Average and deviation for stochastic FitzHugh–Nagumo system

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Abstract

An averaged system for slow–fast stochastic FitzHugh–Nagumo system is derived in this paper. The rate of convergence in probability is obtained as a byproduct. Moreover the deviation between the original equations and the averaged equation is studied. A martingale approach proves that the deviation is described by an Gaussian process. This gives a much better approximation than previous work.

1 Introduction

The FitzHugh–Nagumo system arises as a simplification of the Hodgkin-Huxley model describing the signal transmission across axons in neurobiology [5, e.g.]. If the integral input from the surrounding cells to the given neuron is a random signal, then we have the following stochastic FitzHugh–Nagumo system

$$\partial_t u^\epsilon = \partial_{xx}u^\epsilon + u^\epsilon - (u^\epsilon)^3 + v^\epsilon + \sigma_1\partial_t W_1(t)\,,\ u^\epsilon(0) = u_0 \in L^2(-1,1), \quad (1)$$

$$\partial_t v^\epsilon = \frac{1}{\epsilon}\big[\partial_{xx}v^\epsilon + u^\epsilon - v^\epsilon\big] + \frac{\sigma_2}{\sqrt{\epsilon}}\partial_t W_2(t)\,,\ v^\epsilon(0) = v_0 \in L^2(-1,1) \quad (2)$$

with Dirichlet boundary conditions, where $L^2(-1,1)$ is the Lebesgue space of square integrable real valued functions on the nondimensional interval $(-1,1)$. W_1 and W_2 are mutually independent $L^2(-1,1)$ valued Wiener processes. Figure 1 plots an example solution of the FitzHugh–Nagumo system (1)–(2)

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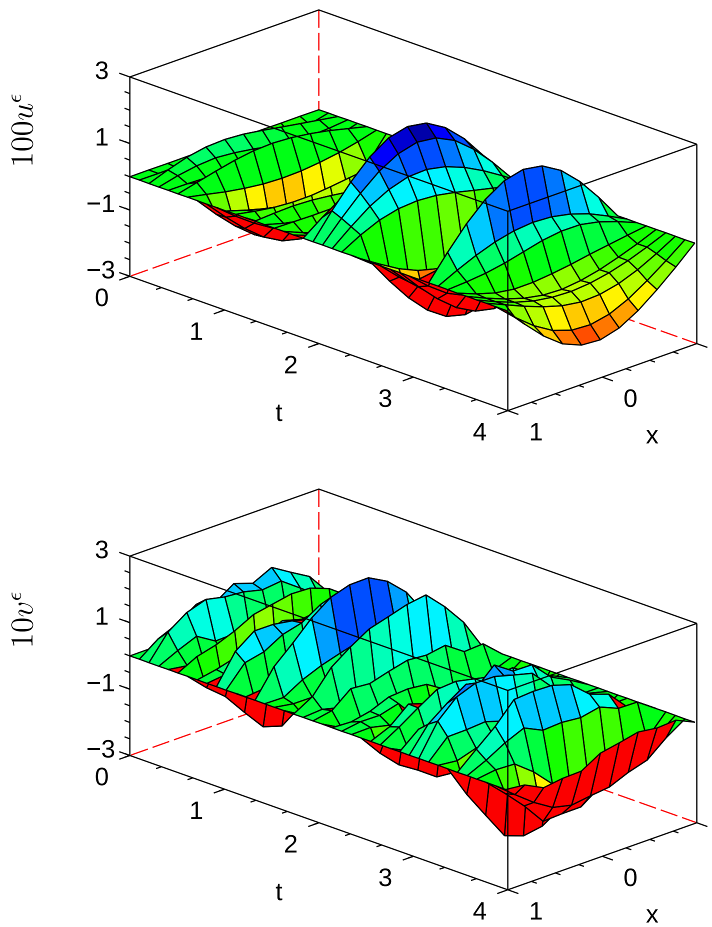

Figure 1: an example realisation of the FitzHugh–Nagumo system (1)–(2) with $\sigma_1 = 0$, $\sigma_2 = 3$, $W_2 = (1 - \partial_{xx})^{-1} Z$ for $Z(t)$ delta-correlated in x, and small parameter $\epsilon = 0.1$.

