ESTIMATION OF ANALYTIC SPECTRAL DENSITY OF GAUSSIAN STATIONARY PROCESSES.

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Abstract. In this paper we consider the problem of estimation of the spectral density $f(\lambda)$ of a gaussian stationary sequence (process) on the base of observations $X_t, 0 \le t \le T$. We suppose that f belongs to the class of spectral densities analytic and bounded inside a bounded region $G \ni [a,b]$. We found the rate of the asymptotic minimax risk in $L_p[a,b], 1 \le p \le \infty$ when $T \to \infty$.

1. Introduction.

Let $\{X_j\}$ be a real-valued stationary Gaussian sequence with mean zero and spectral density $f(\lambda)$. We assume that the spectral density f is unknown and should be estimated on the base of observations

$$X_1, X_2, \dots X_T. \tag{1.1}$$

Assume further that $f \in F$ where F is a given (known) class of spectral densities. Let $[a,b] \subseteq [-\pi,\pi]$ be an interval and assume that we would like to estimate the restriction of f on $[a,b], \{f(\lambda), a \le \lambda \le b\}$. If f_T is an estimator for f, we measure its quality by the average $L_p[a,b]$ distance from f, namely

$$\begin{aligned} \mathbf{E}_f|f-f_T|_p = \mathbf{E}_f\big(\int_a^b |f(\lambda)-f_T(\lambda)|^p d\lambda\big)^{1/p}, 1 \le p < \infty, \\ \mathbf{E}_f|f-f_T|_\infty = \mathbf{E}_f\{\sup_\lambda |f(\lambda)-f_T(\lambda)|\}. \end{aligned} \tag{1.2}$$

We denote by $|.|_p$ the norms in $L_p(a,b)$ spaces; the norms in $L_p(-\pi,\pi)$ are denoted $||.||_p$.

Let us define

$$\Delta_p(T;F) = \Delta_p(T) = \inf\sup \mathbf{E}_f|f-f_T|_p \tag{1.3}$$

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where sup is taken over all $f \in F$ and inf is taken over all possible estimators f_T of f. We are interesting in the asymptotic behaviour of the minimax risk $\Delta_p(T;F)$ when $T \to \infty$. The rate of convergence of $\Delta_p(T)$ to zero depends on F.

For example, let F consist of all spectral densities f with uniformly bounded in L_p fractional derivatives of order β. Then (see, for example, [1], [2], [3] and references there)

$$\begin{aligned} &\Delta_p \asymp \big(T\big)^{-\frac{\beta}{1+2\beta}}, 1 \le p < \infty, \\ &\Delta_\infty \asymp \big(T\big)^{-\frac{\beta}{1+2\beta}} \big(\ln T\big)^{\frac{\beta}{1+2\beta}}. \end{aligned} \tag{1.4}$$

If F consists of all 2π-periodic functions analytic and uniformly bounded inside a strip $|\Im z| < c, z = \lambda + i\mu$, then

$$\begin{aligned} \Delta_p(T) &\asymp \sqrt{\frac{\ln T}{T}},\ 1 \le p < \infty, \\ \Delta_\infty(T) &\asymp \sqrt{\frac{\ln T}{T}} \sqrt{\ln\ln T}. \end{aligned} \tag{1.5}$$

In this paper we study the asymptotic behaviour of $\Delta_p(T)$ for the case when spectral densities f are analytic in a *bounded* region G of the complex plane, $[a,b] \subset G$. Some similar problems of curve estimation for curves analytic in a bounded region has been considered in the author's paper [4]. In this paper we use the same approach as in [4].

Theorem 1.1. *Let the expression $\Delta_p(T;F)$ be defined by (1.2) where the set F consists of all spectral densities analytic in some bounded region $G, [a,b] \subset G$, and bounded there by a common constant M.Then*

$$\begin{aligned} \Delta_p(T) &\asymp \sqrt{\frac{\ln T}{T}}, 1 \le p < 4, \\ \Delta_4(T) &\asymp \sqrt{\frac{\ln T}{T}} (\ln\ln T)^{1/4}, \\ \Delta_p(T) &\asymp \sqrt{\frac{1}{T}} (\ln T)^{1-2/p}, 4 < p \le \infty. \end{aligned} \tag{1.6}$$

The next sect.2 and 3 are devoted to the proof of the theorem. Sect.4 treats continuous time processes.

2. Proof of the theorem 1.1. Upper bounds.

2.1. Without loss of generality we may and will suppose that $[a,b] = [-1,1]$. The only exeption arises in the special case when one or both endpoints of the interval $[a,b]$ coincides with $-\pi$ or π. A spectral density is naturally defined on the unit circle or as a function 2π-periodic on the real line. If $[a,b] \subset (-\pi,\pi)$, the function f is smooth in a vicinity of $[a,b]$, but if a(or b) is $-\pi$ (or π), the function f may have jumps at this points. Sometimes we have to consider this case separately.

If O is a domain in the complex plane we denote by $A(O,M)$ the class of functions analytic in O and uniformly bounded there by a common constant M. The set G

is contained inside an ellipsis E with the foci at the points ± 1 and the sum of half-axes equal to $R > 1$. Evidently $A(E, M) \subseteq A(G, M)$ and hence any function $f \in F$ belongs to $A(E, M)$. Functions $f \in A(E, M)$ can be represented by the Fourier series with respect to the orthonormal Legendre polynomials $P_j(\lambda)$, namely

$$f(\lambda) = \sum_1^\infty a_j P_j(\lambda), a_j = \int_{-1}^1 P_j(\lambda) f(\lambda) d\lambda. \tag{2.1}$$

The coefficients a_j decreases exponentially fast,

$$\Big(\sum_{n+1}^\infty |a_j|^2\Big) \le \Big(\frac{\pi}{R^2 - 1}\Big)^{1/2} \frac{M}{R^n},$$

see, for example, [5]. Hence

$$|a_n| \le c R^{-n} = c e^{-\gamma n}. \tag{2.2}$$

Here and below let c, C represent constants; they may be different even inside the same formula.

