Removable Singularities for Fully Nonlinear Elliptic Equations

Denis A. Labutin
Centre for Mathematics and its Applications
Australian National University
Canberra 0200 ACT Australia
E-mail: labutin@maths.anu.edu.au

Abstract

We obtain a removability result for the fully nonlinear uniformly elliptic equations $F(D^2u) + f(u) = 0$. The main theorem states that every solution to the equation in a punctured ball (without any restrictions on the behaviour near the centre of the ball) is extendable to the solution in the entire ball provided the function f satisfies certain sharp conditions depending on F. Previously such results were known for linear and quasilinear operators F. In comparison with the semi- or quasilinear theory the techniques for the fully nonlinear equations are new and based on the use of the viscosity notion of generalised solution rather than the distributional or the weak solutions.

1 Introduction

In the present paper we prove the existence of removable isolated singularities for equations

$$F(D^2u) + f(u) = 0 \tag{1.1}$$

with fully nonlinear (nonlinear on the second derivatives) uniformly elliptic operators F. None restrictions will be imposed on the behaviour of the solution near the singularity. Starting from Serrin's seminal papers [31]–[34] the main tools in the papers on this topic for semi- and quasilinear equations are the flexible notions of distributional and weak generalised solutions. Recently the notion of generalised *viscosity* solution was introduced for the fully nonlinear elliptic equations. Rather complete viscosity theory (existence, uniqueness, regularity) was developed for a large class of the fully nonlinear elliptic equations, see sec. 2 for references. In this paper we show how the viscosity notions can be used to investigate removable singularities for solutions of (1.1).

Let us first make some historical comments and describe how our results fit in the general picture. Let Ω be an open set in $\mathbf{R}^d$, $d \geq 2$, containing 0. In by now classical papers [33], [34] Serrin investigated the following unconditional removability effect for a large class of quasilinear equations of mean curvature type. Any solution to the equation in $\Omega \setminus K$ can be extended as a solution to the equation in the entire Ω provided the compact set K has the vanishing $(d-1)$-Hausdorff measure. For the minimal surfaces equation such removability was discovered earlier in [2], [10]. It was very important for the future development of the theory that the arguments in [33], [34] had purely PDE's nature and relied only on properties of the weak solutions to the equations under the consideration ($W^{1,1}$-weak solutions in that case), see also [31], [32]. The most interesting part in such removability statements is that none restrictions of any kind are imposed on the behaviour of the solution near the singularities. Later Brezis and Veron [4] turned to similar removability effects for equations with the Laplace operator. In particular they proved [4] that any solution $u \in C^2_{\text{loc}}(\Omega \setminus \{0\})$ of the equation

$$\Delta u - |u|^{q-1}u = 0 \text{ in } \Omega \setminus \{0\}, \quad q > 1, \tag{1.2}$$

can be defined at 0 as a C^2_{loc} solution to (1.2) in the entire Ω provided $q \geq d/(d-2)$, $d \geq 3$. For all $q \in (1, d/(d-2))$ there are radial solutions of (1.2) with nonremovable singularity at 0. In other words the theorem of Brezis and Veron states that *isolated singularities are removable* for solutions of (1.1) if and only if $q \geq d/(d-2)$. The case $q = (d+2)/(d-2)$ related to some geometrical problems was considered earlier by Loewner and Nirenberg [24]. Thereafter these results were generalised for different semi- and quasilinear equations, their parabolic counterparts as well as for singular sets rather than singletons. Up to now there were no results on similar effects for the equations nonlinear on the second derivatives. It should be mentioned however that some partial results are implicit in papers on linear equations in nondivergence form, see Remark 1.4.

The literature on singularities for semi- and quasilinear equations is very extensive and can not be reviewed in a short article. We refer to monograph [40] for the state of art up to 1996 and the rich bibliography. Among the recent results we mention paper [3] on the removability of compact sets with the vanishing Newton capacity for some semilinear equations and the series of papers by Dynkin and Kuznetsov and by Marcus and Veron on removable boundary singularities and boundary traces for semilinear equations. Dynkin and Kuznetsov [11], [12], see also [13], [14], [15] and references therein, utilised the probabilistic interpretation for the solutions of the corresponding semilinear equations, see also [23]. The approach of Marcus and Veron [25]–[27] is purely analytic.

The viscosity solutions for the fully nonlinear equations, sec. 2, do not have the integral nature similar to the distributional or the weak solutions. As a consequence our technique is different from the above papers on the semi- and quasilinear equations. It includes the analysis of the pointwise behaviour of the solution near the singularity and the application of the viscosity notions. As a corollary of our main Theorem 1.1 we obtain necessary and sufficient conditions for the removability of singletons provided f in (1.1) is the same as in (1.2), Remark 1.2. Thus our results can be viewed as an extension of the Brezis-Veron theorem to fully nonlinear operators. An extension to quasilinear operators was done by Vazquez and Veron in [38]. Now we recall some notions and finish this section stating the main results.

Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product in $\mathbf{R}^d$, $d \geq 2$; $B(x,R)$ denotes an open ball in $\mathbf{R}^d$ with centre x and radius R, $B_R = B(0,R)$. By $\mathbf{S}^d$, $d \geq 2$, we will denote the space of real $d \times d$ symmetric matrices equipped with its usual order; that is for $N \in \mathbf{S}^d$ $N \geq 0$ means $\langle Nx, x\rangle \geq 0$ for all $x \in \mathbf{R}^d$; I stands for the identity matrix. In the equation (1.1) $F : \mathbf{S}^d \to \mathbf{R}^1$. We will assume that F is *uniformly elliptic operator*. That is there are two constants $\Lambda \geq \lambda > 0$ (which are called the *ellipticity constants*) such that for any $M \in \mathbf{S}^d$

$$\lambda \mathrm{trace}(N) \leq F(M+N) - F(M) \leq \Lambda \mathrm{trace}(N) \quad \forall N \geq 0,$$

or equivalently $\lambda I \leq [\partial F(M)/\partial M_{ij}] \leq \Lambda I$. Examples of fully nonlinear uniformly elliptic equations arising in applications are the Bellman and Isaacs equations, see [5], [16] for the definitions. Important operators for the viscosity theory (and for our work) are the Pucci extremal operators $\mathcal{P}^{\pm}_{\lambda,\Lambda}$, see sec. 2. By *solution* we always mean the viscosity solution, sec. 2.

