Soliton solutions for KdV equation by homotopy analysis method

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Abstract

In this paper, the well-known KdV equation is solved by homotopy analysis method, an analytical, totally explicit technique. By choosing a proper auxiliary parameter $\hbar$, the new series solution converges very rapidly to the exact solution, with a simple way to adjust the convergence region. In addition, we showed that a significant improvement of the convergence rate and region is achieved by applying Homotopy-Padé Approximants. The present method holds promise in providing soliton solutions for more complicated wave equations.

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1 Introduction

Most of the physical problem in the real world are nonlinear and in turn are described by nonlinear equations. However, it is generally difficult to solve nonlinear equations accurately by analytical means. Therefore, seeking suitable solving methods is an active task in branches of computational physics. Recently, a new analytic approach named homotopy analysis method (HAM) has seen rapid development. It has been successfully applied to many nonlinear problems and logically contains Lyapunov's small parameter method, the δ-expansion method, and Adomian's decomposition method [4]. Without depending on a small parameter such as in a perturbation approach, the HAM has a particular advantage in solving strong nonlinear problems. Other advantages associated with the HAM over the perturbation technique include greater flexibility in the selection of a proper set of basis functions for the solution and a much simpler method in the control of the convergence rate and region. Liao and Cheung [5] successfully applied HAM in fully analytical way to nonlinear waves propagating in deep water and the HAM solution in finite water depth was later obtained by Tao et al. [7]. HAM solution for certain kind of shallow water wave equations could also be found in [1] and [8].

Among many nonlinear equations in physics, KdV equation is a typical, relatively simple and classical one. It is a well-known mathematical model of nonlinear waves on shallow water surfaces. The so-called term “soliton” originated from solving KdV equation. Many researchers are still endeavour to solve KdV-type equations recently (e.g., [2] and [6]). In this paper, the HAM method is applied to obtain the soliton solution for the well-known KdV equation. Explicit solution is presented and compared with the exact solution. Very good agreement is achieved, demonstrating the high efficiency of HAM.

2 Theoretical Consideration

2.1 The basic idea behind HAM

For a nonlinear equation:

$$\mathscr{N}[f(\mathbf{x},t)] = 0, \tag{1}$$

where $\mathscr{N}$ is a nonlinear operator, $f(\mathbf{x},t)$ is the function to be solved, and $\mathbf{x}$ and t are spatial and temporal independent variables respectively.

A homotopy can be constructed as:

$$(1-q)\mathscr{L}[F(\mathbf{x},t;q)-f_0(\mathbf{x},t)] = q\hbar\mathscr{N}[F(\mathbf{x},t;q)], \quad q\in[0,1], \tag{2}$$

where $F(\mathbf{x},t;q)$ is the mapping function of $f(\mathbf{x},t)$, $f_0(\mathbf{x},t)$ is an initial estimate of $f(\mathbf{x},t)$, $\hbar$ is a nonzero auxiliary parameter, and $\mathscr{L}$ is a linear auxiliary operator with the property of $\mathscr{L}[0]=0$.

It is obvious

$$F(\mathbf{x},t;0) = f_0(\mathbf{x},t), \tag{3}$$

$$\mathscr{N}[F(\mathbf{x},t;1)] = 0. \tag{4}$$

Therefore, as the embedding parameter q varies from 0 to 1, $F(\mathbf{x},t;q)$ maps continuously from the initial estimate of $f_0(\mathbf{x},t)$ to the exact solution $f(\mathbf{x},t)$.

By Taylor's theorem, $F(\mathbf{x},t;q)$ can be expanded with respect to the embedding parameter q as

$$F(\mathbf{x},t;q) = f_0(\mathbf{x},t) + \sum_{m=1}^{+\infty} f_m(\mathbf{x},t)q^m, \tag{5}$$

where

$$f_m(\mathbf{x},t) = \frac{1}{m!}\left.\frac{\partial^m F(\mathbf{x},t;q)}{\partial q^m}\right|_{q=0}. \tag{6}$$

Differentiating the zeroth-order deformation equation (2) m-times with respect to q at $q=0$ and then dividing it by $m!$, we have the following mth-order deformation equation

$$\mathscr{L}[f_m(\mathbf{x},t) - \chi_m f_{m-1}(\mathbf{x},t)] = \hbar R_m(\mathbf{x},t), \tag{7}$$

where

$$\chi_m = \begin{cases} 0, & m\le 1 \\ 1, & m>1 \end{cases}, \tag{8}$$

$$R_m(\mathbf{x},t) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathscr{N}[F(\mathbf{x},t;q)]}{\partial q^{m-1}}\right|_{q=0}. \tag{9}$$

If the series (5) converges at $q=1$, we have

$$f(\mathbf{x},t) = f_0(\mathbf{x},t) + \sum_{m=1}^{+\infty} f_m(\mathbf{x},t). \tag{10}$$

The auxiliary parameter $\hbar$ provides us a convenient way to control and adjust the rate and region of the convergence [4]. The dependence of HAM solution on $\hbar$ and detailed discussions are presented in Section 3.