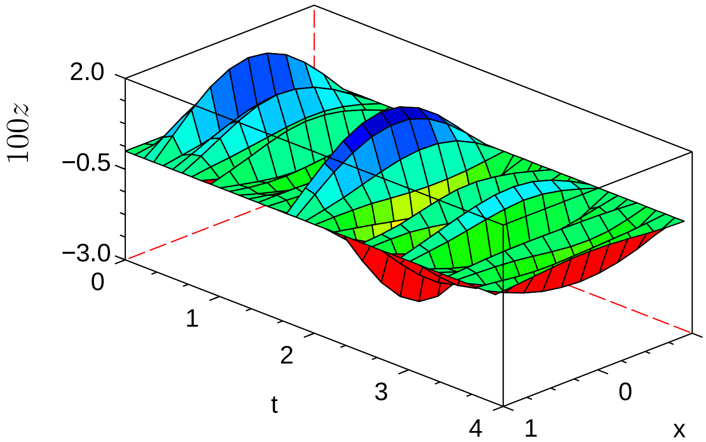

Figure 2: an example realisation of the stochastic deviation (7) for small parameter $\epsilon = 0.1$, $\sigma_2 = 3$ and $Q_2 = (1 - \partial_{xx})^{-1}$ to correspond to Figure 1.

showing that noise forcing of v feeds into the dynamics of u.

Assume $\epsilon > 0$ is small. Then the FitzHugh–Nagumo system (1)–(2) has two widely separated timescales. A simplified equation which governs the evolution of the system over the long time scale is highly desirable. Such a simplified equation, capturing the dynamics of the system at the slow time scale, is often called an averaged equation. There is a great deal work on averaging principle for deterministic ordinary differential equations [1, 2, 13, e.g.] and for stochastic ordinary differential equations [6, 8, 9, e.g.]. But there are few results on averaging for stochastic partial differential equations (SPDEs). Recently an averaged equation for a system of reaction-diffusion equations with stochastic fast component was obtained by a Lipschitz assumption on all nonlinear terms [4]. The resultant averaged equation is deterministic. This article derives the averaged equation (3) modelling (1)–(2) and proves a square-root rate of convergence in probability. Furthermore, the stochastic deviation between the original system and the averaged system is determined, the SPDE (7), as shown in the example solution of Figure 2.

If for any fixed u, the fast system (2) has a unique invariant measure μ_u, then as $\epsilon \to 0$, under some conditions, the solution u^ϵ of (1), converges in distribution to the solution of

$$\partial_t u = \partial_{xx} u + u - u^3 + (-\partial_{xx} + 1)^{-1} u + \sigma_1 \partial_t W_1(t) \,, \tag{3}$$

$$u(0) = u_0 \quad \text{and} \quad u(t)|_{x=\pm 1} = 0 \,. \tag{4}$$

Section 4 proves the convergence rate is 1/2 in the sense that

$$\sup_{0\le t\le T} \mathbb{E}|u^\epsilon(t) - u(t)| \le C_T \epsilon^{1/2} \tag{5}$$

for some positive constant C_T. Furthermore by estimate (5), Section 5 proves that as $\epsilon \to 0$ the limit of $(u^\epsilon(t) - u(t))/\sqrt{\epsilon}$ is a Gaussian process. Including the deviation SPDE (7) gives a much better approximation than the averaged equation (3). In particular, when the initial state $u_0 = 0$ and there is no direct forcing of u, $W_1 = 0$, as used in Figures 1 and 2, then the averaged solution is identically $u(t) = 0$. In such a case, the dynamics of u as seen in Figure 1 are modelled solely by deviations governed by the SPDE (7).