Our main result is Theorem 1.1 below, cf. [4]. All conditions of the theorem are *sharp*, see Remark 1.2. For fixed positive $\lambda \leq \Lambda$, $1 \leq \Lambda/\lambda < d-1$ we define

$$q(\Lambda/\lambda) = \frac{\lambda(d-1)+\Lambda}{\lambda(d-1)-\Lambda}. \tag{1.3}$$

Assume $f : \mathbf{R}^1 \to \mathbf{R}^1$ is a continuous function satisfying

$$\begin{aligned} &\limsup_{t\to+\infty} \frac{f(t)}{t^{q(\Lambda/\lambda)}} < 0 \\ &\liminf_{t\to-\infty} \frac{f(t)}{|t|^{q(\Lambda/\lambda)}} > 0. \end{aligned} \tag{1.4}$$

Theorem 1.1 *Let F be a uniformly elliptic operator in $\mathbf{S}^d$, $d \geq 3$, with the ellipticity constants λ and Λ, $1 < \Lambda/\lambda < d-1$, and let $u \in C_{\mathrm{loc}}(B_R \setminus \{0\})$ be a solution to*

$$F(D^2u) + f(u) = 0 \text{ in } B_R \setminus \{0\}, \tag{1.5}$$

where the continuous function f satisfies (1.4). Then u can be defined at 0 as a solution to the equation in (1.5) in the entire ball B_R.

For general uniformly elliptic F in (1.1), (1.5) the Trudinger $C^{1,\alpha}$ estimates [36] imply that $u \in C^{1,\alpha}_{\mathrm{loc}}(B_R)$. This is the best regularity known in such general situation, see [28] for the failure of C^2_{loc} regularity. If F in the theorem is additionally concave (or convex) on $\mathbf{S}^d$ then we obtain in the proof the viscosity solution which is the a.e. solution from $W^{2,p}_{\mathrm{loc}}(B_R)$ for all $0 < p < +\infty$, according to the Evans-Krylov and Caffarelli regularity results [5]. If in the theorem F is concave (convex) and $f \in C^{0,\alpha}_{\mathrm{loc}}(\mathbf{R}^1)$ then the Trudinger and Evans-Krylov estimates imply that $u \in C^{2,\gamma}_{\mathrm{loc}}(B_R)$ is the classical solution of the equation in the entire B_R.

Theorem 1.1 and its proof are valid also for $\Lambda = \lambda$ and $q(1) = d/(d-2)$ from (1.3). In this case

$$F(D^2u) = \Lambda\Delta u + F(0)$$

and due to the regularity for the viscosity solutions discussed above we recover exactly the Brezis and Veron result for (1.2) [4]. To exclude from our consideration the Laplace operator treated in [4] we will always assume that $\Lambda/\lambda > 1$.

Remark 1.2 *As a corollary of Theorem 1.1 we obtain that isolated singularities are removable for the fully nonlinear equation*

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) - |u|^{q-1}u = 0, \quad q > 1, \tag{1.6}$$

if and only if $1 < \Lambda/\lambda < d-1$ and $q \geq q(\Lambda/\lambda)$ from (1.3). Here $\mathcal{P}^+_{\lambda,\Lambda}$ is Pucci extremal operator (2.2). The same criterion holds for $\mathcal{P}^-_{\lambda,\Lambda}$ from (2.3). To see that the "only if" part holds it is enough to note the following. For $\Lambda/\lambda \geq d-1$ and any

$q > 1$, or for $1 < \Lambda/\lambda < d-1$ and $1 < q < q(\Lambda/\lambda)$ equation (1.6) has a solution of the form

$$u(x) = \frac{A_1}{|x|^{\frac{2}{q-1}}}, \quad A_1 > 0.$$

For $1 < q < (\Lambda(d-1)+\lambda)/(\Lambda(d-1)-\lambda)$ and any $\Lambda/\lambda > 1$ equation (1.6) has a solution of the form

$$u(x) = -\frac{A_2}{|x|^{\frac{2}{q-1}}}, \quad A_2 > 0.$$

Constants $A_{1,2}(\lambda, \Lambda, d, q)$ can easily be calculated explicitly. Let us also remark that for solutions of (1.6) we have $C^{2,\alpha}_{\text{loc}}$ regularity and the comparison principle. Using these tools the proof of Theorem 1.1 can be slightly simplified in the particular case of (1.6). In fact, after Lemma 3.1 is proved we can use the linearisation and argue utilising the comparison with fundamental solutions (2.4), (2.5) in a way similar to [20]. This approach collapses for general (non-convex, non-concave) F or for non-monotone f in (1.5).

Remark 1.3 *As we see from Remark 1.2 the removability does not hold for uniformly elliptic operators with ellipticity constants $\Lambda/\lambda \geq d-1$ under power growth condition of type (1.4). However the ideas from the proof of Theorem 1.1 work for arbitrary uniformly elliptic equations. Utilising them we can give some sufficient conditions for the removability of isolated singularities. To illustrate this let us consider the following example. Let u be a solution to*

$$F(D^2u) + g(x, u, Du) = 0 \quad \text{in} \quad B_R \setminus \{0\}, \tag{1.7}$$

where $g : B_R \times \mathbf{R}^1 \times \mathbf{R}^d \to \mathbf{R}^1$ is, say, bounded and continuous uniformly on Du. If $\Lambda/\lambda = d-1$ and u is uniformly bounded near 0, or if $\Lambda/\lambda > d-1$ and $u \in C_{\text{loc}}(B_R)$, $|u(x) - u(0)| \leq C|x|^\beta$ for some $\beta > 1 - (d-1)(\lambda/\Lambda)$, then u is a solution to (1.7) in the entire ball B_R. In Remark 3.3 we will briefly indicate the proof. Fundamental solutions (2.4) for $\mathcal{P}^+_{\lambda,\Lambda}$, $\Lambda/\lambda > d-1$, show that the Hölder continuity with the exponent β is the sharp condition for such general equations as (1.7). In the case $\Lambda/\lambda = d-1$ we can assume $u(x) = o(\log|x|)$, $x \to 0$, for slightly more restrictive g. Extensions to uniformly elliptic operators $F(x, D^2u)$ (and more general) are also straightforward. The case $\Lambda/\lambda = d-1$ for equation (1.5) is similar to the case of the Laplace operator in $\mathbf{R}^2$. It is quite likely (cf. [39], [40]) that for (1.5) with $\Lambda/\lambda = d-1$ Theorem 1.1 holds provided f has the super-exponential growth instead of the power growth (1.4). To obtain the sharp result similar to our Theorem 1.1 in the exponential scale is an interesting open problem.

