \in (0,1)$, let D^0 be the generator of a continuous-time Markov chain X^0 on G_0, with rates

$$d_0(x,y) = \begin{cases} 1, & \tau(x,y) = h, \\ \omega, & \tau(x,y) = d, \\ 0, & (x,y) \notin E_0, \end{cases} \qquad d_0(x,x) = -\sum_{y\neq x} d_0(x,y).$$

The Dirichlet form for the graph G_0 is then, for $f : G_0 \to \mathbb{R}$,

$$\mathcal{E}_0^{(\omega)}(f,f) = -f^T D^0 f.$$

We can use this to recursively define, for any n, the level n Dirichlet form on G_n, for $f : G_n \to \mathbb{R}$, by

$$\mathcal{E}_n^{(\omega)}(f,f) = \mathcal{E}_{n-1}^{(\omega)}(f \circ \psi_2, f \circ \psi_2).$$

The trace of $\mathcal{E}_n^{(\omega)}$ on G_{n-1} is defined as

$$\tilde{\mathcal{E}}_{n-1}^{(\omega)}(f,f) = \inf\{\mathcal{E}_n^{(\omega)}(g,g) : g|_{G_{n-1}} = f\}.$$

Requiring $\tilde{\mathcal{E}}_{n-1}^{(\omega)}(f,f) = \mathcal{E}_{n-1}^{(\chi)}(f,f)/K$ for some χ and K induces a map $\chi = \alpha(\omega)$ and a resistance scaling $K = R(\chi)$, given by

$$\alpha(\omega) = \frac{\omega(4+6\omega)}{3+6\omega+\omega^2} \text{ and } R(\chi) = \frac{2(2\chi+3)(3\chi+2)}{\chi^3+8\chi^2+15\chi+6}.$$

That is, for $f : G_n \to \mathbb{R}$,

$$\mathcal{E}_n^{(\omega)}(f,f) = \frac{1}{R(\alpha(\omega))}\mathcal{E}_{n-1}^{(\alpha(\omega))}(f \circ \psi_2, f \circ \psi_2).$$

The map α is invertible on $[0,1]$, with fixed points at 0 and 1.

We now fix ω and define a sequence of weights $\omega_n = \alpha^{-n}(\omega)$ and resistance scaling factors $R_n(\omega) = \prod_{i=0}^{n-1} R(\omega_n)$, then $\mathcal{E}_n^{(\omega_n)}(f,f) = -f^T D^n f$, where D^n is a conductivity matrix given by

$$d_n(x,y) = \begin{cases} R_n(\omega), & \tau(x,y) = h, \\ R_n(\omega)\omega_n, & \tau(x,y) = d, \\ 0, & (x,y) \notin E_n, \end{cases} \qquad d_n(x,x) = -\sum_{y \neq x} d_n(x,y).$$

Note that we have

$$\omega_n \equiv \left(\frac{3}{4}\right)^n \text{ and } R_n(\omega) \equiv 2^n.$$

Regarding the weights $d_n(x,y)$ as conductances in an electrical network, this choice of weights ensures that — by definition of α and R — the sequence of networks is compatible, in that they are electrically equivalent when observed at the vertices of any G_n. If we let $\mathcal{E}_n(f,g)$ denote the Dirichlet form $\mathcal{E}_n^{(\omega_n)}(f,g)$, then for all $m < n$

$$\mathcal{E}_m(f,f) = \inf\{\mathcal{E}_n(g,g) : g|_{G_m} = f\}.$$

Thus the forms are a monotone increasing sequence and they converge to a limit Dirichlet form $(\mathcal{E},\mathcal{F})$, on $L^2(G,\mu)$, defined by

$$\begin{aligned} \mathcal{F} &= \{f \in L^2(G,\mu) : \sup_n \mathcal{E}_n(f,f) < \infty\}, \\ \mathcal{E}(f,f) &= \lim_{n\to\infty} \mathcal{E}_n(f,f), \text{ for } f \in \mathcal{F}, \end{aligned}$$

provided the measure μ has full support. Moreover, $(\mathcal{E},\mathcal{F})$ is regular and local. From now on we consider the case where μ is Hausdorff measure, normalised so that $\mu(G^0) = 1$, and any following references to a Dirichlet form refer to this one.

Once we have a Dirichlet form there is an associated transition semigroup P_t, resolvent R_λ and infinitesimal generator $\mathcal{G}$. Moreover, as we have locality and regularity, there is an associated continuous strong Markov process X. This process is what we call the asymptotically one dimensional diffusion on the Sierpinski gasket. Note that this is a one parameter family of diffusions, depending on ω. We will not refer to ω in what follows though the constants obtained will be dependent on ω.

If we regard the Dirichlet form $\mathcal{E}_n^{\omega_n}$ on $L^2(G_n, \mu_n)$, where μ_n converges weakly to the Hausdorff measure on G, then we can consider the continuous-time Markov chain X^n on G_n, which has jump rate $3^n d_n(x,y)$ from x to y. ¿From the convergence of the Dirichlet forms, and using either the continuity of the local time, [Bar98] Theorem 7.2, or a tightness argument, [HK98B] Section 3, we have that

$$X_t^n \text{ converges weakly to } X_t.$$

In [HK98A], the existence of the heat kernel for X was noted. In particular the Dirichlet space is a reproducing kernel Hilbert space and the resolvent has a bounded symmetric density. As in Lemma 2.9 of [FHK94], it follows that P_t has a bounded symmetric density $p_t(x,y)$ with respect to μ, and that $p_t(x,y)$ satisfies the Chapman-Kolmogorov equations.

The Dirichlet form allows us to define the effective resistance between $x, y \in G$ as

$$R(x,y) = \inf\{\mathcal{E}(f,f) : f(x) = 0, f(y) = 1\}^{-1}.$$

As observed in [HK98A], this is a metric on the Sierpinski gasket and gives the following control on functions in the domain $\mathcal{F}$ of the form

$$|f(x) - f(y)|^2 \leq R(x,y)\mathcal{E}(f,f), \quad \forall f \in \mathcal{F}. \tag{4}$$

This shows that $\mathcal{F} \subset C(G)$, the space of continuous functions on the gasket. As in [HK98A] Section 4, the effective resistance between neighbouring points in V_n, is comparable with the resistance of edges in the electrical network on the graph G_n defined by D^n, in that for $(x,y) \in E_n$,

$$\begin{array}{rcll} c_1 2^{-n} \leq & R(x,y) & \leq c_2 2^{-n}, & \tau(x,y) = h, \\ c_3 (3/2)^{-n} \leq & R(x,y) & \leq c_4 (3/2)^{-n}, & \tau(x,y) = d. \end{array} \tag{5}$$

That is, $R(x,y) \approx 1/d_n(x,y)$.

6 Crossing times and the branching process

Let Y^n be the jump chain for X^n, then Y^n is a symmetric nearest-neighbour walk on G_n, with transition probabilities proportional to 1 for horizontal edges and ω_n for diagonal edges. We can always arrange that the X^n and thus the Y^n are 'nested'. That is, for any $m < n$, the sequence of G_m points visited by Y^n is precisely Y^m.

An MTBPVE is obtained by considering the frequencies of the steps each Y^n makes along edges of G_n. Take as our original ancestor at time 0 the first step made by Y^0. The children of this step are the steps made by Y^1 in going from Y_0^0 to Y_1^0. Continuing in this manner, the children of a step (Y_k^n, Y_{k+1}^n) are the corresponding sequence of steps made by Y^{n+1}. As the diagonal weights ω_n vary with n, we have a varying environment. Also, we need to distinguish five types of step, depending on the local geometry of G_n, giving a multi-type process.

Let $\tau(x,y) \in \{1,2,3,4,5\}$ be the type of a step along the directed edge (x,y). Types are assigned as in Figure 2. Types 1 and 3 are horizontal steps and types 2, 4 and 5 diagonal steps. As above, we will write $\tau(x,y) = h$ if $\tau(x,y) \in \{1,3\}$ and $\tau(x,y) = d$ if $\tau(x,y) \in \{2,4,5\}$.