2 Preliminaries and main results

Let $H = L^2(-1, 1)$. Define the abstract operator $A = \partial_{xx}$ with Dirichlet boundary condition which defines a compact analytic semigroup e^{At}, $t \ge 0$ on H. Denote by $\lambda_1 = -\pi^2/4$ the first eigenvalue of A. For any $\alpha > 0$, we introduce the space $H_0^{2\alpha} = D((-A)^\alpha)$, which is compactly embedding into H. And let $H^{-\alpha}$ denote the dual space of H_0^α. The usual norm defined on H^α is written as $|\cdot|_{H^\alpha}$. And for $\alpha = 0$ and 1, the norm is written as $|\cdot|$ and $\|\cdot\|$ respectively. Also we are given H valued Q-Wiener processes $W_1(t)$ and $W_2(t)$, $t \ge 0$, with trace operator (spatial correlation) Q_1 and Q_2 respectively, which are mutually independent on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ [10]. Denote by $\mathbb{E}$ the expectation operator with respect to $\mathbb{P}$. Here we assume $\sigma_1, \sigma_2 > 0$ and make the following hypothesis:

H $\operatorname{tr}[(-A)^{1/2} Q_i] < \infty$, $i = 1, 2$.

Then the first result on the fast component, proved at the end of Section 3, is the following.

Theorem 1. *Assume* **H**. *For any fixed $u \in H$, system (2) has a unique stationary solution, $\eta_u^\epsilon(t)$, with distribution μ_u independent of ϵ. Moreover the stationary measure μ_u is exponentially mixing.*

Then we prove the following averaging result in Section 4.

Theorem 2. *Assume* **H**. *Given $T > 0$, for any $u_0 \in H$, solution $u^\epsilon(t, u_0)$ of (1) converges in probability to u in $C(0, T; H)$ which solves (3)–(4). Moreover the convergence rate is 1/2, that is, for some positive constant $C_T > 0$*

$$\sup_{0\le t\le T} \mathbb{E}|u^\epsilon(t) - u(t)| \le C_T \sqrt{\epsilon}.$$

Furthermore, we consider the deviation between u^ϵ and u. Introduce

$$z^\epsilon = \frac{1}{\sqrt{\epsilon}}(u^\epsilon - u)\,, \tag{6}$$

then Section 5 proves the following.

Theorem 3. *z^ϵ converges in distribution to z in space $C(0,T;H)$ which solves*

$$\partial_t z = \partial_{xx} z + (1-3u^2)z + (-\partial_{xx}+1)^{-1}Q_2\partial_t \bar{W}(t), \tag{7}$$

where $\bar{W}(t)$ is an cylindrical Wiener process.

3 Some a priori estimates

This section gives some a priori estimates for the solution of (1)–(2) which yields the tightness of u^ϵ in space $C(0,T;H)$. For simplicity define $f(u^\epsilon, v^\epsilon) = u^\epsilon - (u^\epsilon)^3 + v^\epsilon$ and $g(u^\epsilon, v^\epsilon) = -v^\epsilon + u^\epsilon$. First we give a wellposedness result.

Theorem 4. *Assume* **H**. *For any $u_0 \in H$, $v_0 \in H$ and any $T > 0$, there is unique solution $(u^\epsilon(t), v^\epsilon(t))$ in $L^2(\Omega, C(0,T;H) \cap L^2(0,T;H_0^1))$ for (1)–(2).*

The above result is derived by a standard method [10] and so is here omitted. Then we have the following estimates for (u^ϵ, v^ϵ) by a series of application of Itô formula and Gronwall lemma.

Theorem 5. *Assume* **H**. *For $u_0 \in H_0^1$ and $v_0 \in H_0^1$, for any $T > 0$, there is a positive constant C_T which is independent of $0 < \epsilon < 1$, such that*

$$\mathbb{E}\sup_{0\le t\le T}\|u^\epsilon(t)\|^2 + \mathbb{E}|\dot{u}^\epsilon|_{L^2(0,T;H^{-1})} + \sup_{0\le t\le T}\mathbb{E}|v^\epsilon(t)|^2 \le C_T(|u_0|^2 + |v_0|^2) \tag{8}$$

and for any positive integer m,

$$\mathbb{E}\int_0^T \|u^\epsilon(s)\|^{2m}ds + \mathbb{E}\int_0^T \|v^\epsilon(t)\|^2 ds \le C_T(|u_0|^2 + |v_0|^2)\,. \tag{9}$$

Taking $E_0 = H^{-1}$, $E = H$ and $E_1 = H_0^1$ and $p_0 = 2$, $\theta = 1/2$, and p_1 is a lagre enough positive integer, then a lemma by Simon [11] implies the following.