2.2 Soliton solution of KdV equation by HAM

The KdV equation is a nonlinear, dispersive partial differential equation in the following form

$$u_t + u_{xxx} + 6uu_x = 0, \tag{11}$$

where x is the space and t is the time and the subscript designates partial derivative with respect to the variable.

Define $\theta = x - ct$ and $u(\theta) = av(\theta)$, Eq. (11) is

$$v(\theta)''' + 6av(\theta)v(\theta)' - cv(\theta)' = 0, \tag{12}$$

where c is the wave speed, a is the wave amplitude, and the prime denotes the derivative with respect to θ.

Suppose

$$v(\theta) \sim B\exp(-\lambda\theta), \quad \text{as} \quad \theta \to +\infty, \tag{13}$$

where $\lambda > 0$ and B are constants. Substituting Eq. (13) into Eq. (12) and balancing the main terms, we have

$$\lambda = \sqrt{c}, \tag{14}$$

Define $\tau = \lambda\theta$, Eq. (12) becomes

$$cv(\tau)''' + 6av(\tau)v(\tau)' - cv(\tau)' = 0, \tag{15}$$

Assuming the nondimensional wave elevation $v(\tau)$ arrives its maximum at the origin, we have the boundary conditions as follows

$$v(0) = 1, \quad v'(0) = 0, \quad v(+\infty) = 0. \tag{16}$$

Then the solution can be expressed by the base functions

$$\{\exp(-n\tau) | n = 1, 2, 3, \ldots\}, \tag{17}$$

in the form

$$v(\tau) = \sum_{n=1}^{+\infty} b_n \exp(-n\tau), \tag{18}$$

where b_n is a coefficient to be determined.

The nonlinear operator $\mathscr{N}$ is chosen as

$$\mathscr{N}\left[\Phi(\tau; q), A(q)\right] = c\frac{\partial^3\Phi(\tau; q)}{\partial\tau^3} + 6a\Phi(\tau; q)\frac{\partial\Phi(\tau; q)}{\partial\tau} - c\frac{\partial\Phi(\tau; q)}{\partial\tau}, \tag{19}$$

and the linear operator $\mathscr{L}$ is chosen as

$$\mathscr{L}\left[\Phi(\tau;q)\right] = \left(\frac{\partial^3}{\partial\tau^3} - \frac{\partial}{\partial\tau}\right)\Phi(\tau;q), \tag{20}$$

with the property

$$\mathscr{L}\left[C_1\exp(-\tau) + C_2\exp(\tau) + C_3\right] = 0, \tag{21}$$

where C_1, C_2 and C_3 are constants.

According to the boundary condition (16) and the *rule of solution expression* (18), the initial guess is chosen as

$$\phi_0(\tau) = 2\exp(-\tau) - \exp(-2\tau). \tag{22}$$

The *zeroth-order deformation equation* is

$$(1-q)\mathscr{L}[\Phi(\tau;q) - \phi_0(\tau;q)] = q\hbar\mathscr{N}\left[\Phi(\tau;q), A(q)\right], \tag{23}$$

subject to the boundary conditions

$$\Phi(0;q) = 1, \quad \Phi_\tau(0;q) = 0, \quad \Phi(+\infty;q) = 0. \tag{24}$$

Expand $\Phi(0,q)$ and $A(q)$ in Taylor series with respect to q, we have

$$\Phi(\tau;q) = \phi_0(\tau) + \sum_{m=1}^{+\infty}\phi_m(\tau)q^m, \tag{25}$$

$$A(q) = a_0 + \sum_{m=1}^{+\infty}a_m q^m, \tag{26}$$

where

$$\phi_m(\tau) = \frac{1}{m!}\left.\frac{\partial^m\Phi(\tau;q)}{\partial q^m}\right|_{q=0}, \tag{27}$$

$$a_m = \frac{1}{m!}\left.\frac{\partial^m A(q)}{\partial q^m}\right|_{q=0}. \tag{28}$$