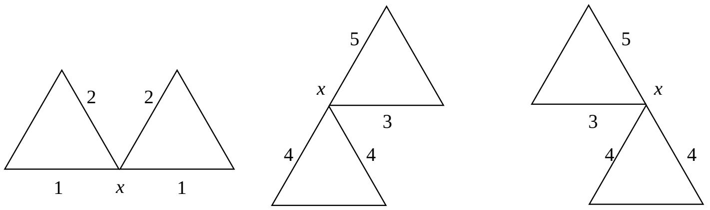

Figure 2: Types of directed edge, starting from x

For fixed $m \geq 0$, let $Z_m = \{Z_{m,n}\}_{n=0}^{\infty}$ be the branching process. Using the notation of Section 2, we have from [Jon97] that Conditions A2 ($\Rightarrow$ A) and B of Section 2.2 hold, with

$$\begin{aligned}
M_n &\equiv \begin{pmatrix} 1 & (3/4)^n & 1 & (3/4)^n & 0 \\ 1 & 1 & (4/3)^n & (3/4)^n & 1 \\ 1 & (3/4)^n & 1 & (3/4)^n & (3/4)^n \\ 1 & (3/4)^n & (4/3)^n & 1 & (3/4)^n \\ 1 & 1 & (4/3)^n & (3/4)^n & 1 \end{pmatrix}, \\
\sigma_n^2 &\equiv \begin{pmatrix} 1 & (3/4)^n & (4/3)^n & (3/4)^n & 0 \\ 1 & (3/4)^n & (4/3)^{2n} & (3/4)^n & 0 \\ 1 & (3/4)^n & (4/3)^n & (3/4)^n & (3/4)^n \\ 1 & (3/4)^n & (4/3)^{2n} & (3/4)^n & (3/4)^n \\ 1 & 1 & (4/3)^{2n} & (3/4)^n & 1 \end{pmatrix}, \\
{}^{m}R_n &\equiv (9/2)^{n-m}\mathrm{diag}((4/3)^n\ 1\ (4/3)^n\ 1\ 1) \text{ for } n \geq m+2, \\
w_n &\equiv ((3/4)^n\ 1\ (3/4)^n\ 1\ 1)^T, \text{ and} \\
\alpha_n^m &\equiv (9/2)^{n-m}.
\end{aligned}$$

Write $L_m 1^T$ for the $\mathcal{L}^2$ limit of $Z_{m,n}\,{}^{m}R_n^{-1}$ as $n \to \infty$.

Now, we define the crossing times of X for a level m of G by

$$\begin{aligned}
T_0^m &= \inf\{t \geq 0 : X_t \in G_m\}, \\
T_{i+1}^m &= \inf\{t > T_i^m : X_t \in G_m \backslash \{X_{T_i^m}\}\},
\end{aligned}$$

so that $X_{T_i^m} = Y_i^m$. Also, let $T_i^m(j)$ be the time of the ith crossing of type j, for $j = 1, \ldots, 5$, and let $j(i)$ be such that $T_i^m(j) = T_{j(i)}^m$. For $i \geq 1$ put

$$W_i^m = T_i^m - T_{i-1}^m \text{ and } W_i^m(j) = T_{j(i)}^m - T_{j(i)-1}^m.$$

These crossing times can be recovered from the branching process Z_m as limit random variables. Let $a_m \equiv 1$ be such that

$$\lim_{n\to\infty} 6^{-n} \sum_k Z_{m,n}(j,k) = a_m (9/2)^{-m} L_m(j),$$

that is, $a_m = (9/2)^m \lim_{n\to\infty} 6^{-n}({}^{m}R_n(1) + {}^{m}R_n(3))$. For $x \in V_m$, let $b = (b_H, b_D) \equiv (1,1)$ be such that

$$\lim_{n\to\infty} \frac{3^n d_n(x,x)}{6^n} = \begin{cases} b_H, & x \in V_m^H, \\ b_D, & x \in V_m^D, \end{cases}$$

where $V_m = V_m^H \cup V_m^D$, with V_m^H equal those vertices with two horizontal edges attached and V_m^D equal those vertices with only one horizontal edge. As in Figure 2, for any ordered $(x,y) \in E_m$, $x \in V_m^H \Rightarrow \tau(x,y) \in \{1,2\}$, and $x \in V_m^D \Rightarrow \tau(x,y) \in \{3,4,5\}$. Then, for $a_m^* = (9/2)^m \lim_{n\to\infty} 6^{-n}(b_H\, {}^mR_n(1) + b_D\, {}^mR_n(3)) \equiv 1$, we have for any j and $i \geq 1$

$$W_i^m(j) \overset{\mathcal{D}}{=} a_m^*(9/2)^{-m} L_m(j). \tag{6}$$

Note that $\mathrm{E}L_m(1), \mathrm{E}L_m(3) \equiv (3/4)^m$ and $\mathrm{E}L_m(2), \mathrm{E}L_m(4), \mathrm{E}L_m(5) \equiv 1$, so that the horizontal crossings are more rapid on average than the diagonal ones. The factors b_H and b_D are required to allow for the different jump rates of X^n at different types of vertex. Because the proportion of each type of step is deterministic in the limit, the only effect of these different rates is to scale L_m. Consequently, we see that, up to a constant time scaling, X_t equals $\lim_{n\to\infty} Y^n_{[6^n t]}$.

The remainder of this section applies our earlier results to get bounds on the tails of the random variables L_m. Given these, we immediately have estimates on the tails of the crossing time distributions. In particular, from (9) and (10) below, we have constants c_1, c_2, c_3 and c_4 such that for all $x \in G$ and $i \geq 1$,

$$\begin{aligned} \mathrm{P}^x(W_i^m(j) < t) &\leq \exp\{-c_1((3/4)^m((9/2)^m t)^{-\gamma_h})^{1/(1-\gamma_h)}\}\ j = 1,3,\ 0 < t < c_2(3/4)^m, & (7)\\ \mathrm{P}^x(W_i^m(j) < t) &\leq \exp\{-c_3((9/2)^m t)^{-\gamma_d/(1-\gamma_d)}\}\ j = 2,4,5,\ 0 < t < c_4, & (8) \end{aligned}$$

where $\gamma_h = \gamma_h^- = \log(3/2)/\log(9/2)$ and $\gamma_d = \log 2/\log(9/2)$.

6.1 Variance and minimal growth

From [Jon97] Lemma 15 we have that for all i and j, $M_{m,n}(i,j)/w_m(i)\, {}^mR_n(j) \to 1$ as $n \to \infty$, uniformly in m. In fact, we get, for $n \geq m+2$,

$$w_m 1^T M_{m,n} \equiv M_{m,n} \equiv \left(\frac{9}{2}\right)^{n-m} \begin{pmatrix} (4/3)^{n-m} & (3/4)^m & (4/3)^{n-m} & (3/4)^m & (3/4)^m \\ (4/3)^n & 1 & (4/3)^n & 1 & 1 \\ (4/3)^{n-m} & (3/4)^m & (4/3)^{n-m} & (3/4)^m & (3/4)^m \\ (4/3)^n & 1 & (4/3)^n & 1 & 1 \\ (4/3)^n & 1 & (4/3)^n & 1 & 1 \end{pmatrix}.$$

From this, and by checking the cases $n = m, m+1$ separately, we get for all i and $k \geq m$,

$$\sup_{j,l} M_{m,k}(i,j)\sigma^2(j,l)w_{k+1}^2(l) \equiv \left(\frac{9}{2}\right)^{k-m} w_m(i).$$

Thus from (3) we have $\mathcal{V}_m \equiv ((3/4)^m\ 1\ (3/4)^m\ 1\ 1)^T$.

Minimal growth for Z_m is simplified by the fact that $K_m = K_n$ for all m and n. For our purposes, it is sufficient to note that for $y_0 = (1\ 1\ 1\ 1\ 1)^T$, $y_1 = (1\ 0\ 1\ 0\ 0)^T$ and $y_2 = (0\ 1\ 0\ 1\ 1)^T$:

$$\mathcal{K}_n(y_0) = 2y_0,\ \mathcal{K}_n(y_1) = 2y_1 \text{ and } \mathcal{K}_n(y_2) = 2y_2.$$

Moreover, in each case $\mathcal{K}_n(y_i) = Ay_i$ where

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

In particular, since $\mathcal{K}_n(y_0) = 2y_0 > 0$, we have that for any $y > 0$, $\mathcal{K}_{m,n}(y) \equiv 2^{n-m}y$.

6.2 Upper bound

Fix $m \geq 0$. From Section 3.1 and the above, we can take

$$t_n \equiv 1,\ y_n \equiv ((3/4)^n\ 1\ (3/4)^n\ 1\ 1)^T \text{ and } z_n \equiv 2^{n-m}((3/4)^n\ 1\ (3/4)^n\ 1\ 1)^T.$$

Now take cases.