Denote $I_T(\lambda)$ the periodogram

$$I_T(\lambda) = \frac{1}{2\pi} \Big| \sum_1^T X_j e^{-ij\lambda} \Big|^2.$$

Because of (2.1)

$$\hat{a}_k = \int_{-1}^1 I_T(\lambda) P_k(\lambda)$$

are natural estimators for a_k. Consider estimators $f_N(\lambda)$ for $f(\lambda)$ defined by the formula

$$f_N(\lambda) = \sum_1^N \hat{a}_k P_k(\lambda).$$

The main result of this section is the following

Theorem 2.1. *The estimators f_N satisfy the folowing inequalities*

$$\begin{aligned} \mathbf{E}_f |f - f_N|_p \le C(N^{1/2} T^{-1/2} + N T^{-3/4} + \\ T^{-1/2} + e^{-\gamma N}), 1 \le p < 4, \\ \mathbf{E}_4 |f - f_N|_4 \le C(N^{1/2} (\ln N)^{1/4} T^{-1/2} + N T^{-3/4} \\ + T^{-1/2} + e^{-\gamma N}), \\ \mathbf{E}_p |f - f_N|_p \le C(N^{1/2} (\ln N)^{1-2/p} T^{-1/2} + N T^{-3/4} \\ + T^{-1/2} + e^{-\gamma N}), p > 4. \end{aligned} \tag{2.3}$$

The upper bounds of the theorem 1.1 now follow easily from (2.3).

Corollary 2.1. *There exist estimators $\hat{f}_T$ such that*

$$\begin{aligned} &\mathbf{E}_f|f-\hat{f}_T|_p \le C\sqrt{\frac{\ln T}{T}}, 1 \le p < 4, \\ &\mathbf{E}_f|f-\hat{f}_T|_4 \le C\sqrt{\frac{\ln T}{T}}(\ln\ln T)^{1/4}, \\ &\mathbf{E}_f|f-\hat{f}_T|_p \le CT^{-1/2}(\ln T)^{1-2/p}. \end{aligned} \tag{2.4}$$

The Corollary follows immediately from (2.3). Indeed, we can define $\hat{f}_T$ as $f_{N(T)}$, where $N(T) = \left[\frac{1}{\gamma}\ln T\right]$.

Proof of the theorem 2.1. Split the difference $f_N(\lambda) - f(\lambda)$ into the bias part

$$b_N(\lambda) = \sum_1^N P_k(\lambda)\int_{-1}^1 \big(\mathbf{E}I_T(\lambda) - f(\lambda)\big)d\lambda - \sum_{N+1}^\infty a_k$$

and the random part

$$Z_N(\lambda) = \sum_1^N (\hat{a}_k - \mathbf{E}\hat{a}_k)P_k(\lambda)$$

and bound them separately.

2.2. It is well known that the expectation $\mathbf{E}I_T(\lambda)$ of the periodogram is the Fejer integral of the spectral density (see, for ex., [6]),

$$\mathbf{E}I_T(\lambda) = \frac{1}{2\pi T}\int_{-\pi}^{\pi} \frac{\sin^2 T\frac{\lambda-l}{2}}{\sin^2\frac{\lambda-l}{2}} f(l)dl. \tag{2.5}$$

Lemma 2.1. *Let $\Phi_T(g)(\lambda)$ denote the Fejer integral of a function g.*

(i) If $g \in L_p(-\pi,\pi)$ and satisfies there the Hölder condition of order α, then

$$\begin{aligned} &||g - \Phi_T(g)||_p \le CT^{-\alpha}, \qquad 0 < \alpha < 1, \\ &||g - \Phi_T(g)||_p \le CT^{-1}\ln T, \qquad \alpha = 1. \end{aligned} \tag{2.6}$$

(ii) If g satisfies a Lipschitz condition on an interval $[a,b]$, then for any interval $[\alpha,\beta] \subset (a,b)$

$$\sup_{[\alpha,\beta]} |g(\lambda) - \Phi_T(g)(\lambda)| \le CT^{-1}. \tag{2.7}$$

The proof of the Lemma 2.1 can be find in books on approximation theory, see, for example, [7].

Lemma 2.2.. *For all $1 \le p \le \infty$ the following inequality holds*

$$|b_N|_p \le C(N^{3/2}T^{-3/4} + T^{-1/2} + e^{-\gamma N}). \tag{2.8}$$

Proof of the Lemma. The derivative $f'(\lambda)$ is bounded in a vicinity of $[-1,1]$ by a constant C which depends on M and G only. Hence by (2.5) and (2.7) for all $\lambda \in [-1,1]$

$$|\mathbf{E}I_T(\lambda) - f(\lambda)| \le CT^{-1}\ln T. \tag{2.9}$$

It folows that

$$|\int_{-1}^{1} P_n(\lambda)(\mathbf{E}I_T(\lambda)-f(\lambda))d\lambda\big| \le C|P_n|_2 T^{-1}\ln T = CT^{-1}\ln T. \tag{2.10}$$

On the interval $[-1,1]$ the Legendre polynomials satisfy the inequalities (see [5])

$$|P_n(\lambda)| \le P_n(1) = (n+1/2)^{1/2}. \tag{2.11}$$

Hence

$$|b_N|_p \le CT^{-1}\ln T\sum_1^N k^{1/2} + \sum_{N+1}^{\infty}|a_k| \le C(N^{3/4}T^{-3/2}+e^{-\gamma N})$$

and the lemma is proved except the special case which has been mentioned above. Indeed if the interval $[a,b]$ of analyticity coincides with the interval $[-\pi,\pi]$,the arguments proving (2.10) do not work becouse the *periodic* function $f(\lambda)$ may have a jump at the points $\pm\pi$. This jump is the only possible singularity of f so in any case it satisfies Holder condition of order 1/2 in the L_2-norm and by Lemma 2.1

$$|\int_{-\pi}^{\pi} P_n(\lambda)(\mathbf{E}I_T(\lambda)-f(\lambda))d\lambda\big| \le ||P_n||_2||\Phi_T(f)-f||_2 \le CT^{-1/2}.$$

(Of course P_n here are the polynomials orthonormal on $[\pi,\pi]$). The lemma is proved.