For brief, define

$$\vec{\phi}_m = \{\phi_0, \phi_1, \phi_2, \ldots, \phi_m\}, \tag{29}$$

$$\vec{a}_m = \{a_0, a_1, a_2, \ldots, a_m\}. \tag{30}$$

Differentiating Eqs. (23) and (24) m times with respect to q, then setting $q = 0$, and finally dividing them by $m!$, the mth-order deformation equation is

$$\mathscr{L}[\phi_m(\tau) - \chi_m \phi_{m-1}(\tau)] = \hbar R_m(\vec{\phi}_m, \vec{a}_m), \tag{31}$$

subject to the boundary conditions

$$\phi_m(0) = \phi_m(+\infty) = \phi'_m(0) = 0, \tag{32}$$

where

$$R_m(\vec{\phi}_{m-1}, \vec{a}_{m-1}) = c\phi'''_{m-1} + 6\sum_{i=0}^{m-1}\sum_{j=0}^{i} a_j \phi_{i-j}\phi'_{m-i-1} - c\phi'_{m-1}. \tag{33}$$

The solution of Eq. (31) is

$$\phi_m(\tau) = C_1 \exp(-\tau) + C_2 \exp(\tau) + C_3 + \hat{\phi}_m(\tau), \tag{34}$$

where $\hat{\phi}_m(\tau)$ is a special solution of Eq. (31) with the unknown term a_{m-1}.

According to the boundary condition (32) and *rule of solution expression* (18), we have

$$C_2 = C_3 = 0, \quad C_1 = -\hat{\phi}_m(0), \tag{35}$$

and

$$\hat{\phi}_m(0) + \hat{\phi}'_m(0) = 0, \tag{36}$$

which determine a_{m-1}.

3 Result and Discussion

The convergence region and rate are controlled by the auxiliary parameter $\hbar$ in HAM. For different value of $\hbar$, a converges to the same value - the approximation of the exact solution. It can be seen in Fig. 1, the nearly horizontal line segments of $a-\hbar$ curves correspond to the convergence regions of the $\hbar$ values. Fig. 1 clearly shows that the first $2m$th-order approximations of the a/c converge in a region around $\hbar \in [-2, -1/2]$. The convergence region enlarges as more high order terms are included in the series. Based on the above arguments, the auxiliary parameter is chosen as $\hbar = -1$ for all the HAM solutions presented in this section.

Since the KdV equation is integrable, the exact solution of (11) is given as

$$u(x,t) = \frac{1}{2}c\frac{1}{\cosh^2\left[\frac{\sqrt{c}}{2}(x - ct - d)\right]}, \tag{37}$$

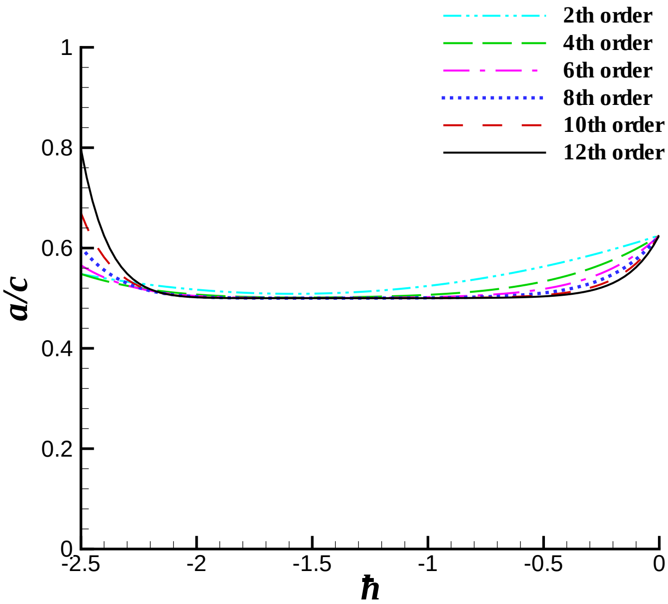

Figure 1: The first $2m$th-order approximations of a/c versus $\hbar$.

where d is an arbitrary constant.

The comparison of the HAM solution and the exact solution is shown in Table 1, where the first $2m$th order solution and $[m, m]$ Homotopy-Padé (HP) approximation of a/c for $\hbar = -1$ are shown. A rapid convergence rate of the series is evident. Table 1 also shows that the present Homotopy-Padé approximation converges more rapidly, giving high accurate results with only a few number of terms (the relative error is 1.74% for $[1, 1]$ HP approximation and 0.15% for [2,2] HP approximation). This is a further demonstration of the excellent convergence rate in the present Homotopy-Padé technique.