Remark 1.4 *Let* $A(x) = [a_{ij}(x)]$ *be a measurable, symmetric,* $d \times d$ *matrix-valued function,* $\theta I \le A(x) \le \Theta I$ *for all* x *and fixed* $\Theta > \theta > 0$. *Results for the linear operator*

$$L_A u = \sum_{i,j=1}^{d} a_{ij}(x) D_{ij} u \tag{1.8}$$

with such A *are important for the fully nonlinear equations. In connection with this we note the following. The Brezis-Veron removability result does not hold for (1.2) with* Δ *replaced by* L_A *with the discontinuous* A *[1], [20], [40]. On the other hand if* $\Theta/\theta < d-1$ *for a measurable* A *then it is easy to prove the the following fact for equation (1.2) with* Δ *replaced by* L_A *and* $q = q(\Theta/\theta) + \varepsilon$, *(see (1.3))* $\varepsilon > 0$. *Any* $W^{2,d}_{\mathrm{loc}}(B_R \setminus \{0\})$*-solution to the equation in* $B_R \setminus \{0\}$ *with zero values on* ∂B_R *is identical zero. This assertion relies only on the comparison with the fundamental solutions (2.4), (2.5), cf. Remark 3.2, and is implicit in [1], [20], [22]. The obstacle towards proving the removability result for this equation (even for* $q > q(\Theta/\theta)$*) is the lack of the suitable notion of the generalised solution for equations with operators (1.8) with measurable coefficients. The existence of such notion is the famous open problem, e.g. [19].*

2 Preliminaries

First we recall some facts from the viscosity theory. Viscosity notion of generalised solution for the fully nonlinear equation

$$F(x, v, Dv, D^2 v) = 0 \tag{2.1}$$

was introduced and initially investigated in [9], [7] (the first order case) and [18], [17] (the second order case). In the present paper we will refer (sometimes implicitly) to surveys [6], [8] and to the book [5] for the exposition of the existence uniqueness and regularity of viscosity solutions. The history of the subject is carefully expounded in [19]. The reader can also find there the rich bibliography.

Let $\Omega \subset \mathbf{R}^d$ be an open set. We say that a polynomial P of degree 2 touches function $f : \Omega \to \mathbf{R}^1$ by above at $x_0 \in \Omega$ if there is a neighbourhood $B(x_0, \varepsilon)$, $\varepsilon > 0$ such that $f \le P$ in $B(x_0, \varepsilon)$ and $f(x_0) = P(x_0)$. Similarly P touches f by below if $P \le f$ in $B(x_0, \varepsilon)$, $f(x_0) = P(x_0)$. Let the function F in (2.1) be continuous in $\Omega \times \mathbf{R}^1 \times \mathbf{R}^d \times \mathbf{S}^d$ and $F(x,r,p,X) \ge F(x,r,p,Y)$ for all $X \ge Y$. A continuous function $v : \Omega \to \mathbf{R}^1$ is a *viscosity subsolution* (resp. *viscosity supersolution*) of (2.1) in Ω when the following condition holds:

if $x_0 \in \Omega$, P is any polynomial of degree 2 touching v by above at x_0 then $F(x_0, v(x_0), DP(x_0), D^2P(x_0)) \geq 0$

(resp. if P touches u by below at x_0 then $F(x_0, v(x_0), DP(x_0), D^2P(x_0)) \leq 0$).

We say that v is a *viscosity solution* of (2.1) when it is a subsolution and a supersolution. We say that $F(x, v, Dv, D^2v) \geq 0$ (resp. ≤ 0, $= 0$) in the viscosity sense in Ω whenever v is a viscosity subsolution (resp. supersolution, solution) of (2.1) in Ω.

The *Pucci extremal operators* $\mathcal{P}^{\pm}_{\lambda,\Lambda}(M)$, $M \in \mathbf{S}^d$, are defined as follows [29], [16] Ch. 17. If μ_j, $j = 1, \ldots, d$ are the eigenvalues of M, and $0 < \lambda \leq \Lambda$ then (see [5] Ch.2)

$$\mathcal{P}^+_{\lambda,\Lambda}(M) = \sup_{\lambda I \leq A \leq \Lambda I} \left(\sum_{i,j=1}^{d} A_{ij} M_{ij} \right) = \Lambda \sum_{\mu_j > 0} \mu_j + \lambda \sum_{\mu_j < 0} \mu_j, \tag{2.2}$$

$$\mathcal{P}^-_{\lambda,\Lambda}(M) = \inf_{\lambda I \leq A \leq \Lambda I} \left(\sum_{i,j=1}^{d} A_{ij} M_{ij} \right) = \lambda \sum_{\mu_j > 0} \mu_j + \Lambda \sum_{\mu_j < 0} \mu_j. \tag{2.3}$$

Obviously, $\mathcal{P}^+_{\lambda,\Lambda}(M) = -\mathcal{P}^-_{\lambda,\Lambda}(-M)$. We will use extensively the following connection between $\mathcal{P}^{\pm}$ and general uniformly elliptic operators. For an arbitrary uniformly elliptic F with the ellipticity constants $0 < \lambda \leq \Lambda$ the inequality $F(D^2u) \geq 0$ (resp. $F(D^2u) \leq 0$) in the viscosity sense implies $\mathcal{P}^+_{\lambda,\Lambda}(D^2u) \geq -F(0)$ (resp. $\mathcal{P}^-_{\lambda,\Lambda}(D^2u) \leq -F(0)$), [5] Ch. 2.