2.2. We pass now to the investigations of the random polynomials Z_N.

Lemma 2.3. *The norms of the polynomials Z_N satisfy the following inequalities:*

$$\begin{aligned} \mathbf{E}|Z_N|_p &\le C(N^{1/2}T^{-1/2}+NT^{-1}), \qquad p<4,\\ \mathbf{E}|Z_N|_4 &\le C(N^{1/2}T^{-1/2}(\ln N)^{1/4}+NT^{-3/4}),\\ \mathbf{E}|Z_N|_p &\le C(N^{1-2/p}T^{-1/2}+NT^{-3/4}), \qquad p>4. \end{aligned} \tag{2.12}$$

Proof. We begin with $p=2$. Evidently

$$\begin{aligned} \mathbf{E}|Z_N(\lambda)|^2 &= \sum_{k,l=1}^{N} P_k(\lambda)P_l(\lambda)\\ &\int_{-1}^{1}\int_{-1}^{1} P_k(\mu_1)P_l(\mu_2)cov(I_T(\mu_1),I_T(\mu_2))d\mu_1 d\mu_2. \end{aligned} \tag{2.13}$$

The covariation of I_T has the following integral representation (see [6], [8])

$$\begin{aligned} cov(I_T(\lambda),I_T(\mu)) = \big[\frac{1}{2\pi T}\int_{-\pi}^{\pi}\frac{\sin(T(l-\lambda)/2)\sin(T(l-\mu)/2)}{\sin((l-\lambda)/2)\sin((l-\mu)/2)}f(l)dl\big]^2+\\ \big[\frac{1}{2\pi T}\int_{-\pi}^{\pi}\frac{\sin(T(l-\lambda)/2)\sin(T(l+\mu)/2)}{\sin((l-\lambda)/2)\sin((l+\mu)/2)}f(l)dl\big]^2. \end{aligned} \tag{2.14}$$

Lemma 2.4. *Let $k, l \leq N$. Then*

$$
\begin{aligned}
I_{kl} = \int_{-1}^{1}\int_{1}^{1} P_k(\lambda)P_l(\mu)cov(I_T(\lambda), I_T(\mu)d\lambda d\mu = \\
\frac{2\pi}{T}\int_{-1}^{1}(P_k(\lambda)P_l(\lambda) + P_k(\lambda)P_l(-\lambda))f^2(\lambda)d\lambda + r_{TN}
\end{aligned}
\tag{2.15}
$$

where $|r_{TN}| \leq CNT^{-1}$.

Proof. Using the representation (2.14) we find that I_{kl} is the sum of two summands

$$I_{kl}^{(1)} = \int_{-1}^{1}\int_{-1}^{1} P_k(\lambda)P_l(\mu)\big(\frac{1}{2\pi T}\int_{-\pi}^{\pi}\frac{\sin T(l-\lambda)/2}{\sin(l-\lambda)/2}\frac{\sin T(l-\mu)/2}{\sin(l-\mu)/2}f(l)dl\big)^2 d\lambda d\mu$$

and

$$I_{kl}^{(1)} = \int_{-1}^{1}\int_{-1}^{1} P_k(\lambda)P_l(\mu)\big(\frac{1}{2\pi T}\int_{-\pi}^{\pi}\frac{\sin T(l-\lambda)/2}{\sin(l-\lambda)/2}\frac{\sin T(l+\mu)/2}{\sin(l+\mu)/2}f(l)dl\big)^2 d\lambda d\mu.$$

It is enough to study the first summand, $I_{kl}^{(1)}$. Uniformly in $\lambda \in [-1, 1]$ the integral

$$
\begin{aligned}
\frac{1}{2\pi T}\int_{-\pi}^{\pi}\frac{\sin T(l-\lambda)/2}{\sin(l-\lambda)/2}\frac{\sin T(l-\mu)/2}{\sin(l-\mu)/2}dl = \\
\frac{f(\lambda)}{2\pi T}\int_{-\pi}^{\pi}\frac{\sin T(l-\lambda)/2}{\sin(l-\lambda)/2}\frac{\sin T(l-\mu)/2}{\sin(l-\mu)/2}dl + \\
O(1)T^{-1}\int_{-\pi}^{\pi}\big|\frac{\sin T(l-\lambda)/2}{\sin(l-\lambda)/2}\big|dl = \\
\frac{f(\lambda)}{T}\frac{\sin T(\lambda-\mu)/2}{\sin(\lambda-\mu)/2} + O(\frac{\ln T}{T}),
\end{aligned}
$$

and we have that

$$I_{kl}^{(1)} = \frac{2\pi}{T}\int_{-1}^{1} P_k(\lambda)f^2(\lambda)d\lambda\frac{1}{2\pi T}\int_{-1}^{1} P_l(\mu)\frac{\sin^2\frac{T(\lambda-\mu)}{2}}{\sin^2\frac{\lambda-\mu}{2}}d\mu.$$

Let us define the 2π-periodic function $\tilde{P}_l(\mu)$ as

$$
\begin{aligned}
\tilde{P}_l(\mu) &= P_l(\mu), \qquad |\mu| \leq 1, \\
\tilde{P}_l(\mu) &= 0, \qquad 1 < |\mu| \leq \pi.
\end{aligned}
$$

It follows easily from properties of the Fejer integral that

$$
\begin{aligned}
\int_{-1}^{1}\big|\frac{1}{2\pi T}\int_{-1}^{1} P_l(\mu)\frac{\sin^2\frac{T(\lambda-\mu}{2}}{\sin^2\frac{\lambda-\mu}{2}}d\mu - P_l(\lambda)\big|^2 d\lambda \leq \\
\int_{-\pi}^{\pi} d\lambda\big|\frac{1}{2\pi T}\int_{-\pi}^{\pi}\big[\tilde{P}_l(\mu) - \tilde{P}_l(\lambda)\big]\frac{\sin^2\frac{T(\lambda-\mu)}{2}}{\sin^2\frac{\lambda-\mu}{2}}d\mu\big|^2 \leq CNT^{-1/2}.
\end{aligned}
$$

Hence

$$I_{kl}^{(1)} = \frac{2\pi}{T} \int_{-1}^{1} P_k(\lambda) P_l(\lambda) f^2(\lambda) d\lambda + O(N^2 T^{-3/2}). \tag{2.16}$$

In the same way

$$I_{kl}^{(2)} = \frac{2\pi}{T} \int_{-1}^{1} P_k(\lambda) P_l(-\lambda) f^2(\lambda) d\lambda + O(N^2 T^{-3/2}). \tag{2.17}$$

The relations (2.16), (2.17) prove the lemma if the endpoints a, b do not coincide with $\pm\pi$. The exceptional case can be treated as in the proof of the Lemma 2.2. We omit the arguments.