(i) $i = 1, 3$. We have

$$\gamma_{m,n}(i) = \frac{\log[c_1(3/2)^{n-m}(3/4)^m]}{\log[c_2(9/2)^{n+1-m}]} \xrightarrow{n\to\infty} \frac{\log(3/2)}{\log(9/2)} =: \gamma_h^- \simeq 0.2696,$$

whence, for $\epsilon = 0$,

$$\epsilon_n(i) \leq \frac{-\log[c_3(3/4)^m]}{\log[c_2(9/2)^{n+1-m}]} \Rightarrow b_m(i) = -\log[c_3(3/4)^m].$$

So, for $s \geq 1$ we have $-\log \phi_m^i(s) \geq c_3(3/4)^m s^{\gamma_h^-}$, and thus for $l \leq l_m(i) = c_4(3/4)^m$,

$$\mathrm{P}(L_m(i) < l) \leq \exp\{-c_5(3/4)^{m/(1-\gamma_h^-)} l^{-\gamma_h^-/(1-\gamma_h^-)}\}. \tag{9}$$

(ii) $i = 2, 4, 5$. We have

$$\gamma_{m,n}(i) = \frac{\log[c_1 2^{n-m}]}{\log[c_2(9/2)^{n+1-m}]} \xrightarrow{n\to\infty} \frac{\log 2}{\log(9/2)} =: \gamma_d \simeq 0.4608,$$

whence, for $\epsilon = 0$,

$$\epsilon_n(i) \leq \frac{c_3}{\log[c_2(9/2)^{n+1-m}]} \Rightarrow b_m(i) = c_3.$$

So, for $s \geq 1$ we have $-\log \phi_m^i(s) \geq e^{-c_3} s^{\gamma_d}$, and thus for $l \leq l_m(i) = c_4$,

$$\mathrm{P}(L_m(i) < l) \leq \exp\{-c_5 l^{-\gamma_d/(1-\gamma_d)}\}. \tag{10}$$

6.3 Lower bound

Fix $m \geq 0$. From Section 4.1 and the above, we can take

$$y_n \equiv t_n((3/4)^n\ 1\ (3/4)^n\ 1\ 1)^T \text{ and } z_n \equiv 2^{n-m}t_n((3/4)^n\ 1\ (3/4)^n\ 1\ 1)^T,$$

and we have ${}^nA_{m,n} = A^{n-m}$ for A as above. Given A, it easy to check that

$${}^n\pi_k \equiv (1/2\ (3/4)^k\ 1/2\ (3/4)^k\ (3/4)^k)^T,$$

whence, for some positive constants $c_1, \ldots, c_4$,

$$\begin{aligned} -\log p_n(i) &\leq \sum_{k=m}^{n-1} A^{k-m}(c_1 y_1 + k c_2 y_2) \\ &\leq c_3 2^{n-m} y_1 + c_4 n 2^{n-m} y_2. \end{aligned}$$

(Here we have used the identity $\sum_{k=m}^{n-1} kx^{k-m} = ([(n-1)(x-1)-1]x^{n-m} - [(m-1)(x-1)-1])/(x-1)^2$.) Now take cases

(i) $i = 1, 3$. Take $t_n \equiv (4/3)^n$, then

$$\gamma_{m,n}(i) = \frac{\log[c_1 2^{n-m}]}{\log[c_2 6^{n-1-m}(4/3)^m]} \xrightarrow{n\to\infty} \frac{\log 2}{\log 6} =: \gamma_h^+ \simeq 0.3869$$

whence, for $\epsilon = 0$,

$$\epsilon_n(i) \leq \frac{\log[2c_1]}{\log[c_2 6^{n-1-m}(4/3)^m]} \Rightarrow b_m(i) = \log[2c_1].$$

So, for $s \geq (4/3)^m$ we have $-\log \phi_m^i(s) \leq 2c_1 s^{\gamma_h^+}$, and thus for $l \leq l_m(i) = c_3(3/4)^{m(1-\gamma_h^+)}$,

$$\mathrm{P}(L_m(i) < l) \geq \tfrac{1}{2} \exp\{-c_4 l^{-\gamma_h^+/(1-\gamma_h^+)}\}$$

(ii) $i = 2, 4, 5$ Take $t_n \equiv 1$, then

$$\gamma_{m,n}(i) = \frac{\log[c_1 n 2^{n-m}]}{\log[c_2 (9/2)^{n-1-m}]} \xrightarrow{n\to\infty} \frac{\log 2}{\log(9/2)} =: \gamma_d \simeq 0.4608,$$

and, for $\epsilon = 0$,

$$\epsilon_n(i) \leq \frac{\log[2c_1 n]}{\log[c_2 (9/2)^{n-1-m}]}.$$

Thus, we must take $\epsilon > 0$ and $\gamma = \gamma_d + \epsilon =: \gamma_{d,\epsilon}$, which gives

$$\epsilon_n(i) \leq \frac{\log[2c_1 n] - \log[(9/2)^{(n-1-m)\epsilon}]}{\log[c_2 (9/2)^{n-1-m}]} \Rightarrow b_m(i) = \log[c_3(\epsilon)(9/2)^{m\epsilon}].$$

So, for $s \geq 1$ we have $-\log \phi_m^i(s) \leq c_3(\epsilon)(9/2)^{m\epsilon} s^{\gamma_{d,\epsilon}}$, and thus for $l \leq l_m(i) = 2c_3(\epsilon)(9/2)^{m\epsilon}$,

$$\mathrm{P}(L_m(i) < l) \geq \tfrac{1}{2} \exp\{-c_4(\epsilon)(9/2)^{m\epsilon/(1-\gamma_{d,\epsilon})} l^{-\gamma_{d,\epsilon}/(1-\gamma_{d,\epsilon})}\}$$

Here $c_3(\epsilon), c_4(\epsilon) \to \infty$ as $\epsilon \to 0$.

7 THE SPECTRAL DIMENSION

We prove that the spectral dimension for the asymptotically one dimensional diffusion on the Sierpinski gasket is given by $d_s = 2\log 3/\log(9/2)$. In particular, if $N(\lambda)$ denotes the eigenvalue counting function for the operator $\mathcal{G}$, then we have that

$$\lim_{\lambda\to\infty} \frac{\log N(\lambda)}{\log \lambda} = \frac{\log 3}{\log(9/2)}.$$

In the case of the Brownian motion on the Sierpinski gasket it is known, [FS92], that there is oscillation in $N(\lambda)$ related to localisation of the eigenfunctions [BK97], in that

$$\liminf_{\lambda\to\infty} N(\lambda)\lambda^{-d_s/2} < \limsup_{\lambda\to\infty} N(\lambda)\lambda^{-d_s/2}.$$

The techniques that we use here involve estimation of the heat kernel, and are too crude to give us such precise results.

Define the m-complexes of G^0 to be the subsets $\{\psi_a(G^0) : a \in A^m\}$. The m-complexes of G are obtained by translation. If $x \in G_m$ then it belongs to precisely two m-complexes, while if $x \notin G_m$ then it only belongs to one. Define $\triangle_m(x)$ to be the maximal subset of G, containing

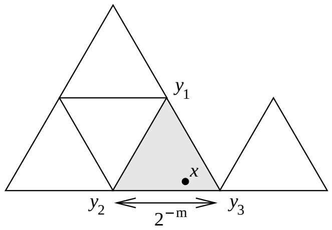

Figure 3: Typical $D_m(x)$ for $x \notin G_m(x)$, with $\triangle_m(x)$ shaded and $N_m(x) = \{y_1, y_2, y_3\}$

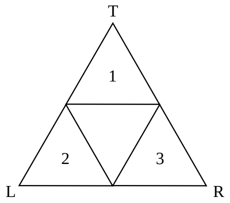

Figure 4: The three $(n+1)$-complexes making up an n-complex, and its three boundary vertices

the point x, with its boundary consisting of neighbouring points in G_m. If $x \notin G_m$ then $\triangle_m(x)$ is an m-complex, else if $x \in G_m$ then $\triangle_m(x)$ is the union of the two m-complexes containing x. Let $N_m(x) = \partial\triangle_m(x)$, then $N_m(x) = \{y \in G_m \backslash \{x\} : |x-y|_G \leq 2^{-m}\}$, where $|x-y|_G$ is the Euclidean shortest path distance between x and y. The level m neighbourhood of x is then given by $D_m(x) = \cup_{y \in N_m(x)} \triangle_m(y)$. Figure 3 illustrates a typical $D_m(x)$ for $x \notin G_m$.

Let $n^d_{m,k}(x)$ denote the number of diagonal steps in the shortest path on G_{m+k} from x to $\partial D_m(x)$. It is easy to see that there is an upper bound of 2^{k+1} on the number of diagonal steps. We obtain an almost everywhere lower bound on the number of diagonal steps of the same order, which holds for all m.