We define the *fundamental solutions* E^+, e^+ of the equation $\mathcal{P}^+_{\lambda,\Lambda}(D^2v) = 0$ in $\mathbf{R}^d$, $d \geq 2$, as

$$E^+(x) = E^+_{\Lambda/\lambda}(x) = \begin{cases} \dfrac{1}{|x|^{(d-1)\lambda/\Lambda - 1}} & \text{if } \; 1 \leq \Lambda/\lambda < d-1 \\ -\log|x| & \text{if } \; \Lambda/\lambda = d-1 \\ -|x|^{1-(d-1)\lambda/\Lambda} & \text{if } \; d-1 < \Lambda/\lambda, \end{cases} \tag{2.4}$$

$$e^+(x) = e^+_{\Lambda/\lambda}(x) = \begin{cases} \dfrac{-1}{|x|^{(d-1)\Lambda/\lambda - 1}} & \text{if } \; \Lambda/\lambda \geq 1 \text{ and } d \geq 3 \\ \dfrac{-1}{|x|^{\Lambda/\lambda - 1}} & \text{if } \; \Lambda/\lambda > 1 \text{ and } d = 2 \\ \log|x| & \text{if } \; \Lambda = \lambda \text{ and } d = 2. \end{cases} \tag{2.5}$$

It is easy to calculate that E^+, e^+ satisfy the equation in $\mathbf{R}^d \backslash \{0\}$. The fundamental solutions for the equation $\mathcal{P}^-_{\lambda,\Lambda}(D^2v)$ are defined as $E^- = -E^+$, $e^- = -e^+$.

We finish this section recalling the Krylov-Safonov Harnack inequality for the viscosity solutions of uniformly elliptic equation

$$F(D^2v) = g(x), \tag{2.6}$$

[5], see also [16], [21], [30], [37]. For general quasilinear divergence form elliptic equations the Harnack inequality was proved in [31]. If $v \in C_{\mathrm{loc}}(B_{3\rho/2})$, $v \geq 0$ in $B_{3\rho/2}$, satisfies (2.6) in $B_{3\rho/2}$, then

$$\sup_{B_\rho} v \leq C(\lambda, \Lambda, d) \left(\inf_{B_\rho} v + \rho \| g - F(0) \|_{L^d(B_{3\rho/2})} \right). \tag{2.7}$$

3 Proofs of the results

The proof of Theorem 1.1 consists of two steps. The first step (Lemma 3.1) is to show that u from the theorem can be defined at 0 by continuity. The second step is to show that this continuous function is a viscosity solution to (1.5) in the entire B_R. In comparison with quasilinear equations each step requires a new approach.

The proofs of the uniform boundness of the solution near the isolated singularity in linear and quasilinear cases rely on integral estimates. Such tools are not available in the fully nonlinear situation. Instead we prove the uniform boundness (3.2) in Lemma 3.1 utilising an observation based on the scale invariance of the equation and the comparison principle. The crucial property is that $r^{2/(q-1)}u(r\cdot)$ is a solution to (1.6) provided u is. For general equation (1.5) more work is needed. The proof of (3.2) under more restrictive conditions than (1.4) is easier, see Remark 3.2 below.

Lemma 3.1 *Under the assumptions of Theorem 1.1 there exists*

$$\lim_{x \to 0} u(x) = u_0, \quad -\infty < u_0 < +\infty. \tag{3.1}$$

Proof. We will show below that

$$u \in L^\infty(B_{R/2}). \tag{3.2}$$

Assuming for a moment that uniform bound (3.2) holds we can conclude the proof utilising the Harnack inequality (2.7), cf. [35]. In fact, from (3.2) it follows that our function $u \in C_{\mathrm{loc}}(B_R \setminus \{0\})$ is a solution to equation (2.6) in $B_R \setminus \{0\}$ with $g = -f \circ u \in L^\infty(B_{R/2}) \cap C_{\mathrm{loc}}(B_R \setminus \{0\})$. From (3.2) we can assume that

$$\liminf_{x \to 0} u(x) = 0. \tag{3.3}$$

We will show that

$$\limsup_{x\to 0} u(x) = 0, \tag{3.4}$$

thus establishing (3.1). In fact, (3.3) implies that for any ε, $0<\varepsilon<1$, the function $u_\varepsilon = u+\varepsilon$ is nonnegative near 0. We consider the shells

$$S_j = B_{r_{j+1}} \setminus B_{r_j}, \quad r_j = 2^{-j}, \quad j = 0,1,\dots.$$

Note that $u_\varepsilon \ge 0$ in S_j for all j sufficiently large. If such j is fixed then S_j can be covered by $\chi(d)$ number of balls

$$B(x_k, r_{j+1}), \quad |x_k| = 3r_{j+1}/2, \quad k = 1,\dots,\chi(d).$$

Applying (2.7) to u_ε in each of $B(x_k, r_{j+1})$ and combining the results we obtain

$$\begin{aligned} \sup_{S_j} u_\varepsilon &\le C_1(\lambda,\Lambda,d)\left(\inf_{S_j} u_\varepsilon + r_j \|g-F(0)\|_{L^d(B_{r_j})}\right) \\ &\le C_2(\lambda,\Lambda,d)\left(\inf_{S_j} u_\varepsilon + r_j^2 C_3\left(F,f,\|u\|_{L^\infty(B_{R/2})}\right)\right). \end{aligned}$$

Consequently, when $j\to\infty$ we get

$$\limsup_{x\to 0} u_\varepsilon(x) \le \liminf_{x\to 0} u_\varepsilon(x) \le C_2\varepsilon,$$

and (3.4) holds when $\varepsilon \to 0$.

Thus it is left to prove (3.2). To do this we will show that

$$u^+ = \max\{u,0\} \in L^\infty(B_{R/2}). \tag{3.5}$$

The boundness of $u^- = -(-u)^+$ can be established similarly, see the very end of the proof. We put

$$q = q(\Lambda/\lambda)$$

(see (1.3)) until the end of the proof. From the viscosity definitions, properties of the extremal operators, and (1.4) we obtain (after a possible scaling depending on f)

$$\mathcal{P}^+_{\lambda,\Lambda}\left(D^2(u^+)\right) - |u^+|^{q-1}u^+ + b \ge 0 \text{ in } B_{R/2}\setminus\{0\}, \tag{3.6}$$

where $b>0$ depends only on equation (1.5). By subtracting $\max_{\partial B_{R/2}}(u^+)$ and again taking the nonnegative part we can assume

$$u^+ = 0 \text{ on } \partial B_{R/2}. \tag{3.7}$$

We *claim* that there is a positive constant $\beta = \beta(q, \lambda, \Lambda, d, b)$ such that

$$u^+(x) \le \frac{\beta}{|x|^{\frac{2}{q-1}}} = \beta E^+(x) \quad \text{for all} \quad x \in B_{R/2} \setminus \{0\}. \tag{3.8}$$

Here E^+ is the fundamental solution (2.4). The proof of inequality (3.8) is the same as for the Laplace operator in [4]. It is based on (3.6) and the Keller-Osserman type barrier. We indicate it at the end of the proof of the lemma. We proceed further accepting estimate (3.8).