Theorem 6. *Assume* **H**. *$\{\mathcal{L}(u^\epsilon)\}_{0<\epsilon<1}$ is tight in space $C(0,T;H)$.*

Proof of Theorem 1. This is a direct result of Theorem 11.7 of [10]. The distribution of the unique stationary solution η_u^ϵ is

$$\mu_u = \mathcal{N}\big(\bar{\eta}_u\ , (-\partial_{xx}+1)^{-1}Q_2^2/2\big) \tag{10}$$

with $\bar{\eta}_u = (-\partial_{xx}+1)^{-1}u$ for any fixed u. And for any $v_0 \in H$

$$\mathbb{E}|v^\epsilon(t) - \eta_u^\epsilon(t)|^2 \leq e^{-2(\lambda_1 - C_g)t/\epsilon}\mathbb{E}|v_0 - \eta_u^\epsilon(0)|^2\,, \tag{11}$$

which yields the exponential mixing. Moreover we also have

$$\Big|\mathbb{E}f(u, v^\epsilon(t)) - \int_H f(u,x)\mu_u(dx)\Big| \leq C(1+|v_0|^2)e^{-\rho t/\epsilon}\,. \tag{12}$$

□

By the time scale transformation $t \to \tau = \epsilon t$, (2) is transformed to

$$\partial_\tau v = \partial_{xx}v + g(u,v) + \sigma_2\partial_\tau \tilde{W}_2\,, \quad v(0) = v_0\,, \tag{13}$$

where $\tilde{W}_2$ is the scaled version of W_2 and with same distribution. Then (13) has a unique stationary solution η_u with distribution μ_u. And

$$\bar{f}(u) := \int_H f(u,v)\mu_u(dv) = \lim_{t\to\infty}\frac{1}{t}\int_0^t f(u, \eta_u(s))ds\,, \tag{14}$$

by the ergodic property of μ_u. Precisely $\bar{f}(u) = u - u^3 + \bar{\eta}_u$. Furthermore, by a generalized theorem on contracting maps depending on a parameter [3, Appendix C], [4], $v_u(t)$ is a differential with respect to u with

$$\sup_{u, v_0\in H, 0\leq t\leq T} |D_u v_u|_{\mathcal{L}(H)} \leq C_T\,. \tag{15}$$

4 Averaged equation

This section gives the averaged equation and, as a byproduct, the convergence rate is obtained. By Theorem 6, for any $\kappa > 0$ there is compact set K_κ in $C(0,T;H)$ such that $\mathbb{P}\{u^\epsilon \in K_\kappa\} > 1-\kappa$. Here K_κ is chosen as a family of decreasing sets with respect to κ. Moreover, by the estimate (8) and Markov inequality, we further choose the set K_κ such that there is some positive constant C_T^κ, for $u^\epsilon \in K_\kappa$ such that $\|u^\epsilon(t)\|^2 \leq C_T^\kappa$, $t \in [0,T]$.

Proof of Theorem 2. Now we prove the rate of convergence. For any $\kappa > 0$ we introduce a new sub-probability space $(\Omega_\kappa, \mathcal{F}_\kappa, \mathbb{P}_\kappa)$ defined by

$$\Omega_\kappa = \{\omega \in \Omega : u^\epsilon(\omega) \in K_\kappa\}\,, \quad \mathcal{F}_\kappa = \{S \cap \Omega_\kappa : S \in \mathcal{F}\}$$

and

$$\mathbb{P}_\kappa(S) = \frac{\mathbb{P}(S \cap \Omega_\kappa)}{\mathbb{P}(\Omega_\kappa)} \quad \text{for} \quad S \in \mathcal{F}_\kappa.$$

Then $\mathbb{P}(\Omega \setminus \Omega_\kappa) \le \kappa$.