Lemma 6 *For any $\alpha > \log 2/\log(3/2)$, we have that for μ-a.e. $x \in G$ there exists $K(x) < \infty$ such that*

$$(mk)^{-\alpha} 2^k \leq n^d_{m,k}(x) \leq 2^{k+1}, \text{ for all } m \geq 0 \text{ and } k \geq K(x).$$

Proof W.l.o.g. we can assume $x \in G^0$. A point $x \in G^0$, chosen according to μ, can be represented by an infinite sequence of independent random variables $\{I_n\}_{n=1}^{\infty}$, taking values in $A = \{1, 2, 3\}$ with the uniform distribution. If we label the subcomplexes of any n-complex as in Figure 4, so that 1 stands for the top triangle, then $I_{n+1} = i$ implies x is in subcomplex i of $\triangle_n(x)$.

Write $(G^0, \mathcal{B}(G^0), \mathbb{P})$ for the probability space determined by the $\{I_n\}$. Clearly $\mathbb{P}(\cup_{n=0}^{\infty} G^0_n) = 0$, so we can consider only $x \in G^0 \backslash \cup_{n=0}^{\infty} G^0_n$, giving a μ-a.e. result.

The sequence of n-complexes, $m \leq n \leq m+k$, traversed by the shortest path from x to $\partial D_m(x)$ is just $\triangle_{m+k}(x) \subset \triangle_{m+k-1}(x) \subset \cdots \subset \triangle_m(x)$. Label the boundary points of an n-complex T, L and R (for top, left and right) as in Figure 4. Let x_n, $n = m+k, m+k-1, \ldots, m$, be the boundary point at which the shortest path from x to $\partial D_m(x)$ exits $\triangle_n(x)$, and let $b_n \in \{T, L, R\}$ be the label of that point. It is easily seen that $b_{m+k} = b_{m+k-1} = \cdots = b_m$. Moreover, $b_m = T, L, R$ if and only if $I_{m+1} = 1, 2, 3$ respectively.

In moving from x_m to $\partial D_m(x)$, we do not necessarily make any diagonal moves, depending on the shape of $D_m(x)$. We make diagonal moves in moving from x_n to x_{n-1} if and only if $b_{n-1} = T$ and $I_n \in \{2,3\}$, or $b_{n-1} \in \{L, R\}$ and $I_n = 1$. In either case we get precisely 2^{m+k-n} diagonal steps on G_{m+k}. Thus

$$n^d_{m,k}(x) \geq \sum_{n=m+2}^{m+k} 2^{m+k-n} \left(1_{(I_{m+1}=1, I_n \in \{2,3\})} + 1_{(I_{m+1} \in \{2,3\}, I_n = 1)}\right).$$

Thus, for $2 \leq l \leq k$,

$$\begin{aligned}
\mathbb{P}(n^d_{m,k}(x) \leq 2^{k-l}) &\leq \mathbb{P}(I_{m+1} = 1)\mathbb{P}(I_{m+2}, \ldots, I_{m+l} \notin \{2,3\}) \\
&\quad + \mathbb{P}(I_{m+1} \in \{2,3\})\mathbb{P}(I_{m+2}, \ldots, I_{m+l} \neq 1) \\
&= (\tfrac{1}{3})^{l-1} + (\tfrac{2}{3})^{l-1} \\
&\leq c_0 (\tfrac{2}{3})^{l-1}.
\end{aligned}$$

Writing the above in terms of a parameter δ we obtain, for $\alpha_0 = \log 2/\log(3/2)$,

$$\mathbb{P}(2^{-k} n^d_{m,k}(x) \leq \delta) \leq c_1 \delta^{1/\alpha_0}, \text{ for } 0 < \delta < 1.$$

In particular, for $\epsilon > 0$

$$\begin{aligned}
\mathbb{P}(2^{-k} n^d_{m,k}(x) \leq \delta m^{-\alpha_0-\epsilon}, \text{ for some } m \geq 0) &\leq \sum_{m=0}^{\infty} \mathbb{P}(2^{-k} n^d_{m,k}(x) \leq \delta m^{-\alpha_0-\epsilon}) \\
&\leq \sum_{m=0}^{\infty} c_1 (\delta m^{-\alpha_0-\epsilon})^{1/\alpha_0} \\
&= c_2 \delta^{1/\alpha_0}.
\end{aligned}$$

Now let $\delta = k^{-\alpha_0-\epsilon}$ and apply Borel-Cantelli to show that $(mk)^{-\alpha_0-\epsilon} 2^k \leq n^d_{m,k}(x)$ for all $m \geq 0$ eventually. Putting $\alpha = \alpha_0 + \epsilon$ gives the result. ▮

Define the exit time from a neighbourhood as

$$T_{D_m(x)} = \inf\{t \geq 0 : X_t \in G \backslash D_m(x)\}.$$

We use our bound (10) to estimate the tail probabilities of $T_{D_m(x)}$.

Lemma 7 *For any* $\alpha > \log 2/\log(3/2)$, *there exists a constant* $c_{7.1}$ *such that, for* μ*-a.e.* $x \in G$, *there exists a* $t_0(x) > 0$ *such that for all* $m \geq 0$ *and* $0 < t < t_0(x) m^{-\alpha/\gamma_d} (9/2)^{-m}$,

$$\mathrm{P}^x(T_{D_m(x)} < t) \leq \exp\{-c_{7.1}(-\log[m^{\alpha/\gamma_d}(9/2)^m t])^{-\alpha/(1-\gamma_d)} (m^{\alpha/\gamma_d}(9/2)^m t)^{-\gamma_d/(1-\gamma_d)}\},$$

where $\gamma_d = \log 2/\log(9/2)$.

Proof Let $n^j_{m,k}(x)$ be the number of type j steps in G_{m+k} on the shortest path from x to $\partial D_m(x)$, so that $n^d_{m,k}(x) = n^2_{m,k}(x) + n^4_{m,k}(x) + n^5_{m,k}(x)$. Then, for any k,

$$\begin{aligned}
T_{D_m(x)} &\geq \sum_{l=1}^{n^2_{m,k}(x)} W^{m+k}_l(2) + \sum_{l=1}^{n^4_{m,k}(x)} W^{m+k}_l(4) + \sum_{l=1}^{n^5_{m,k}(x)} W^{m+k}_l(5) \\
&= T^{d,m+k}_{D_m(x)} \text{ say.}
\end{aligned}$$

The components of this sum are independent given $n^j_{m,k}(x)$ for $j = 2, 4, 5$. Thus if we let $\phi^d_{m,k}(\theta) = \mathrm{E}\exp\{-\theta T^{d,m+k}_{D_m(x)}\}$, then from (6), (10) and Lemma 6, we have for $k \geq K(x)$ and $\theta \geq (9/2)^{m+k}$,

$$-\log \phi^d_{m,k}(\theta) \geq c_1(mk)^{-\alpha}2^k((9/2)^{-(m+k)}\theta)^{\gamma_d} = c_1(mk)^{-\alpha}2^{-m}\theta^{\gamma_d}.$$

Thus, as in Proposition 3, for $0 < t < t_1$, where $t_1 = c_2(mk)^{-\alpha}2^{-m}(9/2)^{-(m+k)(1-\gamma_d)} = c_2(mk)^{-\alpha}(9/2)^{-m}(9/4)^{-k}$, we have

$$\begin{aligned}\mathrm{P}(T^{d,m+k}_{D_m(x)} < t) &\leq \exp\{-c_3(mk)^{-\alpha/(1-\gamma_d)}2^{-m/(1-\gamma_d)}t^{-\gamma_d/(1-\gamma_d)}\} \\ &= \exp\{-c_3((mk)^{\alpha/\gamma_d}(9/2)^m t)^{-\gamma_d/(1-\gamma_d)}\}.\end{aligned}$$

Now take $t_2 = c_2(mK(x))^{-\alpha/\gamma_d}(9/2)^{-m}(9/4)^{-K(x)}$ and $k \geq K(x)$ such that

$$(9/4)^k k^{\alpha/\gamma_d} \leq c_2 m^{-\alpha/\gamma_d}(9/2)^{-m}t^{-1} < (9/4)^{k+1}(k+1)^{\alpha/\gamma_d}, \tag{11}$$

which we can do provided $t < t_2$. Note that for this k, $t \leq c_2(mk)^{-\alpha/\gamma_d}(9/2)^{-m}(9/4)^{-k} < t_1$. Equation (11) now gives

$$(mk)^{\alpha/\gamma_d}(9/2)^m t \leq (9/4)^{-k} \leq c_4 m^{\alpha/\gamma_d}(9/2)^m t(-\log[m^{\alpha/\gamma_d}(9/2)^m t])^{\alpha/\gamma_d}.$$