If

$$\limsup_{x \to 0} \frac{u^+(x)}{E^+(x)} = 0, \tag{3.9}$$

then the comparison arguments show that (3.5) holds. In fact, we define the function

$$v(x) = KE^+(x) - M|x|^2 + N, \; x \ne 0, \tag{3.10}$$

by specifying the positive constants K, M, N as follows. We put $K > 0$,

$$M \ge b/\lambda \quad \text{and} \quad N = MR^2/4. \tag{3.11}$$

Then due to the uniform ellipticity of the Pucci extremal operators we obtain for all $x \in B_{R/2} \setminus \{0\}$

$$\begin{aligned} \mathcal{P}^+_{\lambda,\Lambda}(D^2 v(x)) - |v(x)|^{q-1} v(x) \quad &+ b \le \mathcal{P}^+_{\lambda,\Lambda}(KD^2E^+(x) - 2MI) + b \\ &\le \mathcal{P}^+_{\lambda,\Lambda}(KD^2E^+(x)) - \lambda d 2M + b \\ &\le 0. \end{aligned} \tag{3.12}$$

Let $x \in B_{R/2} \setminus \{0\}$. We recall (3.7), (3.9) and apply the comparison principle to u^+ and v satisfying (3.6) and (3.12) in $B_{R/2} \setminus B_\varepsilon$ for small enough $\varepsilon > 0$. As a result we obtain for any fixed $x \in B_{R/2} \setminus \{0\}$

$$u^+(x) \le KE^+(x).$$

Letting $K \to 0$ then establishes (3.5).

Now we show that (3.9) is always the case. Seeking a contradiction assume that (3.9) does not hold and consequently

$$\limsup_{x \to 0} \frac{u^+(x)}{E^+(x)} > 0. \tag{3.13}$$

Then we can choose $K > 0$ in (3.10) so that

$$\limsup_{x \to 0} \frac{u^+(x)}{v(x)} = 1/2, \tag{3.14}$$

for all M, $N > 0$. Let M and N be defined in (3.11). Thus in particular (3.12) holds. We prove that (3.13) leads to a contradiction in three steps.

First we show that (3.14) and the comparison principle imply that

$$u^+ \leq v/2 \quad \text{in } B_{R/2} \setminus \{0\}. \tag{3.15}$$

In fact, from (3.14) we can find a sequence $\{\delta_j\}$, $\delta_j \downarrow 0$ as $j \to \infty$, such that for all $j \geq 1$

$$1/3 < \max_{\partial B_{\delta_j}} \left(u^+/v\right) < 2/3, \tag{3.16}$$

and

$$\max_{\partial B_{\delta_j}} (u^+/v) \to 1/2, \quad j \to \infty. \tag{3.17}$$

Let us introduce the shells $S_{\delta_j} = \overline{B}_{R/2} \setminus B_{\delta_j}$. Recalling (3.7) we put

$$\gamma_j = \max_{\partial S_{\delta_j}} (u^+/v) = \max_{\partial B_{\delta_j}} (u^+/v) = u(x_j)/v(x_j),$$

for some x_j, $|x_j| = \delta_j$. Thus we have

$$u^+ \leq \gamma_j v \quad \text{on } \partial S_{\delta_j}. \tag{3.18}$$

In S_{δ_j} we have differential inequality (3.6) for u^+. Let us show that $\gamma_j v$ is a supersolution to the same equation in S_{δ_j}. In fact, we have:

$$\begin{aligned} \mathcal{P}^+_{\lambda,\Lambda}(D^2 \gamma_j v) - |\gamma_j v|^{q-1} \gamma_j v \quad &+ b = \gamma_j \mathcal{P}^+_{\lambda,\Lambda}(D^2 v) - \gamma_j^q |v|^{q-1} v + b \\ &= \gamma_j^q \left(\mathcal{P}^+_{\lambda,\Lambda}(D^2 v) - |v|^{q-1} v + b\right) \\ &+ (\gamma_j - \gamma_j^q) \mathcal{P}^+_{\lambda,\Lambda}(D^2 v) + (1 - \gamma_j^q) b \\ &\leq 0, \end{aligned} \tag{3.19}$$

provided we take M and N in (3.11) such that in addition to (3.12) one has

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 v) + \frac{1 - \gamma_j^q}{\gamma_j - \gamma_j^q} b \leq 0.$$

From (3.10), (3.16) and the calculations in (3.12) the latter estimate holds for any

$$M \geq \frac{3^q - 1}{3^{q-1} - 2^{q-1}} \, \frac{b}{\lambda}.$$

Now from (3.6), (3.18), (3.19) due to the comparison principle $u^+ \leq \gamma_j v$ in S_{δ_j}, or

$$\max_{\overline{S}_{\delta_j}} (u^+/v) \leq \max_{\partial B_{\delta_j}} (u^+/v). \tag{3.20}$$

Consequently

$$\max_{\partial B_{\delta_j}}(u^+/v) = u^+(x_j)/v(x_j) \uparrow 1/2, \quad j \to \infty. \tag{3.21}$$

From (3.20) and (3.21) we conclude that (3.15) holds.

The next step towards establishing the contradiction is to construct two sequences of functions. In every shell

$$\Omega_j = B_{2\delta_j} \setminus B_{\delta_j/2}, \quad j = 1, 2, \ldots,$$

we define the function w_j as the solution to the Dirichlet problem

$$\begin{cases} \mathcal{P}^+_{\lambda,\Lambda}(D^2 w_j) - |w_j|^{q-1} w_j + b = 0 & \text{in} \quad \Omega_j \\ w_j = u^+ & \text{on} \quad \partial\Omega_j. \end{cases} \tag{3.22}$$

Due to the comparison principle from (3.6), (3.12), (3.19), (3.22) we obtain

$$u^+ \le w_j \le v/2 \ \text{ in } \ \Omega_j. \tag{3.23}$$

Let $\Omega = \overline{B}_2 \setminus B_{1/2}$. For any fixed j we consider the mapping $\Omega \to \Omega_j$ given by $x \mapsto \delta_j x$ and the corresponding pullbacks

$$v_{\delta_j}(x) = \delta_j^{\frac{2}{q-1}} v(\delta_j x), \quad x \in \Omega,$$

$$w_{\delta_j}(x) = \delta_j^{\frac{2}{q-1}} w_j(\delta_j x), \quad x \in \Omega.$$