Now we restrict $\omega \in \Omega_\kappa$ and introduce an auxiliary process. For any $T > 0$, partition the interval $[0, T]$ into subintervals of length $\delta = \sqrt{\epsilon}$. Then we construct processes $(\tilde{u}^\epsilon, \tilde{v}^\epsilon)$ such that for $t \in [k\delta, (k+1)\delta)$,

$$\begin{aligned} \tilde{u}^\epsilon(t) &= e^{A(t-k\delta)}u^\epsilon(k\delta) + \int_{k\delta}^t e^{A(t-s)} f(u^\epsilon(k\delta), \tilde{v}^\epsilon(s))ds \\ &\quad + \sigma_1 \int_{k\delta}^t e^{A(t-s)} dW_1(s)\,, \quad \tilde{u}^\epsilon(0) = u_0 \end{aligned} \tag{16}$$

$$\begin{aligned} \partial_t \tilde{v}^\epsilon(t) &= \frac{1}{\epsilon}\big[\partial_{xx}\tilde{v}^\epsilon(t) + g(u^\epsilon(k\delta), \tilde{v}^\epsilon(t))\big] + \frac{\sigma_2}{\sqrt{\epsilon}}\partial_t W_2(t)\,, \\ &\quad \tilde{v}^\epsilon(k\delta) = v^\epsilon(k\delta)\,. \end{aligned} \tag{17}$$

By the choice of Ω_κ, there is $C_T > 0$, such that

$$|u^\epsilon(t) - u^\epsilon(k\delta)|^2 \le C_T \delta^2 \tag{18}$$

for $t \in [k\delta, (k+1)\delta)$. Then by the Itô formula and Gronwall lemma

$$|v^\epsilon(t) - \tilde{v}^\epsilon(t)|^2 \le C_T\delta^2\,, \quad t \in [0, T]\,. \tag{19}$$

Moreover by the choice of Ω_κ and the growth of $f(\cdot, v)$, $f(\cdot, v) : H \to H^{-\beta}$ is Lipschitz with $-1/2 \le \beta \le -1/4$. Then we have, by noticing (18),

$$|u^\epsilon(t) - \tilde{u}^\epsilon(t)| \le C_T\delta\,, \quad t \in [0, T]\,. \tag{20}$$

On the other hand, in the mild sense

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}\bar{f}(u(s))ds + \sigma_1 \int_0^t e^{A(t-s)}dW_1(s)\,.$$

Then, using $\lfloor z \rfloor$ to denote the largest integer less than or equal to z,

$$\begin{aligned} |\tilde{u}^\epsilon(t) - u(t)| &\le \int_0^t e^{A(t-s)}\big|f(u^\epsilon(\lfloor s/\delta \rfloor\delta), \tilde{v}^\epsilon(s)) - \bar{f}(u^\epsilon(\lfloor s/\delta\rfloor\delta))\big|ds \\ &\quad + \int_0^t e^{A(t-s)}\big|\bar{f}(u^\epsilon(\lfloor s/\delta\rfloor\delta)) - \bar{f}(u^\epsilon(s))\big|ds \\ &\quad + \int_0^t e^{A(t-s)}\big|\bar{f}(u^\epsilon(s)) - \bar{f}(u(s))\big|ds\,. \end{aligned}$$

Moreover by estimate (15), for any $u_1, u_2 \in H$, we have

$$\begin{aligned} & \frac{1}{\tau}\Big|\int_0^\tau \big[f(u_1, v_{u_1}(s)) - f(u_2, v_{u_2}(s))\big]ds\Big|_H \\ \leq \; & \Big[\|u_1\|^4 + \|u_2\|^4 + \sup_{u, v_0 \in H, 0 \leq t \leq T} |D_u v(t)|_{\mathcal{L}(H)}\Big]|u_1 - u_2|\,. \end{aligned}$$

Then by (14) we have for $t \in [0, T]$

$$|\tilde{u}^\epsilon(t) - u(t)| \leq C_T\big[\delta + \int_0^T |u^\epsilon(s) - u(s)|ds\big]\,. \tag{21}$$

As $|u^\epsilon(t) - u(t)| \leq |u^\epsilon(t) - \tilde{u}(t)| + |\tilde{u}(t) - u(t)|$, by the Gronwall lemma and (18), (20) and (21) we have for $t \in [0, T]$,

$$|u^\epsilon(t) - u(t)| \leq C_T\sqrt{\epsilon}\,. \tag{22}$$

Now by the arbitrariness of κ, the proof of Theorem 2 is complete. □

5 Deviation estimate

The previous section proved that for any $T > 0$, $\sup_{t\in[0,T]} \mathbb{E}|u^\epsilon(t) - u(t)| \leq C_T\sqrt{\epsilon}$ for some positive constant C_T. That is, formally $u^\epsilon(t) = u(t) + \mathcal{O}(\epsilon^{1/2})$. This section determines the coefficient of $\epsilon^{1/2}$, the so-called deviation.