Finally, put $t_0(x) = t_2 m^{\alpha/\gamma_d}(9/2)^m = c_2 K(x)^{-\alpha/\gamma_d}(9/4)^{-K(x)}$ to give the result. ▮

Proposition 8 *If there is an increasing continuous function f on $[0, t_0(x)]$, such that $f(t) \to 0$ as $t \to 0$ and for some $\zeta > 0$ and $\xi > 1$,*

$$\mathrm{P}^x(T_{D_m(x)} < t) \leq f(m^\zeta \xi^m t), \;\; 0 < t < t_0(x)m^{-\zeta}\xi^{-m}, \text{ for all } m \geq 0 \text{ and } \mu\text{-a.e. } x \in G,$$

then there exist constants $c_{7.2}$ and $t_1(x)$ such that for μ-a.e. $x \in G$,

$$p_t(x,x) \geq c_{7.2}t^{-\log 3/\log\xi}|\log t|^{-\zeta/\log\xi}, \;\; 0 < t < t_1(x).$$

Proof We follow [HK98A]. Let $0 < a < t_0(x)$ be such that $f(a) \leq 1/2$. Then, for μ-a.e. $x \in G$, if $t < a/\xi =: t_1(x)$ and $m = \sup\{k : k^\zeta\xi^k t < a\}$, then

$$\mathrm{P}^x(X_t \in D_m(x)) \geq \mathrm{P}^x(T_{D_m(x)} > t) \geq 1/2.$$

For this value of m we have $m + 1 \geq \log(a/t)/\log\xi - (\zeta/\log\xi)\log\log(a/t)$. By Cauchy-Schwarz

$$\begin{aligned}1/4 &\leq \mathrm{P}^x(X_t \in D_m(x))^2, \\ &= \left(\int_{D_m(x)} p_t(x,y)\mu(dy)\right)^2, \\ &\leq \mu(D_m(x))p_{2t}(x,x).\end{aligned} \tag{12}$$

Thus, as there is a constant c_1 such that $\mu(D_m(x)) \leq c_1 3^{-m}$ for all $m \in \mathbb{N}$, we have $\mu(D_m(x)) \leq c_2(t/a)^{\log 3/\log\xi}(\log(a/t))^{\zeta/\log\xi}$. Substituting this in (12) gives the result. ▮

Thus our control on the exit distribution allows us to find a lower bound on the on-diagonal heat kernel for μ-a.e. $x \in G$.

Corollary 9 *For any* $\beta > 1/\log(3/2)$*, there exists a constant* $c_{7.3}$ *such that, for* μ*-a.e.* $x \in G$*, there exists a* $t_1(x) > 0$ *such that for all* $0 < t < t_1(x)$*,*

$$p_t(x,x) \geq c_{7.3} t^{-d_s/2} |\log t|^{-\beta},$$

where $d_s = 2\log 3/\log(9/2)$*.*

Proof We combine the exit time estimate of Lemma 7 for μ-a.e. $x \in G$, with Proposition 8. Here $\beta = \alpha/(\gamma_d \log(9/2))$ for $\alpha > \log 2/\log(3/2)$. ∎

In addition, from [HK98A] Lemma 5.1, we know that a Nash inequality holds, and hence we also have a pointwise upper bound for the transition density. By writing the exponent d_β from [HK98A] as d_s we have the following Theorem.

Theorem 10 *There exists a constant* $c_{7.4}$ *such that*

$$p_t(x,y) \leq c_{7.4} t^{-d_s/2}, \ \ \forall x, y \in G, \ \ 0 < t < 1.$$

As in [FHK94] Lemma 4.6, we can now use the symmetry of $p_t(x,y)$, and the fact that it satisfies the Chapman-Kolmogorov equations, to deduce that $p_t(x,y)$ is jointly continuous in x, y for each t. In fact, $p_t(x,y)$ is uniformly continuous on $G \times G$ for each $t > 0$, as the following lemma establishes.

Lemma 11 *There exist a constant* $c_{7.5}$ *such that*

$$\begin{aligned} |p_t(x,y) - p_t(x',y)| &\leq c_{7.5} R(x,x')^{1/2} \sqrt{t^{-1} p_t(y,y)} \\ &\leq c_{7.5} R(x,x')^{1/2} t^{-(1+d_s/2)/2}, \ \ \forall x, x', y \in G, \ \ 0 < t < 1. \end{aligned} \tag{13}$$

Proof Recall that the resistance metric, $R(x,y)$, provides a bound (4) on functions in the domain $\mathcal{F}$. In particular for the transition density we have

$$|p_t(x,y) - p_t(x',y)|^2 \leq R(x,x') \mathcal{E}(p_t(\cdot,y), p_t(\cdot,y)). \tag{14}$$

As in [FHK94] Lemma 6.4, we have, writing $u(x) = p_{t/2}(x,y)$,

$$\mathcal{E}(P_{t/2}u, P_{t/2}u) \leq c_1 (t/2)^{-1} \|u\|_2^2 \leq 2c_1 t^{-1} p_t(y,y) \leq c_2 t^{-1-d_s/2}.$$

and putting this into (14) gives (13). ∎

As in [BH97] Section 7, the uniform continuity of $p_t(\cdot,\cdot)$ implies that P_t is a compact operator on $L^2(G,\mu)$, so that P_t and thus the infinitesimal generator have discrete spectra. Write the eigenvalues of the infinitesimal generator in ascending order as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, and let $N(\lambda) = \#\{\lambda_i : \lambda_i \leq \lambda\}$ denote the eigenvalue counting function for this operator.

Theorem 12 *For any* $\beta > 1/\log(3/2)$*, there exist constants* $c_{7.6}, c_{7.7}, c_{7.8}$ *such that*

$$c_{7.6} \lambda^{d_s/2} (\log \lambda)^{-\beta} \leq N(\lambda) \leq c_{7.7} \lambda^{d_s/2}, \ \ \lambda > c_{7.8}, \tag{15}$$

where $d_s = 2\log 3/\log(9/2)$*.*

Proof As in [BH97] Section 7, we have the following relationship between the integrated heat kernel and the Laplace transform of $N(\lambda)$:

$$\int_G p_t(x,x)\mu(dx) = \int_0^\infty e^{-st} dN(s).$$

Our upper bound on the heat kernel from Theorem 10 gives

$$\int_0^\infty e^{-st} dN(s) \leq c_2 t^{-d_s/2}.$$

Let $B_r = \{x : t_1(x) > r\}$. Then the almost sure lower bound of Corollary 9 gives us

$$c_{7.3}\mu(B_r) t^{-d_s/2} |\log t|^{-\beta} \leq \int_0^\infty e^{-st} dN(s), \quad 0 < t < r,$$

Note that $c_{7.3} > 0$ and, by construction of $t_1(x)$, the set B_r has positive measure for small enough r. Standard Tauberian ideas, as used in [BH97], allow us to invert the transform to obtain (15) for $\lambda > 1/r$, as required. ∎

By taking logarithms in (15), dividing by $\log\lambda$, and letting $\lambda \to \infty$ we see that the spectral dimension is indeed d_s.

Corollary 13 *The spectral dimension for the asymptotically one dimensional diffusion process is given by* $d_s = 2\log 3/\log(9/2)$ *in that*

$$\lim_{\lambda\to\infty} \frac{\log N(\lambda)}{\log\lambda} = \frac{d_s}{2}.$$

8 HEAT KERNEL UPPER BOUNDS

The next aim is to prove almost sure heat kernel bounds. From the previous work on this process [HK98A] we have on-diagonal upper and lower bounds. In the last section we obtained an on-diagonal lower bound which agreed with the upper bound, up to logarithmic corrections, for almost every point in G under μ. This indicates that the typical heat kernel behaviour should correspond to a walk dimension given by

$$d_w = \frac{2d_f}{d_s} = \frac{\log(9/2)}{\log 2} = \frac{1}{\gamma_d}.$$

However, we will show that there are points in the fractal — lying on horizontal line segments — where the heat flow is much more rapid and there is a different walk dimension, given by

$$d'_w = \frac{\log(9/2)}{\log(3/2)} = \frac{1}{\gamma_h}.$$

In what follows we extend the heat kernel upper bounds off the diagonal, and show that, for most points, the decay is determined by d_w, but for some points is determined by d'_w.

