On Gödel's Proof

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13 June 2008

Abstract

This paper demonstrates that Gödel's proof may be modified to demonstrate that Mendelson's first-order theory of Peano Arithmetic (S) is inconsistent. The result appears to be due to a defect in simple type theory rather than some peculiarity of S itself and thus poses a challenge for the view that every primitive recursive number-theoretic function is effectively computable.

Contents

1 Introduction

This paper demonstrates that Gödel's proof may be modified to demonstrate that Mendelson's first-order theory of Peano Arithmetic (S) is inconsistent. The result appears to be due to a defect in simple type theory rather than some peculiarity of S itself. Before entering into details I present an informal sketch of the main argument of my proof .

Gödel's proof [1] implies that there exists a (primitive recursive) class of natural numbers (R) such that: for each natural number n, the proposition $R(n)$ is provable in P, however the proposition that R holds for all natural

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numbers [$(\forall x)R(x)$] is not provable in P (assuming that the system is consistent). But this implies that the following method of proof (α), illustrated in Euclid's proof of the infinitude of primes, is not formalised in P (or else the system is inconsistent). Suppose that we are able to prove (β) that each natural number exhibits a certain property (F); that is, we can prove $F(0)$, ..., $F(n)$, etc. To refute the hypothesis that there exists a number that fails to exhibit F [$(Ex)\overline{F(x)}$], we asssume to derive a contradiction that b is such a number [$\overline{F(b)}$]. From β however we have $F(b)$, hence (proof by contradiction): $\overline{(Ex)\overline{F(x)}}$, that is, $(\forall x)F(x)$. (Thus if a system T (subject to [1] Theorem VI) formalises all methods of proof commonly used in mathematics then: if T is not ω consistent it is not consistent.)

For a precise statement of the proof Section 2 examines the case of Mendelson's [3] first-order theory of Peano Arithmetic (S). Since the method of proof discussed above (α) is formalised in S the result for this case is that S is inconsistent. 'Recursive' is used in [1] to mention a class of functions now known as the primitive recursive functions; whereas 'recursive' is used in [3] to mention a wider class of functions. This raises some minor, easily resolved complications which for brevity I do not address herein. Section 3 addresses the bearing of these results on the claim that every primitive recursive number-theoretic function is effectively computable.

2 The Proof

In this section I prove the following proposition regarding Mendelson's [3] (S):

Proposition 2.1. *If S exhibits the properties such that [1] Theorem VI applies to S then S is inconsistent.*

Proof. To simplify notation the following two conventions are observed:

1. We use corners (' $\ulcorner$ ',' $\urcorner$ ') for quasi-quotation [4]. For example if B is ' $x_1 = x_1$ ', then $\ulcorner Subst \left(B \begin{matrix} x_1 \\ 0 \end{matrix} \right) \urcorner$ is ' $Subst \left(x_1 = x_1 \begin{matrix} x_1 \\ 0 \end{matrix} \right)$ ' or ' $(0 = 0)$ '.

2. I use informal notation for the arithmetised syntax of S from [3], with the following modifications.

    (a) In (β) ' $\mathrm{Pf}(r,q)$. Hence $\vdash_K Pf(\overline{r},\overline{q})$ ' ([3]: 207) ' Pf ' is used informally to mention a relation of numbers, whereas '*Pf*' is a metamathematical name for the K-wff that expresses this relation of

numbers (Pf) in K (in the sense of [3]:§ 3.2). To maintain consistency with the more general convention that informal mathematics is italicised, I use ' *Pf* ' to informally mention a relation of numbers (with similarly usage for ' *Neg* ' etc.), whereas 'Pf' is used as a metamathematical name for the K-wff that expresses this relation of numbers in K (in the sense of [3]:§ 3.2). (In [3] K is an arbitrary first-order theory, whereas K is always S herein unless stated otherwise.) Thus β would be written:
' $Pf(r, q)$. Hence $\vdash_K \ulcorner Subst \left(\mathrm{Pf} \begin{array}{cc} x_1 & x_2 \\ \bar{\mathrm{r}} & \bar{\mathrm{q}} \end{array} \right) \urcorner$'.
Nonitalicised text is reserved in such contexts for metamathematical names for S expressions. S particles ('x_1', '$x_1 = x_1$' etc) are left italicised since it is always clear from context whether such an expression is informal or belongs to S.