It follows from the lemma 2.4 that

$$\mathbf{E}|Z_N|_2^2 = \sum_1^N \mathbf{E}|\hat{a}_k - \mathbf{E}\hat{a}_k|^2 \le$$
$$\frac{4\pi}{T} \int_{-1}^{1} \Big(\sum_1^N P_k^2(\lambda)\Big) f^2(\lambda) d\lambda + O(N^3 T^{-3/2})$$
$$\le CNT^{-1} + O(N^3 T^{-3/2}).$$

Hence for all $1 \le p \le 2$

$$\mathbf{E}|Z_N|_p \le CN^{1/2} T^{-1/2} + O(N^2 T^{-3/4}).$$

To consider the case $p > 2$ we need the following lemma.

Lemma 2.5. *Let ξ be a Gaussian random variable with values in an Euclidian or Hilbert space H and $\mathbf{E}\{\xi\} = 0$. Let $A : H \to H$ be a self adjoint bounded linear operator. Then for any $p > 2$*

$$\mathbf{E}|(A\xi, \xi) - \mathbf{E}(A\xi, \xi)|^p \le c_p \mathbf{E}^{p/2} |(A\xi, \xi) - \mathbf{E}(A\xi, \xi)|^2.$$

Proof. The quadratic form $(A\xi, \xi)$ has the same distribution as the sum $\sum \mu_i \xi_i^2$, where ξ_j are iid normal random variables with mean 0 and variance 1, while μ_j are the eigenvalues of the operator RA where R is the correlation operator of ξ (see [8], Appendix). Hence

$$\mathbf{E}|(A\xi, \xi) - \mathbf{E}(A\xi, \xi)|^p = \mathbf{E}\Big|\sum \mu_j (\xi_j^2 - 1)\Big|^p \le$$
$$c_p \Big(\sum \mu_j^2\Big)^{p/2} = c_p \big(2^{-1}(Var(A\xi, \xi)\big)^{p/2}.$$

The lemma is proved.

Notice now that the value $Z_N(\lambda)$ of the polynomial Z_N at a point λ is a quadratic form of the variables $X_1, \dots X_T$. Applying the Lemma 2.5 we find that

$$\mathbf{E}|Z_N|_p \le \Big(\int_{-1}^{1} \mathbf{E}|Z_N(\lambda)|^p d\lambda\Big)^{1/p} \le$$
$$\Big(\int_{-1}^{1} \big(\mathbf{E}|Z_N(\lambda)|^2\big)^{p/2} d\lambda\Big)^{1/p}. \tag{2.18}$$

It follows from (2.13) and lemma 2.4 that

$$\mathbf{E}|Z_N(\lambda)|^2 \le \frac{C}{T}\int_{-1}^{1}\Big|\sum_1^N P_k(\lambda)P_k(\mu)\Big|^2 d\mu + O(N^5T^{-3/2}) = \tag{2.19}$$
$$CT^{-1}\sum_1^N P_k^2(\lambda) + O(N^5T^{-3/2}).$$

Substitute the last estimate into (2.18). We find that

$$\mathbf{E}|Z_N|_p \le CT^{-1/2}\Big(\int_{-1}^{1}\big(\sum_1^N P_k^2(\lambda)\big)^{p/2}d\lambda\Big)^{1/p} + O(N^3T^{-3/4}). \tag{2.20}$$

The integrals on the right can be bounded as follows.

Lemma 2.6. *For* $2 \le p < 4$

$$\int_{-1}^{1}\big(\sum_1^N P_k^2(\lambda)\big)^{p/2}d\lambda \le c_pN^{p/2}.$$

For $p = 4$

$$\int_{-1}^{1}\big(\sum_1^N P_k^2(\lambda)\big)^{2}d\lambda \le cN^2\ln N.$$

For $p > 4$

$$\int_{-1}^{1}\big(\sum_1^N P_k^2(\lambda)\big)^{p/2}d\lambda \le c_pN^{p-2}.$$

Proof. The inequalities of the lemma follow imediatly from the inequalities (see [5])

$$|P_k(\lambda)| \le (k+1/2)^{1/2}, |P_k(\lambda)| \le \sqrt{\frac{3}{\pi}}(1-\lambda^2)^{-1/4}.$$

The inequalities of the Lemma 2.5 together with (2.20) prove the bounds (2.12) for $p < \infty$. To treat the case $p = \infty$ we apply the following result (see [7]).

Lemma 2.7. *Let* Q *be an algebraic polynomial of degree* n. *Then*

$$|Q|_p \le (p+1)^{1/p}n^{2/p}|Q|_p.$$

The last lemma gives that for any $p > 4$

$$\mathbf{E}|Z_N|_\infty \le CN^{2/p}\mathbf{E}|Z_N|_p \le CN^{2/p}(T^{-1/2}N^{1-2/p} + O(N^3T^{-3/4})) \le$$
$$C(NT^{-1/2} + N^4T^{-3/4}).$$

The inequalities (2.12) are thus proved and we have finished the proof of the theorem 2.1.

3. LOWER BOUNDS.

3.1. Let E be an ellipsis with the foci at the points ± 1 and such that $G \subset E$. Then $A(G, M) \supset A(E, M)$ and any spectral density $f \in A(E, M)$ will also belong to F. Hence it is enough to establish the lower bounds for the case $G = E$ and below we suppose that $G = E$.