The final step to establish that (3.13) is impossible is to analyse the behaviour of v_{δ_j} and w_{δ_j} in Ω. We have

$$\delta_j^{\frac{2}{q-1}} E^+(\delta_j x) = E^+(x), \quad x \ne 0.$$

Consequently, from definition (3.10) the sequence $\{v_{\delta_j}\}$ converges in $C(\overline{\Omega})$ to KE^+, $K > 0$. We put

$$V = KE^+, \quad \|v_{\delta_j} - V\|_{L^\infty(\Omega)} \to 0, \ j \to \infty.$$

Next we analyse the sequence $\{w_{\delta_j}\}$. From (3.10), (3.23) we conclude that

$$0 \le w_{\delta_j} \le V/2 + \delta_j^{\frac{2}{q-1}} C(M, N, R) \ \text{ in } \ \Omega. \tag{3.24}$$

In other words, the sequence $\{w_{\delta_j}\}$ is uniformly bounded in Ω. Moreover, the sequence is locally equicontinuous in Ω. In fact, for any j we have

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 w_{\delta_j}) - |w_{\delta_j}|^{q-1} w_{\delta_j} + b\delta_j^{\frac{2q}{q-1}} = 0 \ \text{ in } \ \Omega. \tag{3.25}$$

The equicontinuity is now a consequence of (3.24) and Trudinger's $C^{1,\alpha}$ estimates [36] (see also [5]) applied to (3.25). Passing to a subsequence we can assume that $w_{\delta_j} \to w$ in $C^{0,1}_{\mathrm{loc}}(\Omega)$ as $j \to \infty$ for some $w \in C^{0,1}_{\mathrm{loc}}(\Omega)$. From (3.25) due to the stability of the viscosity solutions with respect to uniform convergence [5]

$$\mathcal{P}^{+}_{\lambda,\Lambda}(D^2 w) - |w|^{q-1}w = 0 \quad \text{in} \quad \Omega. \tag{3.26}$$

Moreover, the Evans-Krylov $C^{2,\alpha}$ estimates imply that $w \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ satisfies (3.26) in the classical sense. From (3.24) we have

$$w - V/2 \le 0 \text{ in } \Omega. \tag{3.27}$$

From (3.26) by the linearisation we see that the function $(w - V/2) \in C^{2}_{\mathrm{loc}}(\Omega)$ satisfies

$$\sum_{i,j=1}^{d} A_{ij}(x) D_{ij}(w - V/2) + c(x)(w - V/2) \ge 0 \quad \text{a.e. in} \quad \Omega, \tag{3.28}$$

where $\lambda I \le A(\cdot) \le \Lambda I$ and

$$c(x) = \begin{cases} -\dfrac{|V/2|^{q-1}V/2 - |w|^{q-1}w}{V/2 - w} & \text{if} \quad w(x) < V(x)/2 \\ 0 & \text{if} \quad w(x) = V(x)/2, \end{cases}$$

$c \le 0$ in Ω. At the same time from (3.21), (3.23) we can find a sequence $\{z_k\}$, $|z_k| = 1$, such that

$$\frac{w_{\delta_k}(z_k)}{v_{\delta_k}(z_k)} \ge \frac{u^{+}(\delta_k z_k)}{v(\delta_k z_k)} = \max_{\partial B_{\delta_k}} \left(u^{+}/v\right) \to 1/2.$$

Consequently for some z, $|z| = 1$,

$$0 \le w(z) - V(z)/2. \tag{3.29}$$

From the strong maximum principle [16] and (3.27)–(3.29) we obtain that $w - V/2 \equiv$ *const*. But this contradicts (3.26). We can conclude that (3.9) always holds and the lemma is proved.

Estimate (3.5) will be proved completely provided we show (3.8). As in the semilinear case [4] to prove (3.8) it suffices to show that the Keller-Osserman function

$$U(x) = \mu \frac{1}{(\rho^2 - |x|^2)^{\frac{2}{q-1}}} + \nu$$

satisfies $\mathcal{P}^{+}_{\lambda,\Lambda}(D^2 U) - |U|^{q-1}U + b \le 0$ in B_ρ, $\rho > 0$, for some $\mu, \nu > 0$. To check this note that U is convex. Consequently $\mathcal{P}^{+}_{\lambda,\Lambda}(D^2 U) = \Lambda \Delta U$. Thus we can use the calculations made in [4] for the Laplacian. Following that paper we obtain that, for example,

$$\mu = \left(\frac{8\Lambda}{q-1} \left(d + \frac{q+1}{q-1} \right) \right)^{\frac{1}{q-1}} R^{\frac{2}{q-1}}, \quad \nu = (2b)^{1/q}$$

will do the job.

The proof that $u^- \in L^\infty(B_{R/2})$ is exactly the same. Similar to (3.6) from (1.4) the function u^- satisfies

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2(u^-)) - |u^-|^{q-1}u^- - \tilde{b} \le 0 \ \text{ in } \ B_{R/2} \setminus \{0\}. \tag{3.30}$$

Then we proceed as above using (3.30) instead of (3.6) and the fundamental solution E^- (sec. 2) instead of E^+. □

Remark 3.2 *If the function* f *in (1.5) satisfies (1.4) with* $q_\varepsilon = q(\Lambda/\lambda) + \varepsilon$, $\varepsilon > 0$, *instead of the sharp exponent* $q(\Lambda/\lambda)$, *then the proof of (3.2) is substantially easier. In fact, for such* q_ε *Keller-Osserman estimate (3.8) implies (3.9) immediately.*

Proof of Theorem 1.1. From Lemma 3.1 the function u can be defined at 0 as a continuous function in the entire B_R. We show that this function is a viscosity solution to (1.5) in B_R. In sec. 1 we have already discussed when this viscosity solution is the strong or the classical solution.

Let us show that u is a viscosity *subsolution.* The proof that u is a supersolution is the same. Thus we need to establish that

$$F(D^2P(0)) + f(u(0)) \ge 0 \tag{3.31}$$

for any quadratic polynomial P touching u at 0 by above. We make the following statement concerning the behaviour of u near 0. Let l be an affine function in $\mathbf{R}^d$,

$$l(x) = a_0 + \sum_{i=1}^{d} a_i x_i. \tag{3.32}$$

We *claim* that if $l(0) = u(0)$, then there exists a sequence of points $\{z_j\}$, $z_j \in B_R$, $z_j \to 0$ as $j \to \infty$, such that

$$u(z_j) \ge l(z_j) + o(|z_j|) \text{ when } j \to \infty. \tag{3.33}$$

The existence of a sequence for which the opposite inequality holds is needed to prove that u is a supersolution in B_R. It can be established in the same way as (3.33) is established below.