It is necessary to consider the short paths in the fractal. As observed previously, these paths are composed of steps of two different types. Label the G_m points of the shortest path from x to y by

$$\pi_m(x,y) = \{x_i : x_0 = x,\ x_i \in G_m, i = 1,\dots,n_m,\ x_{n_m+1} = y\},$$

where $n_m = n_m(x,y) = |\pi_m(x,y)| - 2$. We partition the edges of the shortest path from x to y into horizontal and diagonal components. Noting that (x_0, x_1) and (x_{n_m}, x_{n_m+1}) are not generally edges in G_m, we put

$$\epsilon_m^h(x,y) = \{(x_i, x_{i+1}) : \tau(x_i, x_{i+1}) = h, i = 1, \ldots, n_m - 1\}, \;\; n_m^h = |\epsilon_m^h|,$$

$$\epsilon_m^d(x,y) = \{(x_i, x_{i+1}) : \tau(x_i, x_{i+1}) = d, i = 1, \ldots, n_m - 1\}, \;\; n_m^d = |\epsilon_m^d|.$$

So $n_m = n_m^h + n_m^d + 1$. Let $|x-y|_G$ denote the Euclidean shortest path distance between two points $x, y \in G$, where the path is restricted to lie in the set G. This distance can be decomposed into a horizontal and a diagonal part. We define

$$|x-y|_d = \lim_{m\to\infty} 2^{-m} n_m^d(x,y), \;\; |x-y|_h = \lim_{m\to\infty} 2^{-m} n_m^h(x,y),$$

then $|x-y|_G = |x-y|_h + |x-y|_d$. The distance in the set G is comparable to the Euclidean distance.

We are now ready to state our upper bound.

Theorem 14 *There exist constants* $c_{8.1}, c_{8.2}, c_{8.3}, c_{8.4}, c_{8.5}$ *such that if* $|x-y|_h > 0$ *and* $0 < t < \min\{c_{8.1}, c_{8.2}|x-y|_h^{\log(4/3)/\log 2}\}$, *or if* $|x-y|_h = 0$ *and* $0 < t < c_{8.2}$, *then*

$$p_t(x,y) \le c_{8.3} t^{-d_s/2} \min\{\exp(-c_{8.4}(|x-y|_d^{d_w} t^{-1})^{1/(d_w-1)}),\ \exp(-c_{8.5}(|x-y|_h^{d'_w} t^{-1})^{1/(d'_w-1)})\}.$$

Proof This result follows from an extension of standard techniques. Fix x, y, t and let $\epsilon > 0$ be sufficiently small. We let ν_x, ν_y be the Hausdorff measure restricted to $B_\epsilon(x), B_\epsilon(y)$, the ball of radius ϵ about x, y respectively, using the shortest path metric $|\cdot - \cdot|_G$. We consider the set $A(x) = \{z : |x-z|_i > \frac{1}{3}|x-y|_i, i = d, h\}$ and assume that $\epsilon < \frac{1}{6}\min(|x-y|_i : i = h, d)$ unless $|x-y|_i = 0$, in which case we take the maximum. We write

$$\begin{aligned} \mathrm{P}^{\nu_x}(X_t \in B_\epsilon(y)) &= \mathrm{P}^{\nu_x}(X_t \in B_\epsilon(y), X_{t/2} \in A(x)) + \mathrm{P}^{\nu_x}(X_t \in B_\epsilon(y), X_{t/2} \in G \backslash A(x)), \\ &= I_1 + I_2. \end{aligned} \tag{16}$$

For the first term

$$\begin{aligned} \mathrm{P}^{\nu_x}(X_{t/2} \in A(x)) &\le \max_{z \in B_\epsilon(x)} \mathrm{P}^z(X_{t/2} \in A(x)) \mu(B_\epsilon(x)) \\ &= \max_{z \in B_\epsilon(x)} \mathrm{P}^z(|X_{t/2} - x|_d > \tfrac{1}{3}|x-y|_d, |X_{t/2} - x|_h > \tfrac{1}{3}|x-y|_h) \mu(B_\epsilon(x)) \\ &\le \max_{z \in B_\epsilon(x)} \min\big\{\mathrm{P}^z(|X_{t/2} - x|_d > \tfrac{1}{3}|x-y|_d), \\ &\qquad\qquad \mathrm{P}^z(|X_{t/2} - x|_h > \tfrac{1}{3}|x-y|_h)\big\} \mu(B_\epsilon(x)) \end{aligned}$$

We consider each term in the minimum separately. For the diagonal estimate, let $m = \inf\{k : D_k(x) \subset B_{|x-y|_d/3}(x)\}$. Then for $t < c_1$

$$\begin{aligned} \mathrm{P}^z(|X_{t/2} - x|_d > \tfrac{1}{3}|x-y|_d) &\le \mathrm{P}^z(T_{B_{|x-y|_d/3}(x)} < t/2) \\ &\le \mathrm{P}^z(T_{D_m(x)} < t/2) \\ &\le \max_{j=2,4,5} \mathrm{P}^z(W_1^m(j) < t/2) \\ &\le \exp(-c_2((9/2)^m t)^{-1/(d_w-1)}). \end{aligned} \tag{17}$$

By our choice of m we have $2^{-m} \leq |x-y|_d$ whence $(9/2)^m \leq |x-y|_d^{-d_w}$. Applying this in (17) we have

$$\mathrm{P}^z(|X_{t/2} - x|_d > \tfrac{1}{3}|x-y|_d) \leq \exp(-c_3(|x-y|_d^{d_w} t^{-1})^{1/(d_w-1)}) = J_1.$$

For the horizontal estimate we choose $m = \inf\{k : D_k(x) \subset B_{|x-y|_h/3}(x)\}$. In the same way as above, by choice of m we get the existence of constants c_4 and c_5 such that, for $t < c_4|x-y|_h^{\log(4/3)/\log 2}$,

$$\mathrm{P}^z(|X_{t/2} - x|_h > \tfrac{1}{3}|x-y|_h) \leq \exp(-c_5(|x-y|_h^{d'_w} t^{-1})^{1/(d'_w-1)}) = J_2.$$

Thus

$$\mathrm{P}^{\nu_x}(X_{t/2} \in A(x)) \leq \min\{J_1, J_2\}\mu(B_\epsilon(x)).$$

Let $q(z) = \mathrm{P}(X_t \in B_\epsilon(y) \,|\, X_{t/2} = z)$, then using the on-diagonal bound we have

$$q(z) = \int_{B_\epsilon(y)} p_{t/2}(z,w)\mu(dw) \leq c_6 t^{-d_s/2}\mu(B_\epsilon(y)).$$

We now use these two expressions in (16). For I_1 we have

$$\begin{aligned} I_1 &= \mathrm{E}^{\nu_x}(q(X_{t/2}) : X_{t/2} \in A(x)) \\ &\leq c_6\mu(B_\epsilon(x))\mu(B_\epsilon(y))t^{-d_s/2}\min\{J_1, J_2\}. \end{aligned}$$

For I_2, by the symmetry of $p_t(x,y)$,

$$\mathrm{P}^{\nu_x}(X_t \in B_\epsilon(y), X_{t/2} \in G\backslash A(x)) = \mathrm{P}^{\nu_y}(X_t \in B_\epsilon(x), X_{t/2} \in G\backslash A(x))$$

which is bounded in exactly the same way as I_1. Adding the bounds for I_1 and I_2 gives

$$\mathrm{P}^{\nu_x}(X_t \in B_\epsilon(y)) \leq 2c_6\mu(B_\epsilon(x))\mu(B_\epsilon(y))t^{-d_s/2}\min\{J_1, J_2\}. \tag{18}$$

Now divide by $\mu(B_{\epsilon(x)})\mu(B_\epsilon(y))$ and use the continuity of $p_t(x,y)$ to get the result. ▮

Remark 15 1. The diagonal term will dominate this bound in the sense that if we fix x, y with $|x-y|_d > 0$, then there exists a $t_2(x,y) > 0$ such that,

$$p_t(x,y) \leq c_{8.3} t^{-d_s/2}\exp(-c_{8.4}(|x-y|_d^{d_w} t^{-1})^{1/(d_w-1)}), \;\; t < t_2(x,y).$$

2. If we consider points which are connected horizontally, in that $|x-y|_d = 0$, then for $0 < t < c_{8.2}|x-y|_h^{\log(4/3)/\log 2}$,

$$p_t(x,y) \leq c_{8.3} t^{-d_s/2}\exp(-c_{8.5}(|x-y|_h^{d'_w} t^{-1})^{1/(d'_w-1)}).$$

As a corollary to our earlier estimate on the exit time distribution for a triangle neighbourhood, we can obtain an almost everywhere upper bound.