Proof of Theorem 3. We approximate the deviation z^ϵ defined by (6). The deviation z^ϵ satisfies

$$\partial_t z^\epsilon = \partial_{xx} z^\epsilon + \frac{1}{\sqrt{\epsilon}}[f(u^\epsilon, v^\epsilon) - \bar{f}(u)] \tag{23}$$

with $z^\epsilon(0) = 0$. By the property of f and Gronwall lemma we have that for any $T > 0$,

$$\mathbb{E}\sup_{0\leq t\leq T} |z^\epsilon(t)|^2 + \mathbb{E}\int_0^T \|z^\epsilon(t)\|^2 dt \leq C_T(|u_0|^2 + |v_0|^2)\,. \tag{24}$$

In the mild sense we write

$$z^\epsilon(t) = \frac{1}{\epsilon}\int_0^t e^{A(t-r)}[f(u^\epsilon(r), v^\epsilon(r)) - \bar{f}(u(r))]dr.$$

Noticing that by Theorem 5 and (24)

$$\mathbb{E}\frac{1}{\epsilon}\big|f(u^\epsilon, v^\epsilon) - \bar{f}(u)\big|_{L^2(0,T;H)} \leq C_T(|u_0|^2 + |v_0^2|).$$

Then by the property of e^{At}, we have for some positive $1 > \delta > 0$ and $1 > \alpha > 0$ such that

$$\mathbb{E}|z^\epsilon(t)|_{C^\delta(0,T;H)} \leq C_T(|u_0|^2 + |v_0|^2) \tag{25}$$
$$\text{and} \quad \mathbb{E}\sup_{0\leq t\leq T}|z^\epsilon(t)|_{H^\alpha} \leq C_{T,\alpha}(|u_0|^2 + |v_0^2|)\,. \tag{26}$$

Here $C^\delta(0,T;H)$ is the Hölder space with exponent δ. Then by the compact embedding of $C^\delta(0,T;H)\cap C(0,T;H^\alpha) \subset C(0,T;H)$, $\{\nu^\epsilon\}_\epsilon$, the distribution of $\{z^\epsilon\}_\epsilon$, is tight in $C(0,T;H)$.

Divide z^ϵ into $z_1^\epsilon + z_2^\epsilon$ which solves

$$\partial_t z_1^\epsilon = \partial_{xx} z_1^\epsilon + \frac{1}{\sqrt{\epsilon}}(\eta_u^\epsilon - \bar{\eta}_u)\,, \quad z_1(0)^\epsilon = 0 \tag{27}$$
$$\text{and} \quad \partial_t z_2^\epsilon = \partial_{xx} z_2^\epsilon + \frac{1}{\sqrt{\epsilon}}[f(u^\epsilon, v^\epsilon) - f(u, \eta_u^\epsilon)]\,, \quad z_2^\epsilon(0) = 0 \tag{28}$$

respectively and consider z_1^ϵ and z_2^ϵ separately. Here $\bar{\eta}_u$ is defined in (10). We follow a martingale approach [7, 14]. Denote by ν_1^ϵ be the probability measure of z_1^ϵ induced on space $C(0,T;H)$. And for $\gamma > 0$ denoted by $UC^\gamma(H,\mathbb{R})$ the space of all functions from H to $\mathbb{R}$ which are uniformly continuous on H together with all the order till to γ order Fréchlet derivatives. We have the following lemma.