(b) For informal reasoning about the arithmetic image of S I use the symbolism of [2].

For brevity I assume the validity of certain proofs presented in [1] and [3] (specified in Conjectures A, B and C) and results obtained thereby.

Conjecture A. *The validity of Gödel's [1] proofs of Theorem's I-VI is assumed.*

Conjecture B. *I assume that S exhibits the following properties (proofs from [3] are noted for each item):*

1. *The formation rules and rules of inference of S are recursive [3] §3.4 (once the primitive symbols are replaced by numbers [1]);*

2. *All recursive relations may be defined in S, in the sense of Theorem V [1], modified as required ([3] Corollary 3.25) ;*

3. *If $f(p_1, p_2, \ldots, p_n)$ is a tautologous S schemata in the syntactical variables $p_1, p_2, \ldots, p_n$ and $\mathrm{a_1}$, $\mathrm{a_2}$, ..., $\mathrm{a_n}$ are S formulae, then* $Subst \left[f(p_1, p_2, \ldots, p_n) \begin{array}{cccc} p_1 & p_2 & \cdots & p_n \\ \mathrm{a_1} & \mathrm{a_2} & \ldots & \mathrm{a_n} \end{array} \right]$ *is an S theorem ([3] Proposition 2.1).*

Conjecture C. *Each of the number-theoretic relations / functions 1-22 defined by Mendelson ([3]:§ 3.4) correspond to the appropriate metamathematical notion concerning S in the required sense.*

To illustrate Conjecture C:

Example 2.1. *The relation of numbers Pf ([3]:§ 3.4 Equation 22) corresponds to the metamathematical relation* ⌜a *is an S proof of* b⌝ *in the required sense if and only if, for every 2-tuple of natural numbers* $< x, y >$ *:*

$< x, y >$ *is a member of the relation of numbers Pf [* $Pf(x, y)$ *] if and only if there exists a sequence of sequences of primitive S signs (* $<< \mathrm{a}_{1,1}, \ldots \mathrm{a}_{1,\mathrm{n}} >, \ldots, < \mathrm{a}_{\mathrm{m},1}, \ldots, \mathrm{a}_{\mathrm{m,o}} >>$ *) such that: (a) x is associated (via the arithmetisation function g) with* $<< \mathrm{a}_{1,1}, \ldots \mathrm{a}_{1,\mathrm{n}} >, \ldots, < \mathrm{a}_{\mathrm{m},1}, \ldots, \mathrm{a}_{\mathrm{m,o}} >>$ *and y is associated via g with the sequence* $< \mathrm{a}_{\mathrm{m},1}, \ldots, \mathrm{a}_{\mathrm{m,o}} >$ *; and (b)* $< \mathrm{a}_{\mathrm{m},1}, \ldots, \mathrm{a}_{\mathrm{m,o}} >$ *is an S formula (sentence) and* $<< \mathrm{a}_{1,1}, \ldots \mathrm{a}_{1,\mathrm{n}} >, \ldots, < \mathrm{a}_{\mathrm{m},1}, \ldots, \mathrm{a}_{\mathrm{m,o}} >>$ *is an S-proof of* $< \mathrm{a}_{\mathrm{m},1}, \ldots, \mathrm{a}_{\mathrm{m,o}} >$ *.*

For the proof of Proposition 2.1 some further definitions are required. [To facilitate comparison with [1] informal variables used therein to mention numbers associated with P formulae (and relations of such numbers) are used in these definitions to mention numbers associated with S formulae defined below (and relations of such numbers).] Q shall be the relation of numbers such that:

$$Q(x, y) \ \sim \ \overline{Pf\{x, Sub[y, Num(y), 29]\}} \tag{1}$$

Since the relation Q is (primitive) recursive ([3] §3.4), by [3] Corollary 3.25 there exists an S formula q (associated via the arithmetisation function g with a number q) with the free variables x_1, x_2, such that, for all 2-tuples of natural numbers (x, y) :

$$\overline{Pf\{x, Sub[y, Num(y), 29]\}} \ \rightarrow$$
$$(Eu)Pf\{u, Sub[Sub(q, Num\{x\}, 21), Num(y), 29]\} \tag{2}$$
$$Pf\{x, Sub[y, Num(y), 29]\} \ \rightarrow$$
$$(Eu)Pf\{u, Neg[Sub(Sub\{q, Num[x], 21\}, Num\{y\}, 29)]\} \tag{3}$$