Corollary 16 *Let $\mu \otimes \mu$ be the product measure on $G \times G$. For any $\gamma > d'_w$, there exist constants $c_{8.6}, c_{8.7}$ such that for $\mu\otimes\mu$-a.e. (x,y) we have, for $0 < t < (t_0(x) \wedge t_0(y))|x-y|^{d_w}(-\log|x-y|)^{-\gamma}$,*

$$\begin{aligned} p_t(x,y) \leq c_{8.6} t^{-d_s/2}\exp\{&-c_{8.7}((-\log|x-y|)^{-\gamma}|x-y|^{d_w}t^{-1})^{1/(d_w-1)} \\ &\times(\log[(-\log|x-y|)^{-\gamma}|x-y|^{d_w}t^{-1}])^{-\gamma/(d_w-1)}\}. \end{aligned}$$

Proof We follow the same steps as before, but just use the almost sure estimate for the exit time distribution obtained in Lemma 7. Here $\gamma = \alpha d_w$ for $\alpha > \log 2/\log(3/2)$. ▮

9 HEAT KERNEL LOWER BOUNDS

The next issue is to obtain some control on the lower bound. The fact that there is different behaviour for different types of points in the fractal means that we cannot hope for a uniformly tight lower bound in the Euclidean metric. Again we have to consider points depending on how close they are to the horizontal. We have already demonstrated an almost everywhere lower bound for the on diagonal. We will now control this for all points. Note that if the points x, y are sufficiently close to a horizontal line, we would expect that for fixed time the decay of the heat kernel with distance should be slower than the typical behaviour.

For μ-a.e. $x \in G$, we have a lower bound on $p_t(x,x)$. In [HK98A] a worst case index, d_γ, was derived which gave a uniform lower bound on the diagonal. However by using our branching process bounds we find a better worst case index, which we write as

$$d_s' = \frac{2\log 3 \log(3/2)}{\log 2 \log(9/2)} = \frac{2d_f}{d_w'} < d_s.$$

Lemma 17 *There exist constants* $c_{9.1}$ *and* $c_{9.2}$ *such that for* $0 < t < c_{9.1} < 1$,

$$p_t(x,x) \geq c_{9.2} t^{-d_s'/2}, \quad x \in G. \tag{19}$$

Proof As a horizontal crossing is possible within $D_m(x)$ we have $\{T_{D_m(x)} < t\} \subset \{W_1^m(j) < t,\ j = 1 \text{ or } 3\}$. Thus, using the crossing time estimate (7), for $t < c_0(3/4)^m$,

$$\mathrm{P}^x(T_{D_m(x)} < t) \leq c_1 \exp(-c_2((3/4)^{-m/\gamma_h}(9/2)^m t)^{-\gamma_h/(1-\gamma_h)}).$$

Now, by taking $\xi = (3/4)^{-1/\gamma_h}(9/2) = (9/2)^{\log 2/\log(3/2)}$ and $\zeta = 0$ in Proposition 8, noting that $c_0(3/4)^m > c_0 m^{-\zeta}\xi^{-m}$, we have our uniform lower bound on the transition density. ∎

Combining this with Corollary 9 and Lemma 11, we get the following.

Lemma 18 *For any* $\beta > 1/\log(3/2)$, *there exist* $c_{9.3}, c_{9.4} > 0$ *such that for* μ*-a.e.* $x \in G$, *if* $0 < t < t_1(x)$, *then*

$$p_t(x,y) \geq c_{9.3} t^{-d_s/2} |\log t|^{-\beta} \text{ whenever } R(x,y) \leq c_{9.4} t^{1/d_w'} |\log t|^{-2\beta}. \tag{20}$$

There exist constants $c_{9.5}, c_{9.6}, c_{9.7} > 0$ *such that for all* $x, y \in G$, *if* $0 < t < c_{9.5}$, *then*

$$p_t(x,y) \geq c_{9.6} t^{-d_s'/2} \text{ whenever } R(x,y) \leq c_{9.7} t^{1/\eta}, \tag{21}$$

where $1 > 1/\eta = 1 - d_s'/2 > 1 - d_s/2 = 1/d_w'$.

Proof To prove (20), suppose $R(x,y) \leq c_{9.4} t^{1/d_w'} |\log t|^{-2\beta}$, and $t < t_1(x)$, then by Corollary 9 and Lemma 11,

$$\begin{aligned} p_t(x,y) &\geq p_t(x,x) - |p_t(x,y) - p_t(x,x)| \\ &\geq c_1 t^{-d_s/2} |\log t|^{-\beta} - R(x,y)^{1/2} t^{-(1+d_s/2)/2} \\ &= t^{-d_s/2} |\log t|^{-\beta} \big(c_1 - R(x,y)^{1/2} t^{-(1-d_s/2)/2} |\log t|^{\beta}\big) \\ &\geq c_2 t^{-d_s/2} |\log t|^{-\beta}, \end{aligned}$$

since $1 - d_s/2 = 1/d'_w$, where $c_{9.4}$ is chosen so that $c_1 - c_{9.4}^{1/2} = c_2 > 0$. For (21) we apply the same ideas but use (13) and Lemma 17 in place of Corollary 9.

$$\begin{aligned} p_t(x,y) &\geq p_t(x,x) - |p_t(x,y) - p_t(x,x)| \\ &\geq p_t(x,x)\left(1 - c_{7.5}\sqrt{\frac{R(x,y)}{tp_t(x,x)}}\right) \\ &\geq c_3 t^{-d'_s/2}\big(1 - c_{7.5}R(x,y)^{1/2}t^{-(1-d'_s/2)/2}\big) \\ &\geq c_4 t^{-d'_s/2}, \end{aligned}$$

if we choose y such that $R(x,y) \leq c_{9.7}t^{1-d'_s/2}$, with $c_3(1 - c_{7.5}c_{9.7}^{1/2}) = c_4 > 0$. ∎

The final step is to construct a chaining argument. For this we require a path between arbitrary points x, y, with resistance between adjacent points such that we can apply the previous lemma. Note that we cannot get the lower bound uniformly the same as the upper bound, but instead get worst case bounds.

In order to chain we need to understand the shortest paths in the fractal. As we have near diagonal estimates in terms of resistances, and the resistance depends on direction, we introduce paths which move on different levels in different directions. If we put

$$\epsilon_{m,m+k}(x,y) = \epsilon^h_m(x,y) \cup \epsilon^d_{m+k}(x,y),$$

then $\epsilon_{m,m+k}(x,y)$ is the set of horizontal edges in G_m and diagonal edges in G_{m+k} which make up the shortest path from x to y. That is, we run the path at finer levels in the diagonal direction. For the points on the path we have

$$\begin{aligned} \pi_{m,m+k}(x,y) = \{x\} \quad &\cup \quad \{x_i, x_{i+1} \,:\, (x_i, x_{i+1}) \in \epsilon^h_m(x,y)\} \\ &\cup \quad \{x_i, x_{i+1} \,:\, (x_i, x_{i+1}) \in \epsilon^d_{m+k}(x,y)\} \cup \{y\} \end{aligned}$$

(repeated points are discarded), and put $n_{m,m+k} = n_{m,m+k}(x,y) = |\pi_{m,m+k}(x,y)| - 2 = n^h_m + n^d_{m+k} + 1$. Label the points of $\pi_{m,m+k}$, $x_0 = x$, $x_1, \ldots, x_{n_{m,m+k}}$, $x_{n_{m,m+k}+1} = y$, ordered in the order they appear on the path. Also, for $j = 1, \ldots, n^h_m$, let $x^h_j = x_{h(j)}$ be the jth point ($\neq x_0$) which begins a horizontal edge on the path, and for $j = 1, \ldots, n^d_{m+k}$, let $x^d_j = x_{d(j)}$ be the jth point which begins a diagonal edge. Clearly, $n^h_m \equiv 2^m$ and $n^d_{m+k} \equiv 2^{m+k}$.

We can now derive a uniform worst case lower bound.

Theorem 19 *There exist constants* $c_{9.8}, c_{9.9}, c_{9.10}, c_{9.11}$ *such that for all* $x, y \in G$ *and* $0 < t < c_{9.8}$ *we have*

$$\begin{aligned} p_t(x,y) \quad \geq \quad c_{9.9}t^{-d'_s/2}\min\Big\{&\exp(-c_{9.10}(|x-y|_h^{\eta}t^{-1})^{1/(\eta-1)}\log(|x-y|_h t^{-1})), \\ &\exp(-c_{9.11}(|x-y|_d^{\eta'}t^{-1})^{1/(\eta'-1)}\log(|x-y|_d t^{-1}))\Big\}, \end{aligned}$$

where $1/\eta = 1 + d_s/2 - d'_s$ *and* $1/\eta' = \log 2/\eta\log(3/2)$.