Lemma 7. *Assume* **H**. *Any limiting measure of ν_1^ϵ, denote by P^0, solves the following martingale problem on $C(0,T;H)$: $P^0\{z_1(0) = 0\} = 1$,*

$$h(z_1(t)) - h(z_1(0)) - \int_0^t \langle h'(z_1(\tau)), \partial_{xx} z_1(\tau)\rangle d\tau - \frac{1}{2}\int_0^t \operatorname{tr}\left[h''(z_1(\tau))B(u)\right] d\tau$$

is a P^0-martingale for any $h \in UC^2(H,\mathbb{R})$. Here

$$B(u) = 2\int_0^\infty \mathbb{E}\big[\big(\eta_u(t) - \bar{\eta}_u\big)(\eta_u(0) - \bar{\eta}_u)\big]\,dt.$$

Proof. For any $0 < s \leq t < \infty$ and $h \in UC^\infty(H)$ we have

$$h(z_1^\epsilon(t)) - h(z_1^\epsilon(s)) = \int_s^t \langle h'(z_1^\epsilon(\tau)), \frac{dz_1^\epsilon}{dt}\rangle d\tau$$
$$= \int_s^t \langle h'(z_1^\epsilon(\tau)), \partial_{xx} z_1^\epsilon(\tau)\rangle d\tau + \frac{1}{\sqrt{\epsilon}}\int_s^t \langle h'(z_1^\epsilon(\tau)), \eta_u^\epsilon(\tau) - \bar{\eta}_u\rangle d\tau\,.$$

Rewrite the second term as

$$\begin{aligned}
&\frac{1}{\sqrt{\epsilon}}\int_s^t \langle h'(z_1^\epsilon(\tau)), \eta_u^\epsilon(\tau)-\bar{\eta}_u\rangle \, d\tau = \frac{1}{\sqrt{\epsilon}}\int_s^t \langle h'(z_1^\epsilon(t)), \eta_u^\epsilon(\tau)-\bar{\eta}_u\rangle \, d\tau \\
&\quad - \frac{1}{\sqrt{\epsilon}}\int_s^t\int_\tau^t h''(z_1^\epsilon(\delta))\Big(\eta_u^\epsilon(\tau)-\bar{\eta}_u\,,\, Az_1^\epsilon(\delta)\Big)\, d\delta\, d\tau \\
&\quad - \frac{1}{\epsilon}\int_s^t\int_\tau^t h''(z_1^\epsilon(\delta))\Big(\eta_u^\epsilon(\tau)-\bar{\eta}_u\,,\, \eta_u^\epsilon(\delta)-\bar{\eta}_u\Big)\, d\delta\, d\tau \\
= \;& L_1 + L_2 + L_3\,.
\end{aligned}$$

Denote by $A^\epsilon(\delta,\tau) = (\eta_u^\epsilon(\tau)-\bar{\eta}_u)(\eta_u^\epsilon(\delta)-\bar{\eta}_u)$. Then we have

$$\begin{aligned}
L_3 &= -\frac{1}{\epsilon}\sum_{ij}\int_s^t\int_\tau^t \partial_{ij}h(z_1^\epsilon(\delta))\langle A^\epsilon(\delta,\tau)e_i, e_j\rangle \, d\delta\, d\tau \\
&= -\frac{1}{\epsilon}\sum_{ij}\int_s^t\int_\tau^t\int_\delta^t \Big\langle \partial_{ij}h'(z_1^\epsilon(\lambda)), \big[z_1^\epsilon(\lambda) + \frac{1}{\sqrt{\epsilon}}(\eta_u^\epsilon(\lambda)-\bar{\eta}_u)\big]\Big\rangle \\
&\qquad\qquad \times \langle \tilde{A}^\epsilon(\delta,\tau)e_i, e_j\rangle \, d\lambda\, d\delta\, d\tau \\
&\quad + \frac{1}{\epsilon}\sum_{ij}\int_s^t\int_\tau^t \partial_{ij}h(z_1^\epsilon(t))\langle \tilde{A}^\epsilon(\delta,\tau)e_i, e_j\rangle \, d\delta\, d\tau \\
&\quad + \frac{1}{\epsilon}\sum_{ij}\int_s^t\int_\tau^t \partial_{ij}h(z_1^\epsilon(\delta))\langle \mathbb{E}[A^\epsilon(\delta,\tau)]e_i, e_j\rangle \, d\delta\, d\tau \\
&= L_{31} + L_{32} + L_{33}
\end{aligned}$$