We prove at first the necessary lower bounds for $p \neq 4$ utilizing methods developped by Ibragimov and Khasminskii ([9], [10]) and based on the use of Fano's lemma (see [9]). Namely, assume that there is a set $S = \{f_1, \dots f_M\}$ of spectral densities from F such that $|f_i - f_j|_p \geq 2\delta$ for any $i \neq j$. Then evidently for any estimator $\hat{f}$ for f

$$\begin{aligned} &\sup_{f \in F} \mathbf{P}_f\{|\hat{f} - f|_p \geq \delta\} \geq \sup_{1 \leq i \leq M} \mathbf{P}_{f_i}\{|\hat{f} - f|_p \geq \delta\} \geq \\ &\qquad \frac{1}{M} \sum_{1 \leq i \leq M} \mathbf{P}_{f_i}\{|\hat{f} - f|_p \geq \delta\}. \end{aligned} \tag{3.1}$$

Let θ be a random variable taking values $1, \dots M$ with equal probabilities $1/M$. It follows from the inequality (3.1) that

$$\begin{aligned} &\inf_{\hat{f}} \sup_{f \in F} \mathbf{P}_f\{|\hat{f} - f|_p \geq \delta\} \geq \\ &\inf_{\hat{\theta}} \frac{1}{M} \sum_{i=1}^{M} \mathbf{P}\{\hat{\theta} \neq i | \theta = i\} = p_e, \end{aligned} \tag{3.2}$$

where the last inf is taken over all estimators $\hat{\theta} = \hat{\theta}(X_1, \dots X_T)$. By Fano's lemma (see [9])

$$p_e \geq 1 - \frac{I(\theta; X) + \ln 2}{\ln M - 1}, \tag{3.3}$$

where $I(\theta; X)$ denotes Shannon's information in $X = (X_1, \dots X_T)$ about θ.

The estimates (3.1) -(3.3) show that

$$\begin{aligned} &\inf_{\hat{f}} \sup_{f \in F} \mathbf{E}_f |\hat{f} - f|_p \geq \inf_{\hat{f}} \sup_{f \in S} \mathbf{E}_f |\hat{f} - f|_p \geq \\ &\qquad \delta \inf_{\hat{f}} \sup_{f \in S} \mathbf{P}_f\{|\hat{f} - f|_p \geq \delta\} \geq \\ &\qquad \delta p_e \geq \delta\big(1 - \frac{\sup_\theta I(\theta, X) + 1}{\ln M - 1}\big). \end{aligned} \tag{3.4}$$

This inequality will serve as an initial points in our arguments for the cases $p \neq 4$. To use it we study at first the asymptotic behaviour of $I(\theta, X)$ when $T \to \infty$.

3.2. By the definition Shannon information (see [11])

$$I(\theta, X) = \mathbf{E}\big(\ln \frac{dP_{\theta, X}}{dP_\theta \times dP_X}(\theta, X)\big).$$

Let now P_0 denote the probability distribution generated by the stationary sequence $\{X_j\}$ of iid Gaussian random variables with the spectral density equal to 1. Then

$$I(\theta, X) = \mathbf{E} \ln \frac{dP_{X|\theta}}{dP_X} = \mathbf{E} \ln \frac{dP_{X|\theta}}{dP_0} + \mathbf{E} \ln \frac{dP_0}{dP_X}.$$

By Jensen's inequality

$$\mathbf{E}\ln\frac{dP_0}{dP_X} \leq \ln\mathbf{E}\frac{dP_0}{dP_X} \leq 0.$$

Hence

$$I(\theta, X) \leq \mathbf{E}\ln\frac{dP_{X|\theta}}{dP_0}. \tag{3.5}$$

Lemma 3.1. *Let $K(m,M)$ denote the class of all spectral densities $f(\lambda)$ such that $f(\lambda) \geq m > 0, f(\lambda) \leq M, \int_{-\pi}^{\pi}|f'(\lambda)|^2 \leq M$. Let $P_f^T = P_f$ be the distribution of the sequence $X_1, \dots X_T$ with the spectral density $f(\lambda)$. Then*

$$\mathbf{E}\ln\frac{dP_f^T}{dP_o^T}(X) = -\frac{T}{4\pi}\int_{\pi}^{\pi}\ln f(\lambda d\lambda + \frac{T}{4\pi}\int_{-\pi}^{\pi}I_T(\lambda)(\frac{1}{f(\lambda)} - 1)d\lambda + R_T,$$

where $\sup_{f\in K}|R_T| < \infty$.

For the proof of the lemma see [12], pp 50-59. If we apply the result of the lemma 3.1 to the inequality (3.5), we find that for $f_j \in K$

$$I(\theta, X) \leq CT\sup_j\{|\int_{-\pi}^{\pi}\ln f_j(\lambda)d\lambda| + \int_{-\pi}^{\pi}|\frac{f_j(\lambda)-1}{f_j(\lambda)}|I_T(\lambda)d\lambda\}. \tag{3.6}$$

3.3. We begin with the case $1 \leq p < 4$, construct the set S and prove the lower bounds for this case. Introduce the function $\phi(\lambda) = \frac{\sin^3\lambda}{\lambda^2}$. Evidently

$$\int_{-c}^{c}\phi(\lambda)d\lambda = 0, \int_{-\infty}^{\infty}|\phi(\lambda)|d\lambda < \infty. \tag{3.7}$$

The function $\phi(z)$ is analytic in the whole comlex plane and on any strip $|\Im z| \leq R$ it satisfies the inequality

$$|\phi(z)| \leq Ce^{3R}. \tag{3.8}$$

Define the number κ from the relation

$$\int_{-\kappa}^{\kappa}|\phi(\lambda)|d\lambda = 100\int_{|\lambda|\geq\kappa}|\phi(\lambda)|d\lambda. \tag*{((3.9))}$$

Take an integer N and introduce the functions

$$\phi_{jN}(\lambda) = \phi_j(\lambda) = \{\frac{\sin^3 N(\lambda - 2j\kappa/N)}{(\lambda - 2j\kappa/N)^2} - \frac{\sin^3 N(\lambda + 2j\kappa/N)}{(\lambda + 2j\kappa/N)^2}\} - c_{jN},$$

where the indices j run through the integers of the interval $1 \leq j \leq M, M = [\frac{\pi N}{4\kappa}]$ and the constants $c_{jN} = c_j$ are defined in a such way that the integrals

$$\int_{-\pi}^{\pi}\phi_j(\lambda)d\lambda = 0, j = 1, \dots, M.$$

Notice that

$$|c_j| \leq \big|2N \int_{N\pi-2j\kappa}^{N\pi+2j\kappa} \frac{\sin^3(\lambda)}{\lambda^2} d\lambda\big| \leq \frac{C}{N}.$$

Consider vectors $\mathbf{a} = (a_1, \dots a_M)$ where a_j takes the values $-1, 1$. Define spectral densities $f_{\mathbf{a}}(\lambda)$ as

$$f_{\mathbf{a}}(\lambda) = \theta + e^{-\gamma N} \sum_1^M a_j \phi_{jN}(\lambda).$$

The functions $f_{\mathbf{a}}$ are analytic and in the strip $|\Im z| \leq R$ they are bounded by $\theta + Ce^{-(\gamma-3R)}$. Hence one can always choose θ and γ in a such way that all $f_{\mathbf{a}}$ will be spectral densities of a real stationary Gaussian sequence($f_{\mathbf{a}}(\lambda) \geq 0, \lambda \in [-\pi, \pi], f_{\mathbf{a}}(\lambda) = f_{\mathbf{a}}(-\lambda)$) from F ($|f_{\mathbf{a}}(z)| \leq M, |\Im z| \leq R$). Below for the sake of simplicity we take $\theta = 1$.