Proof We fix x, y and t. If $R(x,y) \leq c_{9.7}t^{1/\eta}$, then we have the bound by (21). Therefore we assume that $R(x,y) > c_{9.7}t^{1/\eta}$.

Let $B_r(x)$ denote the ball of radius r at x, using the resistance metric rather than the shortest path metric. From (5) we know that if $x_i, x_{i+1} \in \pi_{m,m+k}(x,y)$ with $\tau(x_i, x_{i+1}) = h$, then $R(y_i, y_{i+1}) \leq c_1 2^{-m}$ for $y_i \in B_{2^{-m}/3}(x_i), y_{i+1} \in B_{2^{-m}/3}(x_{i+1})$. Similarly, if $\tau(x_i, x_{i+1}) = d$, then $R(y_i, y_{i+1}) \leq c_2 (3/2)^{-(m+k)}$ for $y_i \in B_{2^{-(m+k)}/3}(x_i), y_{i+1} \in B_{2^{-(m+k)}/3}(x_{i+1})$.

From Lemma 18 we can deduce a lower bound for all points within neighbouring balls at a given level. Firstly we consider the subsequence of vertices $\{x_{h(j)} : j = 1, 2, \ldots, n_m^h\}$ which correspond to the first vertex of a horizontal edge in the path. If $|x-y|_h > 0$, then since $1/\eta < 1$, we can always choose an m such that $c_1 2^{-m} \leq c_{9.7}(t/2n_m^h)^{1/\eta} \leq c_3 2^{-m}$, whence

$$p_{t/2n_m^h}(y_{h(j)}, y_{h(j)+1}) \geq c_{9.6}(t/2n_m^h)^{-d_s'/2},$$

for $y_{h(j)} \in B_{2^{-m}/3}(x_{h(j)})$, $y_{h(j)+1} \in B_{2^{-m}/3}(x_{h(j)+1})$. Also, for this m,

$$\mu(B_{2^{-m}/3}(x_{h(j)})) \geq c_4 3^{-m \log 2/\log(3/2)} = c_4 2^{-m \log 3/\ \log(3/2)} \geq c_5 (t/2n_m^h)^{d_f/\eta'},$$

noting that from (5), we can find a c_6 such that $R(x,y) \leq c_6 |x-y|_G^{\log(3/2)/\log 2}$ for all $x, y \in G$ with $|x-y|_G \leq 1$.

Next, we consider the vertices $\{x_{d(j)} : j = 1, \ldots, n_{m+k}^d\}$, which correspond to the first vertex of a diagonal edge in the path. If $|x-y|_d > 0$, then as $1/\eta < \log(3/2)/\log 2$, given m we can choose a k such that $c_2(3/2)^{-(m+k)} \leq c_{9.7}(t/2n_{m+k}^d)^{1/\eta} \leq c_7 (3/2)^{-(m+k)}$, whence

$$p_{t/2n_{m+k}^d}(y_{d(j)}, y_{d(j)+1}) \geq c_{9.6}(t/2n_{m+k}^d)^{-d_s'/2},$$

for $y_{d(j)} \in B_{2^{-(m+k)}/3}(x_{d(j)})$, $y_{d(j)+1} \in B_{2^{-(m+k)}/3}(x_{d(j)+1})$. Also, for this m,

$$\begin{aligned} \mu(B_{2^{-(m+k)}/3}(x_{d(j)})) &\geq c_4 3^{-(m+k)\log 2/\log(3/2)} \\ &= c_4 (3/2)^{-(m+k)\log 2 \log 3/(\log(3/2))^2} \geq c_8 (t/2n_{m+k}^d)^{d_f/\eta''}, \end{aligned}$$

where $1/\eta'' = \log 2/\eta' \log(3/2) = (\log 2)^2/\eta(\log(3/2))^2$.

We associate with each edge (x_i, x_{i+1}) in the path $\pi_{m,m+k}(x,y)$ the mass

$$\tilde{\mu}(x_i, x_{i+1}) = \sqrt{\mu(B(i))\mu(B(i+1))},$$

where $B(i) = B_{r(i)}(x_i)$ where $r(i) = 2^{-m}/3$ if $\tau(x_i, x_{i+1}) = h$, $r(i) = 2^{-(m+k)}/3$ if $\tau(x_i, x_{i+1}) = d$, and for the penultimate point in the path, $r(n_{m,m+k}) = r(n_{m,m+k} - 1)$. Also write $n(i) = n_m^h$ if $\tau(x_i, x_{i+1}) = h$, $n(i) = n_{m+k}^d$ if $\tau(x_i, x_{i+1}) = d$, and $n(n_{m,m+k}) = n(n_{m,m+k} - 1)$. Putting $N = n_{m,m+k}$, we bound the transition density below by

$$\begin{aligned} &p_t(x,y) \\ &\quad \geq \int_{B(1)} \cdots \int_{B(n_{m,m+k})} p_{t/2n(1)}(x, y_1) p_{t/2n(1)}(y_1, y_2) \cdots p_{t/2n(N-1)}(y_{N-1}, y_N) \\ &\qquad\qquad \times p_{t/2n(N)}(y_N, y)\mu(dy_1) \cdots \mu(d_{y_N}) \\ &\quad \geq \xi_t \prod_{j=1}^{n_m^h} p_{t/2n_m^h}(y_{h(j)}, y_{h(j)+1}) \prod_{j=1}^{n_{m+k}^d} p_{t/2n_{m+k}^d}(y_{d(j)}, y_{d(j)+1}) \\ &\qquad\qquad \times \prod_{j=1}^{n_m^h} \tilde{\mu}(x_{h(j)}, x_{h(j)+1}) \prod_{j=1}^{n_{m+k}^d} \tilde{\mu}(x_{d(j)}, x_{d(j)+1}) \end{aligned}$$

where $\xi_t = p_{t/2n(1)}(x, y_1) p_{t/2n(N)}(y_N, y) \sqrt{\mu(B(1))\mu(B(N))}$. Thus

$$\begin{aligned} & p_t(x,y) && (22) \\ \geq \; & \xi_t \left[\left[\frac{t}{2n^h_m} \right]^{-d'_s/2} \right]^{n^h_m} \left[\left[\frac{t}{2n^d_{m+k}} \right]^{-d'_s/2} \right]^{n^d_{m+k}} \left[\left[\frac{t}{2n^h_m} \right]^{d_f/\eta'} \right]^{n^h_m} \left[\left[\frac{t}{2n^d_{m+k}} \right]^{d_f/\eta''} \right]^{n^d_{m+k}} \\ \geq \; & \xi_t \exp\{-c_9(n^h_m \log(2n^h_m/t) + n^d_{m+k} \log(2n^d_{m+k}/t)\}, && (23) \end{aligned}$$

since $d_f/\eta'' > d_f/\eta' > d_f/\eta > d'_s/2$.

Next, with our choices of m and k we get,

$$\begin{aligned} n^h_m & \leq |x-y|_h 2^m \leq c_{10}(|x-y|^\eta_h t^{-1})^{1/(\eta-1)}, \text{ and} \\ n^d_{m+k} & \leq |x-y|_d 2^{m+k} \leq c_{11}(|x-y|^{\eta'}_d t^{-1})^{1/(\eta'-1)}. \end{aligned}$$

Finally, we can estimate ξ_t using the previous lemma and the given values of m and k. Replacing the terms in (23) we then get the result. ∎

Remark 20 1. Note that the bounds of Theorems 14 and 19 are not tight, with the exponents different in the upper and lower bounds for the leading term. Even if we could use the near diagonal lower bound of (20), we would still obtain different exponents for the upper and lower bounds due to the fact we are working in the Euclidean metric.
2. In [HK98A] it was proved that there is homogenization, in the sense that on the infinite fractal lattice, rescaling the asymptotically one dimensional process by the Brownian time scaling, we obtain the Brownian motion in the limit. Using this result it is clear that the long time heat kernel bounds will be the usual estimates for Brownian motion on the Sierpinski gasket.
3. If we write $d_c = \log 2/\log(3/2)$, we have the following relationships between the exponents,

$$d'_s = d_s/d_c, \; d'_w = d_c d_w, \; \eta' = \eta/d_c \text{ and } d'_w > d_w > \eta > \eta'.$$

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