Let p be the number associated via the arithmetisation function g with the S formula ⌜$(\forall x_1)$q⌝. Put

$$r = Sub[q, Num(p), 29] \tag{4}$$

For the proof of Proposition 2.1 I establish Lemmas 2.1-2.3.

Lemma 2.1. *The assumption that S is consistent implies:*

$$(x)(Eu)Pf\{u, Sub[r, Num(x), 21]\} \tag{5}$$
$$\overline{(Eu)}Pf\{u, Sub[p, Num(p), 29]\} \tag{6}$$

Proof. Lemma 2.1 follows from Conjecture B, since this implies that [1] Theorem VI applies to S, hence:

1. (Proof of Equation 6): If S is consistent, then for every natural number n, $\overline{Pf\{n, Sub[p, Num(p), 29]\}}$ hence $\overline{(Eu)}Pf\{n, Sub[p, Num(p), 29]\}$. Otherwise for some x:

   (a) Equation 3 yields: $(Eu)Pf\{u, Neg[Sub(Sub\{q, Num[x], 21\}, Num\{p\}, 29)]\}$, that is (Equation 4): $(Eu)Pf\{u, Neg[Sub(r, Num[x], 21)]\}$.

   (b) Conjecture C implies $(Eu)Pf\{u, Sub[r, Num(x), 21]\}$ (from $(Eu)Pf\{u, Sub[p, Num(p), 29]\}$).

2. (Proof of Equation 5):

   (a) From Equation 2 and the definition of r:

$$(x)(\{\overline{Pf[x, Sub(p, Num\{p\}, 29)]}\} \rightarrow (Eu)Pf\{u, Sub[r, Num(x), 21]\}) \tag{7}$$

   (b) Equation 7, ($\ulcorner (x)[F(x) \rightarrow G(x)] \rightarrow [(x)F(x) \rightarrow (x)G(x)]\urcorner$), and MP yield:

$$(x)\{\overline{Pf[x, Sub(p, Num\{p\}, 29)]}\} \rightarrow (x)(Eu)Pf\{u, Sub[r, Num(x), 21]\} \tag{8}$$

   (c) Equation 5 thus follows from Equations 8 and 6 via MP.

□

Lemma 2.2. *If* $(x)(Eu)Pf\{u, Sub[r, Num(x), 21]\}$ *then:*

$$\ulcorner\neg(\forall x_1)\mathrm{r}\urcorner \vdash_C \ulcorner(\forall x_1)\mathrm{r}\urcorner \tag{9}$$

Proof. By hypothesis:

$$\ulcorner\neg(\forall x_1)\mathrm{r}\urcorner \tag{10}$$

Let $\overline{a}$ be an S numeral such that Equation 10 holds at a (Rule C [3]:§ 2.6):

$$\ulcorner\neg Subst\begin{bmatrix}\mathrm{r} & \begin{matrix}x_1\\ \overline{a}\end{matrix}\end{bmatrix}\urcorner \tag{11}$$

$(x)(Eu)Pf\{u, Sub[r, Num(x), 21]\}$ yields (instantiation at a):

$$\ulcorner Subst\begin{bmatrix}\mathrm{r} & \begin{matrix}x_1\\ \overline{a}\end{matrix}\end{bmatrix}\urcorner \tag{12}$$

Equations 11-12 yield (conjunction introduction):

$$\ulcorner \neg \{Subst \left[\mathrm{r} \ \frac{x_1}{\overline{a}} \right] \} \ \wedge \ Subst \left[\mathrm{r} \ \frac{x_1}{\overline{a}} \right] \urcorner \tag{13}$$