Denote by $\mathbf{A}$ the subset of the vectors $\{\mathbf{a}\}$ such that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{A}$

$$\sum_1^M |a_j - b_j| \geq M/2.$$

Define the set $S = \{f_{\mathbf{a}}, \mathbf{a} \in \mathbf{A}\}$.

Lemma 3.3. *(see* [10], [4]*)The cardinality of the set S is*

$$card(S) = card(\mathbf{A}) \geq 2^{M/8}. \tag{3.10}$$

The L_1-distance between any two functions $f_{\mathbf{a}}, f_{\mathbf{b}}$ from S

$$\begin{aligned} |f_{\mathbf{a}} - f_{\mathbf{b}}|_1 = e^{-\gamma N} \int_{-1}^{1} |\sum_j (a_j - b_j)\phi_{jN}(\lambda)| d\lambda \geq \\ e^{-\gamma N} \sum_j \Big(\int_{(2j-1)\kappa/N}^{(2j+1)\kappa/N} |\phi_j(\lambda)| d\lambda - \\ \sum_{r \neq j} \int_{(2j-1)\kappa/N}^{(2j+1)\kappa/N} |\phi_r(\lambda)| d\lambda \Big) \geq \\ ce^{-\gamma N} N \Big(\int_{-\kappa}^{\kappa} |\phi(\lambda)| d\lambda - \int_{\lambda \geq \kappa} |\phi(\lambda)| d\lambda \Big) \sum_j |a_j - b_j|, c > 0. \end{aligned} \tag{3.11}$$

Taking into account inequalities (3.9) and (3.10) we find from (3.11) that there exists a positive constant $c > 0$ such that

$$|f_{\mathbf{a}} - f_{\mathbf{b}}|_1 \geq cN^2 e^{-\gamma N}. \tag{3.12}$$

For $f_{\mathbf{a}} \in S$ the Shannon information $I(\theta, X)$ is bounded by (see (3.6))

$$CT\Big(\int_{-\pi}^{\pi} (|f_{\mathbf{a}} - 1|^2 d\lambda + O(\sup_\lambda |f_{\mathbf{a}}(\lambda) - 1|^3)\Big) + C.$$

Now

$$\begin{aligned}
&\int_{-\pi}^{\pi} |1-f_{\mathbf{a}}(\lambda)|^2 d\lambda \leq \\
&Ce^{-\gamma N}N^4 \sum_{-(N-1)\leq j\leq (N-1)} \int_{-\infty}^{\infty} \frac{|\sin^3(\lambda)|}{\lambda^2}\frac{|\sin^3(\lambda-j)|}{(\lambda-j)^2} d\lambda \leq \\
&CN^4e^{-2\gamma N}.
\end{aligned} \tag{3.13}$$

Following the program outlined above, we return to (3.4) and use the bounds (3.12) for δ in (3.4) and the estimates of $I(\theta, X)$ given by (3.13). We find that

$$\Delta_1(T) \geq cN^2e^{-\gamma N}(1 - C\frac{\sup_{\mathbf{a}} I(\theta, X)}{N} \geq cN^2e^{-\gamma N}(1 - CTe^{-2\gamma N}N^3 + O(N^{-1})).$$

Now take $N \asymp \ln T$ in a such way that $CTN^3e^{-2\gamma N} \leq 1/2$. We find then that for sufficiently large N

$$\Delta_1(T) \geq c\left(\frac{\ln T}{T}\right)^{1/2}, c > 0.$$

Hence for $1 \leq p < 4$

$$\Delta_p(T) \geq 2^{-3/4}\Delta_1(T) \geq c\left(\frac{\ln T}{T}\right)^{1/2}.$$

3.4. Consider now the case $p > 4$. Define the set S of spectral densities f_j. Let τ be a small positive number. Define $f_j(\lambda)$ for $|\lambda| \leq 1$ as follows

$$\begin{aligned}
f_0(\lambda) &= 1 + e^{-\gamma N} \sum_{N/2\leq j\leq N} P_{2j}(\lambda), \\
f_1(\lambda) &= 1 + e^{-\gamma N} \sum_{N/2\leq j\leq N-N\tau} P_{2j}(\lambda) \\
&\qquad\qquad \cdots\cdots\cdots,
\end{aligned} \tag{3.14}$$

where as above P_n are the Legendre polynomials. Notice that $P_{2k}(\lambda)$ and hence $f_j(\lambda)$ are even functions. To any f_j we associate the numbers

$$\lambda_+(j) = |f_j(1) - 1|(\sqrt{2}+1), \lambda_-(j) = |f_j(-1) - 1|(\sqrt{2}+1) \tag{3.15}$$

and set

$$f_j(\lambda) = 1, \lambda \in [-\pi, \pi] \setminus [-1-\lambda_-(j), 1+\lambda_+(j)].$$

At the small intervals $[-1-\lambda_-(j), -1], [1, 1+\lambda_+(j)]$ the function $f_j(\lambda)$ is defined as the broken line with slopes 1 and the knots at the points

$$(-1, f_j(-1)), (-1-|f(-1)-1|(1+\frac{1}{\sqrt{2}}), 1-\frac{1}{\sqrt{2}}(f_j(-1)-1)), (-1-\lambda_-(j), 1) \tag{3.16}$$

and

$$(1, f_j(1)), (1+|f_j(1)-1|(1+\frac{1}{\sqrt{2}}), 1-\frac{1}{\sqrt{2}}(f_j(1)-1)), (1+\lambda_+(j), 1) \tag{3.17}$$

respectively.