with $\tilde{A}^\epsilon(\delta,\tau) = A^\epsilon(\delta,\tau) - \mathbb{E}[A^\epsilon(\delta,\tau)]$. Here $\{e_i\}_{i=1}^\infty$ is one eigenbasis of H and $\partial_{ij} = \partial_{e_i}\partial_{e_j}$ with ∂_{e_i} is the directional derivative in direction e_i. For our purpose, for any bounded continuous function Φ on $C(0,s;H)$, let $\Phi(\cdot,\omega) = \Phi(z_1^\epsilon(\cdot,\omega))$. Then by (12), we have the estimate $|\mathbb{E}[(L_{31}+L_{32})\Phi]| \to 0$ as $\epsilon \to 0$. As $\eta_u(t)$ is stationary correlated, by the exponential mixing property, if $\epsilon_n \to 0$ as $n \to \infty$, $\nu^{\epsilon_n} \to P^0$,

$$\lim_{n\to\infty}\mathbb{E}[L_3\Phi] = \frac{1}{2}\int_s^t \mathbb{E}^{P^0}\Big(\operatorname{tr}\big[h''(z_1(\tau))B(u)\big]\Phi\Big)d\tau\,.$$

Similarly by (12), $\mathbb{E}[L_1\Phi + L_2\Phi] \to 0$ as $\epsilon \to 0$. By the tightness of z^ϵ in $C(0,T;H)$, the sequence $z_1^{\epsilon_n}$ has a limit process, denoted by z_1, in a weak sense. Then

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_s^t \langle h'(z_1^{\epsilon_n}(\tau)), Az_1^{\epsilon_n}(\tau)\rangle\Phi\, d\tau\Big] = \mathbb{E}\Big[\int_s^t \langle h'(z_1(\tau)), Az_1(\tau)\rangle\Phi\, d\tau\Big]$$

and

$$\lim_{n\to\infty} \mathbb{E}\big[\big(h(z_1^{\epsilon_n}(t)) - h(z_1^{\epsilon_n}(s))\big)\Phi\big] = \mathbb{E}\big[\big(h(z_1(t)) - h(z_1(s))\big)\Phi\big]\,.$$

At last we have

$$\begin{aligned} & \mathbb{E}^{P^0}\big[\big(h(z_1)(t) - h(z_1(s))\big)\Phi\big] \qquad (29)\\ = \ & \mathbb{E}^{P^0}\Big[\int_s^t \langle h'(z_1(\tau)), Az_1(\tau)\rangle \Phi\, d\tau\Big] + \frac{1}{2}\mathbb{E}^{P^0}\left\{\int_s^t \operatorname{tr}\big[h''(z_1(\tau))B(u)\big]\Phi\, d\tau\right\}. \end{aligned}$$

By an approximation argument we can prove (29) holds for all $h \in UC^2(H)$. This completes the proof. □

Now by the relation between martingale problem and weak solution of stochastic differential equations [12], The limit of ν_1^ϵ, denote by P^0, is unique and solves the martingale problem related to the following stochastic partial differential equation

$$\partial_t z_1 = \partial_{xx} z_1\, dt + \sqrt{B(u)}\,\partial_t \bar{W}\,, \qquad (30)$$

where $\bar{W}(t)$ is cylindrical Wiener process with trace operator being the identity operator on H, define on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that z_1^ϵ converges weakly to z_1 in $C(0,T;H)$.

On the other hand, the distribution of z_2^ϵ on $C(0,T;H)$ is also tight. Suppose z_2 is one weak limit point of z_2^ϵ in $C(0,T;H)$. By the property of f and (11) we derive that z_2 is unique solution of the following equation

$$\partial_t z_2 = \partial_{xx} z_2 + (1-3u^2)z\,, \quad z_2(0) = 0 \qquad (31)$$

with $z = z_1 + z_2$. And by the wellposedeness of the above problem, we have that z^ϵ uniquely converges weakly to z which solves (7). This proves Theorem 3. □

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