We postpone the proof of (3.33). Accepting it for a moment we show now that (3.31) holds. In fact, let the quadratic polynomial P touches u at 0 by above. Let us fix any $\delta > 0$. There exists $r_0 = r_0(\delta) > 0$ such that the polynomial $P_\delta(x) = P(x) + \delta|x|^2/2$ satisfies

$$P_\delta(0) = u(0), \quad P_\delta(x) > u(x) \text{ for any } x \in B_{r_0}.$$

Consequently for the polynomial

$$P_{\delta,\varepsilon}(x) = P_\delta(x) - \varepsilon(x_1 + \cdots + x_d) = P(x) + \delta|x|^2/2 - \varepsilon(x_1 + \cdots + x_d)$$

we have

$$u(0) - P_{\delta,\varepsilon}(0) = 0, \quad (u - P_{\delta,\varepsilon}) < 0 \text{ in } B_{r_0} \setminus B_{\rho(\varepsilon)}, \tag{3.34}$$

where $0 \le \rho(\varepsilon) \le r_0$, $\rho(\varepsilon) \to 0$ when $\varepsilon \to 0$. Next we utilise (3.33) for

$$l(x) = \langle DP_\delta(0), x\rangle + P_\delta(0).$$

Passing to a subsequence we can assume that all coordinates of each z_j in (3.33) are nonnegative. Then from (3.33) we obtain $u(z_j) - P_{\delta,\varepsilon}(z_j) > 0$ for z_j with sufficiently large j. Thus from (3.34) we can find a point x^ε, $x^\varepsilon \neq 0$, such that

$$u(x^\varepsilon) - P_{\delta,\varepsilon}(x^\varepsilon) = \max_{B_{r_0}} (u - P_{\delta,\varepsilon}) > 0. \tag{3.35}$$

From (3.34), (3.35) the polynomial

$$P_{\delta,\varepsilon}(x) + u(x^\varepsilon) - P_{\delta,\varepsilon}(x^\varepsilon)$$

touches u at x^ε, $x^\varepsilon \neq 0$, by above. The function u satisfies (1.5) in the viscosity sense. Consequently

$$F(D^2P + \delta I) + f\left(u(x^\varepsilon)\right) \ge 0.$$

From (3.34), (3.35) $x^\varepsilon \in B_{\rho(\varepsilon)}$ and in particular $|x^\varepsilon| \to 0$ when $\varepsilon \to 0$. When $\varepsilon \to 0$ we obtain using the continuity of $f \circ u$

$$F(D^2P + \delta I) + f(u(0)) \ge 0.$$

Letting $\delta \to 0$ we obtain (3.31).

To complete the proof of the theorem we need to establish the statement contained in (3.33). Seeking a contradiction suppose that for l in (3.32), $l(0) = u(0)$, there exist $m > 0$, $\rho_0 > 0$ such that

$$u(x) < l(x) - m|x| \quad \text{for any } x \in B_{\rho_0}. \tag{3.36}$$

For positive A, B, t we introduce the function

$$w(x) = l(x) - A|x|^2 + tE^+(x) - B, \quad x \in \mathbf{R}^d \setminus \{0\}. \tag{3.37}$$

First we choose $A > 0$ such that for all $B, t > 0$ in (3.37)

$$\begin{aligned} \mathcal{P}^+_{\lambda,\Lambda}(D^2 w) + |F(0)| \quad & + |f(u(0))| + 1 = \mathcal{P}^+_{\lambda,\Lambda}(tD^2 E^+ - 2AI) \\ & + |F(0)| + |f(u(0))| + 1 \\ & \leq -\lambda d 2A + |F(0)| + |f(u(0))| + 1 \\ & \leq 0 \ \text{ in } \ \mathbf{R}^d \setminus \{0\}. \end{aligned} \tag{3.38}$$

Next we choose $a > 0$ such that

$$|f(s) - f(u(0))| \leq 1 \ \text{ for } \ |s - u(0)| < a. \tag{3.39}$$

Now for these A and a we fix the ball B_r, $0 < r < \rho_0$, so small that

$$A|x|^2 \leq m|x|/4 \ \text{ for all } \ x \in B_r, \tag{3.40}$$

$$|u(x) - u(0)| < a \ \text{ for all } \ x \in B_r. \tag{3.41}$$

Finally, we put $B = mr/4$ in (3.37).

Then from (3.36), (3.37), (3.40) we have

$$u < l - mr/2 < w \ \text{ on } \ \partial B_r. \tag{3.42}$$

From (1.5), (3.39), (3.41)

$$\begin{aligned} 0 = F(D^2 u) + f(u) \quad & \leq \mathcal{P}^+_{\lambda,\Lambda}(D^2 u) + |F(0)| + |f(u(0))| + 1 \\ & + (f(u) - |f(u(0))| - 1) \\ & \leq \mathcal{P}^+_{\lambda,\Lambda}(D^2 u) + |F(0)| + |f(u(0))| + 1 \\ & \text{in } \ B_r \setminus \{0\}. \end{aligned} \tag{3.43}$$

Note that for any $t > 0$ in (3.37) $w(x) \to +\infty$ when $x \to 0$. Thus from (3.38), (3.42), (3.43) we can apply the comparison principle to u and w in $B_r \setminus B_{\tilde{r}}$ for all sufficiently small $\tilde{r}$, $0 < \tilde{r} < r$. As a result we obtain

$$u(x) \leq l(x) - A|x|^2 + tE^+(x) - mr/2 \ \text{ in } \ B_r \setminus B_{\tilde{r}}.$$