The collection of functions $\{f_j\}$ will constitute the set S. The number of points in the set is close to $N/2\tau$. For large N all these functions are positive and thus they are spectral densities. The restriction of f_j on the interval $[-1,1]$ can be continued analytically into the whole complex plane as the function $\psi_j(z) = 1+e^{-\gamma N}\sum P_k(z)$. The Legendre polynomials $P_n(z)$ for $z \notin [-1,1]$ satisfy the inequality (see [5])

$$|P_n(z)| \leq Cn^{1/2}|z + (z^2-1)^{1/2}|^n.$$

Hence it is possible to choose γ in a such way that all $f_j \in F$, in other words, $|\psi_j(z)| \leq M, z \in E$ (if M is too small we change 1's in the definition (3.14) of f_j to small δ's etc).

For any two functions f_i, f_j the distance $|f_i - f_j|_p$ satisfies

$$|f_i - f_j|_p \geq e^{-\gamma}\Big(\int_{-1}^{1} |\sum_{k\in I(i,j)} P_k(\lambda)|^p d\lambda\Big)^{1/p}, \tag{3.18}$$

where $I(i,j)$ is an interval of the type $[xN, yN]$ and $y - x > \tau$.

By V.Markov's inequality

$$|P_n'|_\infty \leq cn^2|P_n|_\infty \leq cn^{5/2}. \tag{3.19}$$

It follows from (2.11) and (3.19) that

$$\int_{-1}^{1} |\sum_{k\in I(i,j)} P_k(\lambda)|^p d\lambda \geq \int_{1-N^{-2}}^{1} |\sum_{k\in I(i,j)} P_k(\lambda)|^p d\lambda \geq c\tau N^{3p/2-2}, c > 0.$$

Hence

$$|f_i - f_j|_p \geq c\tau N^{3/2-2/p}e^{-\gamma N}. \tag{3.20}$$

By the definition of f_j,

$$\int_{-\pi}^{\pi}(f_j(\lambda)-1)d\lambda = \sum_k \int_{-1}^{1} P_k(\lambda)d\lambda = 0.$$

Hence by formula (3.6) Shannon's information satisfies

$$I(\theta, X) \leq CTe^{-2\gamma N}\sup_j \big(\int_{-\pi}^{\pi} |f_j(\lambda)-1|^2 d\lambda + O(\sup_\lambda |f_j(\lambda)-1|^3) + C\big).$$

It follows that

$$I(\theta, X) \leq CTe^{-2\gamma N}N + C. \tag{3.21}$$

Returning to (3.4), we find with the help of (3.20) and (3.21) that

$$\Delta_p(T) \geq c\tau N^{3/2-2/p}e^{-\gamma N}\big(1 - \frac{C_1Te^{-2\gamma N} + C_2}{|\ln\tau|}\big), c > 0.$$

Take at first τ so small that $C_2(|\ln\tau|)^{-1} \leq 1/4$ and then choose $N \asymp \ln T$ in a such way that $C_1 T e^{-2\gamma N}|\ln\tau|^{-1} \leq 1/4$. We then obtain

$$\Delta_p(T) \geq cT^{-1/2}(\ln T)^{1-2/p}, c > 0.$$

3.5. The case $p = 4$. This case is more complicated and needs a special treatment. We will only outline the proof omitting technical details concerning the estimation of remainders. Consider the set $\mathbf{A}$ of vectors $\mathbf{a} = (a_1, \dots a_N)$ such that $\mathbf{A} = \{\mathbf{a} : |\mathbf{a}| \leq N^2\}$. Set up in correspondence to a vector $\mathbf{a} \in \mathbf{A}$ the spectral density $f_{\mathbf{a}}$ defined in the following way. In the interval $[-1, 1]$

$$f_{\mathbf{a}}(\lambda) = 1 + T^{-1/2}\sum_1^N a_j P_{2j}(\lambda).$$

Define now the numbers $\lambda_-(\mathbf{a}), \lambda_+(\mathbf{a})$ as in (3.15) and set $f_{\mathbf{a}}(\lambda) = 1$ outside the interval $[-1-\lambda_-(\mathbf{a}), 1 = \lambda - +(\mathbf{a})]$. In the small intervals $f_{\mathbf{a}}$ is defined by (3.16) and (3.17). We denote the set $\{f_{\mathbf{a}}, \mathbf{a} \in \mathbf{A}\}$ also $\mathbf{A}$. If we relate N and T by $N \sim 1/\gamma \ln T$, one can choose γ in a such way that $\mathbf{A} \subset F$.

We have then that

$$\begin{aligned}\inf_{\hat f}\sup_{f\in F}\mathbf{E}_f|f-\hat f|_4 &\geq \inf_{\hat f}\sup_{f\in\mathbf{A}}\mathbf{E}_{\mathbf{a}}|f_{\mathbf{a}}-\hat f|_4 \geq \\ T^{-1/2}\inf_t\frac{1}{mes\mathbf{A}}&\int_{\mathbf{A}}\mathbf{E}_{\mathbf{a}}|\sum_1^N(a_j-t_j)P_{2j}|_4 d\mathbf{a}.\end{aligned}\tag{3.22}$$

Let P_0 denote the distribution corresponding to the Gaussian statationary sequence with the spectral density 1. It follows from (3.22) that

$$\Delta_4(T) \geq T^{-1/2}\inf_t\frac{1}{mes\mathbf{A}}\int_{\mathbf{A}}\mathbf{E}_0\{|\sum_1^N(a_j-t_j)P_{2j}|_4\frac{dP_{\mathbf{a}}}{dP_0}\}d\mathbf{a}.\tag{3.23}$$

By Lemma 3.1 the likelihood ratio can be rewritten as

$$\ln\frac{dP_{\mathbf{a}}}{dP_0} = -T/4\pi\int_{-\pi}^{\pi}\ln f_{\mathbf{a}}(\lambda)d\lambda - T/4\pi\int_{-\pi}^{\pi}I_T(\lambda)(\frac{1}{f_{\mathbf{a}}(\lambda)}-1)d\lambda + R_T.\tag{3.24}$$

The integrals

$$\int_{-\pi}^{\pi}(1-f_{\mathbf{a}}(\lambda)d\lambda = 0.$$

Thus the main part of the right hand side in (3.24) is equal to

$$\begin{aligned}-T/8\pi\int_{-\pi}^{\pi}(1-f_{\mathbf{a}}(\lambda)^2 d\lambda + T/4\pi\int_{-\pi}^{\pi}(I_T(\lambda)-\mathbf{E}I_T(\lambda))(f_{\mathbf{a}}(\lambda)-1)d\lambda = \\ -\frac{|\mathbf{a}|^2}{8\pi} - T/8\pi\int_{-\pi}^{\pi}(I_T(\lambda)-\mathbf{E}I_T(\lambda)(f_{\mathbf{a}}(\lambda)-1)d\lambda.\end{aligned}$$

The last integral is asymptotically normal with mean zero and variance $|\mathbf{a}|^2/8\pi$.