Table 1: The $2m$th-order solution and $[m, m]$ Homotopy-Padé approximation of a/c for $\hbar = -1$.

Order	a/c	Error	$[m, m]$	a/c	Error
2	0.524389	4.8778%	[1,1]	0.508711	1.7422%
4	0.506689	1.3378%	[2,2]	0.500734	0.1468%
6	0.502126	0.4252%	[3,3]	0.500055	0.0110%
8	0.500737	0.1474%	[4,4]	0.500005	0.0010%
10	0.500271	0.0542%	[5,5]	0.500000	0.0000%
Exact			1/2		

The first $2m$th order solution and $[m, m]$ Homotopy-Padé approximation of u versus θ for $\hbar = -1$ is shown in Figure 2. It can be seen in Figure 2 that the present HAM solution is almost identical with the exact solution especially for Homotopy-Padé solutions. Even $[1, 1]$ HP approximation yields very good agreement. It is also evident that the higher-order approximations clearly result in better accuracy in the HAM solutions.

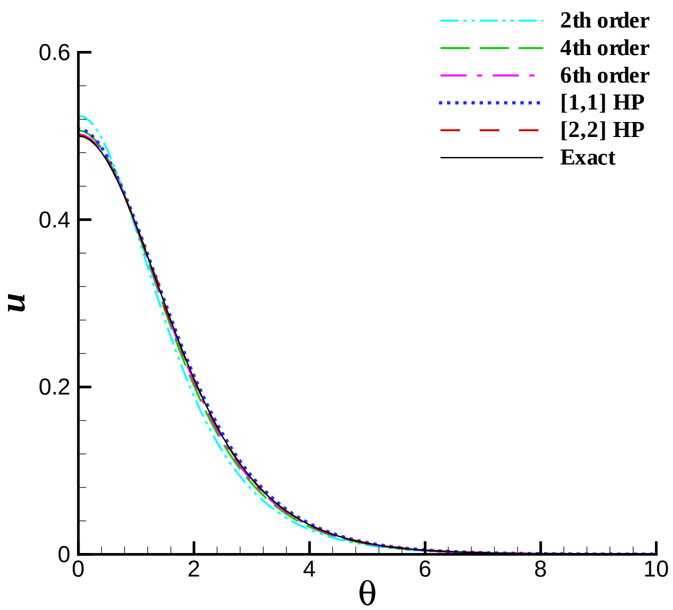

Figure 2: The first $2m$th-order solution and $[m, m]$ Homotopy-Padé approximation of u versus θ for $\hbar = -1$.

4 Conclusion

Explicit solutions of the well-known KdV equation were derived using the Homotopy analysis method. Different from the perturbation technique, the present HAM approach is not dependent on any small parameter, and is particularly suitable for solving nonlinear problems. The convergence region is controlled by the non-zero parameter $\hbar$, providing us a simple way to adjust convergence. Furthermore, a significant improvement of the convergence rate and region is achieved by applying Homotopy-Padé technique. The present method holds promise in providing soliton solutions for more complicated wave equations.

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References

[1] Abbasbandy, S., Homotopy analysis method for generalized Benjamin-Bona-Mahony equation, *Z. angew Math. Phys.* **59**, 2008, 51–62.

[2] Fan, E., A new algebraic method for finding the line soliton solutions and doubly periodic wave solution to a two-dimensional perturbed KdV equation, *Chaos Soliton Fract.* **15**, 2003, 567–574.

[3] Liao, S. J., An approximate solution technique not depending on small parameters: a special example, *Int. J. Nonlinear Mech.* **30**, 1995, 371–380.

[4] S. J. Liao. *Beyond Perturbation: Introduction to the Homotopy Analysis Method*. Chapman & Hall/CRC, Florida, 2004.

[5] Liao, S. J. and Cheung, K. F., Homotopy analysis of nonlinear progressive waves in deep water, *J. Eng. Math.* **45**, 2003, 105–116.

[6] Ping, H., Zheng, C. and Jun, F., Solution of KdV equation by computer algebra, *Appl Math Comput*, **136**, 2003, 511–515.

[7] Tao, L., Song, H. and Chakrabarti, S., Nonlinear progressive waves in water of finite depth an analytic approximation, *Coast. Eng.* **54**, 2007, 825–834.

[8] Wu, W. and Liao, S. J., Solving solitary waves with discontinuity by means of the homotopy analysis method, *Chaos Soliton Fract.* **26**, 2005, 177–185.