Equations 10-13 yield (proof by contradiction):

$$\ulcorner (\forall x_1) \mathrm{r} \urcorner \tag{14}$$

□

Lemma 2.3. *Equation 9 implies* $(Eu)Pf\{u, Sub[p, Num(p), 29]\}$

Proof. Applying [3] Proposition 2.10 and [3] Corollary 2.7, Equation 9 yields:

$$\vdash \ulcorner \neg(\forall x_1)\mathrm{r} \Rightarrow \ (\forall x_1)\mathrm{r} \urcorner \tag{15}$$

[3] Proposition 2.1 and the tautology $\ulcorner(\neg p \ \Rightarrow \ p) \ \Rightarrow \ p\urcorner$ yield:

$$\vdash \ulcorner [\neg(\forall x_1)\mathrm{r} \Rightarrow \ (\forall x_1)\mathrm{r}] \ \Rightarrow \ (\forall x_1)\mathrm{r} \urcorner \tag{16}$$

Equations 15-16 yield via MP:

$$\vdash \ulcorner (\forall x_1)\mathrm{r} \urcorner \tag{17}$$

□

Thus Proposition 2.1 follows from Lemmas 2.1-2.3, since:

1. $\{(x)(Eu)Pf[u, Sub(r, Num\{x\}, 21)]\} \rightarrow \ (Eu)Pf\{u, Sub[p, Num(p), 29]\}$ follows from Lemmas 2.2-2.3.

2. This, in conjunction with Lemma 2.1 (Equation 6), implies that S is inconsistent.

□

3 Discussion

As the methods of proof used in the proof of Proposition 2.1 are commonly used in classical mathematics it appears likely that the above result generalises to a much broader class of formal systems:

Conjecture D. *If a formal system T is such that:*

1. *The formation rules and rules of inference of T are recursive, once the primitive symbols are replaced by numbers [1];*

2. *All recursive relations may be defined in T (in the sense of Theorem V [1] modified as required).*

3. *All methods of proof commonly used in (classical) mathematics are formalised in T;*

Then T is inconsistent.

The generalisation of the above proof proposed at Conjecture D may however be doubted for various reasons. (For example, the above proof for the case of S exploits meta-theorems concerning the system that require complex proofs not exhibited above. Doubts may arise concerning the validity of these meta-theorems.) Irrespective of the extent to which other systems are affected by the proof, the result raises the question of how the formation rules / rules of inference of S should be revised so as to prevent the expression of paradox within the system. This formulation of the problem assumes (γ) that the classical view that there are no true contradictions is correct. Ockham's razor justifies assuming that this view is true until proven false.

A promising hypothesis to be investigated here is that Whitehead and Russell's ([5]: Chapter II) theory of logical types accounts for this paradox. Since the theory is complex and the subject of much debate I restrict myself to simply sketching a line of inquiry for future research. The theory justifies the assertion that the system of *Principia Mathematica* (PM) is not recursive in Gödel's sense. This theory prohibits the use of numbers as the primitive symbols of the system, for if the primitive symbols of PM are replaced by numbers then the system contains examples of sentences which, under the intended (fixed) interpretation assert that a PM expression which (names a) propositional function exhibits the property defined by this function. If the type theory of PM is correct, the sentences of other systems which, informally, are said to assert of themselves that they not provable (within a certain system) are as meaningless as sentences which purport to assert (of themselves) that they are false. If Conjecture D is true this peculiarity of PM may be very important. It perhaps keeps open the possibility that there exists a consistent system that does in fact formalise all the methods of proof commonly used in classical mathematics.

Whatever explanation for the inconsistency of S one finally accepts, the result is a shot in the arm for logic. For if simple type theory is adequate to prevent the expression of paradox in such systems, and such systems are incomplete in Gödel's [1] sense, the working mathematician might reasonably wonder whether the detour through such formal theories is worth the effort. From the above result we learn that informal reasoning about primitive recursive number-theoretic functions may also be subject to paradox in ways not previously recognised. Thus the claim that all such functions are effectively calculable is also subject to challenge, since the consistency of various systems commonly used to investigate the properties of all such functions is now in doubt. The class of such functions, if well defined at all can only be meaningfully defined (assuming γ) within a system resistant to expression of the above paradox. Assuming γ and Conjecture D are correct, standard definitions of this class are meaningless in the sense that they refer to entities that simply do not exist.

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