Letting $t \to 0$ we conclude that

$$u \leq l - mr/2 \ \text{ in } \ B_r \setminus \{0\}.$$

But this contradicts the continuity of u at 0. □

Remark 3.3 *Note that the constructions from the proof of Theorem 1.1 can be used to prove the removability of singletons for equation (1.7) under the conditions stated in Remark 1.3. Under those conditions* u *coincides with a function continuous in* B_R*. As in the proof of Theorem 1.1 for any affine function* l*,* $l(0) = u(0)$*, there exists a sequence of points* $\{z_j\}$*,* $z_j \in B_R$*,* $z_j \to 0$ *as* $j \to \infty$*, such that (3.33) holds. If the claim holds we conclude that* u *is a subsolution exactly as in the proof of Theorem 1.1 utilising the tilted polynomials* $P_{\varepsilon,\delta}$ *with properties (3.34), (3.35). To see that (3.33) holds we can follow the lines (3.36)–(3.43) in the proof of Theorem 1.1 using the function*

$$w(x) = l(x) - A|x|^2 + tE^+(x).$$

In the case $\Lambda/\lambda > d - 1$ $w(0) = u(0)$ *and we obtain under the suitable choice of the constants that* $u - u(0) \leq w - w(0)$ *near* 0*. This contradicts the assumption of the Hölder continuity of* u*.*

Acknowledgements. I wish to thank my PhD supervisor Neil Trudinger for discussions and support, and Ben Andrews and Xu-Jia Wang for useful remarks.

References

[1] Aviles, P., *A study of the singularities of solutions of a class of nonlinear elliptic partial differential equations*, Comm. Partial Differential Equations **7** (1982), 609–643.

[2] Bers. L., *Isolated singularities of minimal surfaces*, Ann. of Math. **53** (1951), 364–386.

[3] Brezis, H., Nirenberg, L., *Removable singularities for nonlinear elliptic equations*, Topol. Methods Nonlinear Anal. **9** (1997), 201–219.

[4] Brezis, H., Veron, L., *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980/81), 1–6.

[5] Cabre, X., Caffarelli, L. A., *Fully nonlinear elliptic equations*, Providence, RI, 1995.

[6] Crandall, M. G., *Viscosity solutions: a primer*, 1–43, Lecture Notes in Math., 1660, Springer, Berlin, 1997.

[7] Crandall, M. G., Evans, L. C., Lions, P.-L., *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **282** (1984), 487–502.

[8] Crandall, M. G., Ishii, H., Lions, P.-L., *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[9] Crandall, M. G., Lions, P.-L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.

[10] DeGiorgi, E., Stampacchia, G., *Sulle singolarita eliminabili delle ipersuperficie minimali*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **38** (1965), 352–357.

[11] Dynkin, E. B., *A probabilistic approach to one class of nonlinear differential equations*, Probab. Theory Related Fields **89** (1991), 89–115.

[12] Dynkin, E. B., Kuznetsov, S. E., *Superdiffusions and removable singularities for quasilinear partial differential equations*, Comm. Pure Appl. Math. **49** (1996), 125–176.

[13] Dynkin, E. B., Kuznetsov, S. E., *Trace on the boundary for solutions of nonlinear differential equations*, Trans. Amer. Math. Soc. **350** (1998), 4499–4519.

[14] Dynkin, E. B., Kuznetsov, S. E., *Fine topology and fine trace on the boundary associated with a class of semilinear differential equations*, Comm. Pure Appl. Math. **51** (1998), 897–936.

[15] Dynkin, E. B., Kuznetsov, S. E., *Extinction of superdiffusions and semilinear partial differential equations*, J. Funct. Anal. **162** (1999), no. 2, 346–378.

[16] Gilbarg, D., Trudinger, N. S., *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin-New York, 1983.

[17] Ishii, H., *On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs*, Comm. Pure Appl. Math. **42** (1989), 15–45.

[18] Jensen, R., *The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations*, Arch. Rational Mech. Anal. **101** (1988), 1–27.

[19] Jensen , R. R., *Viscosity solutions of elliptic partial differential equations*, Doc. Math., Extra Vol. III (1998), 31–38.

[20] Kondratjev, V. A., Landis, E. M., *Qualitative properties of the solutions of a second-order nonlinear equation*, Math. USSR-Sb. **63** (1989), 337–350.

[21] Krylov, N. V., Safonov, M. V., *A property of the solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 161–175, 239.

[22] Landis, E. M., *Uniqueness theorems for the solution of the Dirichlet problem for second-order elliptic equations*, Proc. Moscow Math. Soc. **42** (1981), 50–63.

[23] LeGall, J.-F., *The Brownian snake and solutions of* $\Delta u = u^2$ *in a domain*, Probab. Theory Related Fields **102** (1995), 393–432.

[24] Loewner, C., Nirenberg, L., *Partial differential equations invariant under conformal or projective transformations*, in Contributions to Analysis, 245–272, New York-London: Academic Press 1974.

[25] Marcus, M., Veron, L., *Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 237–274.

[26] Marcus, M., Veron, L., *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, J. Math. Pures Appl. (9) **77** (1998), 481–524.

[27] Marcus, M., Veron, L., *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. Rational Mech. Anal. **144** (1998), 201–231.

[28] Nadirashvili, N., *Nonclassical solutions of fully nonlinear equations*, preprint.

[29] Pucci, C., *Operatori ellittici estremanti*, Ann. Mat. Pura Appl., **72** (1966), 141–170.

[30] Safonov, M. V., *Harnack's inequality for elliptic equations and Hlder property of their solutions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **96** (1980), 272–287, 312.

[31] Serrin, J., *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302.

[32] Serrin, J., *Isolated singularities of solutions of quasi-linear equations*, Acta Math. **113** (1965), 219–240.

[33] Serrin *Removable singularities of solutions of elliptic equations*, Arch. Rational Mech. Anal. **17** (1964), 67–78.

[34] Serrin, J., *Removable singularities of solutions of elliptic equations. II*, Arch. Rational Mech. Anal. **20** (1965), 163–169.

[35] Serrin, J., *Singularities of solutions of nonlinear equations*, 1965 Proc. Sympos. Appl. Math., Vol. XVII, 68–88, Amer. Math. Soc., Providence, R.I.

[36] Trudinger, N. S., *Hölder gradient estimates for fully nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), 57–65.

[37] Trudinger, N. S., *Lectures on nonlinear elliptic equations of second order*, Lecture notes published by Tokyo university, 1995.

[38] Vazquez, J. L., Veron, L., *Removable singularities of some strongly nonlinear elliptic equations*, Manuscripta Math. **33** (1980/81), 129–144.

[39] Vazquez, J. L., Veron, L., *Singularities of elliptic equations with an exponential nonlinearity*, Math. Ann. **269** (1984), 119–135.

[40] Veron, L., *Singularities of solutions of second order quasilinear equations*, Longman, Harlow, 1996.