It follows then from (3.23) that for large T

$$\Delta_4(T) \geq \frac{c}{\sqrt{T}} \int_{\mathbf{A}} \mathbf{E}\{exp\{-|\mathbf{a}|^2 + (\mathbf{a}, \xi)\}| \sum_1^N (a_j - t_j) P_{2j}|\},$$

where $c > 0$ is a positive constant and $\xi = (\xi_1, \dots \xi_N)$ is the standard Gaussian random vector.

It is shown in [4] that the expression on the right is bigger than

$$\frac{c}{\sqrt{T}} \mathbf{E}\{| \sum_1^N \xi_j P_{2j}|_4\} \geq c\sqrt{\frac{N}{T}}(\ln)^{1/4} \geq c\sqrt{\frac{\ln T}{T}}(\ln\ln T)^{1/4}.$$

Theorem 3.1 is thus proved. The main theorem 1.1 follows from Corollary 2.1 and the theorem 3.1.

4. Processes with continuous time.

In this section we suppose that $X(t)$ is a stationary (generalized) Gaussian process with mean zero and the spectral density $f(\lambda)$ observed at the interval $[0, T]$. In the case of generalized processes it means that the set of observables (=statistics) consists of random variables $X(\phi)$, where ϕ runs over the set of all test functions with the support in $[0, T]$. We estimate the restriction of the spectral density f on an interval $[a, b]$ and measure the deviation of estimates from f in $L_p(a, b)$ norms $|.|_p$. As above we define the minimax risk function $\Delta_p(T; F)$ as

$$\Delta_p(T; F) = \Delta_p(T) = \inf \sup \mathbf{E}_f |\hat{f}_T - f|_p,$$

where sup is taken over all $f \in F$ and inf is taken over all possible estimators $\hat{f}_T$.

Theorem 4.1. *Let the set F consist of all spectral densities of generalized processes analytic in some bounded region $G, [a, b] \subset G$ and bounded there by a common constant M. Then there exist estimators $\hat{f}_T$ such that*

$$\sup_{f \in F} \mathbf{E}_f |\hat{f}_T - f|_p \leq C\sqrt{\frac{\ln T}{T}}, 1 \leq p < 4,$$

$$\sup_{f \in F} \mathbf{E}_f |\hat{f}_T - f|_4 \leq C\sqrt{\frac{\ln T}{T}}(\ln\ln T)^{1/4}, \tag{4.1}$$

$$\sup_{f \in F} \mathbf{E}_f |\hat{f}_T - f|_p \leq T^{-1/2}(\ln T)^{1-2/p}, 4 < p \leq \infty.$$

The constants C depend on G, M, p only.

Theorem 4.2. *Let the set F consist from all spectral densities of real-valued (not generalized) processes analytic in some bounded region $G, [a, b] \subset G$ and bounded there by a common constant M. Then there exist positive constants $c > 0$ such that*

$$\Delta_p(T) \geq c\sqrt{\frac{\ln T}{T}}, 1 \leq p < 4$$

$$\Delta_4(T) \geq c\sqrt{\frac{\ln T}{T}}(\ln\ln T)^{1/4}), \tag{4.2}$$

$$\Delta_p(T) \geq cT^{-1/2}(\ln T)^{1-2/p}, 4 < p \leq \infty.$$

The constants c depend on G, M, p only.

Proofs. As above for the sake of simplicity we suppose that $[a,b]=[-1,1]$. To prove Theorem 4.1 we again expend $f(\lambda)$ in the interval $[-1,1]$ into the Fourier series with respect to the Legendre polynomials

$$f(\lambda)=\sum_{0)}^{N} a_j P_j(\lambda),$$

estimate the coefficients a_j by $\hat{a}_j$ and then consider as estimators of $f(\lambda)$ the sums

$$f_N(\lambda)=\sum_{0}^{N} \hat{a}_j P_j(\lambda).$$

Now, to estimate a_j we proceed in the following way. If the process $X(t)$ is not a generalized one, we define

$$\hat{a}_k=\int_{-1}^{1} I_T(\lambda)P_k(\lambda)d\lambda,$$

where the periodogram

$$I_T(\lambda)=\frac{1}{2\pi T}\Big|\int_0^T e^{-it\lambda}X(t)dt\Big|^2.$$

The other arguments coincide absolutely with the proof of the theorem 2.2.

If $X(\phi)$ is a generalized process, we define an analogue of the periodogram as

$$I_T(\lambda;\phi_T)=\frac{1}{2\pi T}|X(e^{-it\lambda}\phi_T)|^2,$$

where ϕ_T is a test function with the support in $[0,T]$ and set again

$$\hat{a}_k=\int_{-1}^{1} I_T(\lambda;\phi_T)P_k(\lambda).$$

We have

$$\mathbf{E}I_T(\lambda;\phi_T)=\frac{1}{2\pi T}\int_{-\infty}^{\infty}|\tilde{\phi}_T(\mu-\lambda)|^2 f(\mu)d\mu,$$

where $\tilde{\phi}$ denotes the Fourier transform of ϕ. One can now choose ϕ_T in a such way that $|\phi_T|^2$ is a function with a sharp maximum at zero and

$$\frac{1}{2\pi T}\int_{-\infty}^{\infty}|\phi_T(\lambda)|^2 d\lambda=1$$

and proceed as in the case of not generalized process. We omit the details.

The proof of Theorem 4.2 coincides up to technical details with the proof of the theorem 3.1 and we omit it.

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