The Convergence of the Immersed Interface Method for Two Channels Dissipation Model with Discontinue Constants Velocity

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Abstract

The Immersed Interface Method can approximate the exact solution one dimensional two channels dissipation model in heterogeneous media. The method obtained two kinds continuous solution that: density and flux. The idea of the standard scheme Lax-Wendroff is used for regular grids, but the immersed interface methods is applied on a set of modified values deduced from numerical values and from jump conditions on interfaces. Numerical examples show that we can compute solutions to these equations with second order accuracy. The purpose of this paper is to investigate the convergence of the immersed interface method for two channels dissipation model with discontinue constants velocity. We analyze the stability using generalized Von Neumann analysis and the convergence of the method using Lax Equivalence Theorem.

Contents

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1 Introduction

In this paper let us consider $u(x,t)$ and $v(x,t)$ that depend on single spatial variable x and time $t \geq 0$. We assume that $u(x,t)$ and $v(x,t)$ are densities or concentrations measured in amount per unit volume that flow with speeds that are piecewise constants $c_1(x)$ and $c_2(x)$ respectively. Two groups of quantity have interactions, and then van Beckum (2003) arrived at the following set of equations for the concentrations $u(x,t)$ in the first group and $v(x,t)$ in the second group:

$$\frac{\partial u(x,t)}{\partial t} + c_1(x)\frac{\partial u(x,t)}{\partial x} = \alpha(v(x,t) - u(x,t)) \tag{1}$$

$$\frac{\partial v(x,t)}{\partial t} + c_2(x)\frac{\partial v(x,t)}{\partial x} = \alpha(u(x,t) - v(x,t)) \tag{2}$$

The constant α is the coefficient of exchanging concentrations.
The initial values the system given by

$$u(x,0) = f(x) \tag{3}$$

$$v(x,0) = g(x) \tag{4}$$

The approximation of the classical solution equations (1)-(4) that have boundary conditions has been investigated [3] by using finite difference method in case $c_1(x) = -c_2(x)$. Sumardi (2006) has solved the exact solution by Fourier transform in case $c_1(x)$ and $c_2(x)$ constant, and Sumardi(2008) also have solved by difference ways. Now we would like to present in case the heterogeneous media with interface . The velocities that change at $x = \beta$ in the interface are written:

$$c_1(x) = \begin{cases} c_1^+ & \text{if } x > \beta \\ c_1^- & \text{if } x < \beta \end{cases}, \; c_2(x) = \begin{cases} c_2^+ & \text{if } x > \beta \\ c_2^- & \text{if } x < \beta \end{cases}$$

The immersed interface method (IIM) has been developed recently by Leveque and Li (1994) to solve interface problem in which the differential equations have discontinuities and singularities in the coefficients and solutions. Zhang (1998) also used IIM for acoustic wave equations with discontinuities coefficients. Sumardi (2007) also applied the IIM for two scalar transport-dissipation model and two channels dissipation model with discontinue constants velocity. Here we investigate the convergence of the algorithms.

2 The explicit scheme for two channels dissipation model

We applied the Immersed Interface Method for two channels dissipation model. To solve this problem (1)-(4) let

$$U = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}, C = \begin{pmatrix} c_1(x) & 0 \\ 0 & c_2(x) \end{pmatrix}, A = \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix}.$$

This yields a system of first order equations

$$U_t + CU_x = AU \tag{5}$$

Here U represent some physical quantities, C are the speeds of the transport, with some jump condition specified at $x = \beta$. The jump condition will depend on the physical problem. The first we can impose the flux CU should be continuous that is, the number of item crossing at the interface per unit time must be the same on both sides, and therefore we have jump conditions:

$$[CU] = 0 \text{ or } [c_1 u] = 0, [c_2 v] = 0 \tag{6}$$

The others are that the quantities $U(x,t)$ should be continuous, so the jump conditions are:

$$[U] = 0 \text{ or } [u] = 0, [v] = 0 \tag{7}$$

To solve this problem by the Immersed Interface method, we need finite difference method approximation for regular grids.

To get second order accuracy for regular grids, it is used the idea of the Lax-Wendroff method that is based on the Taylor method series expansion. We discretize the $x - t$ plane by choosing a mesh width $h \equiv \Delta x$ and a time step $k \equiv \Delta t$ and define discrete mesh points (x_j, t_n) by:

$$\begin{aligned} x_j = jh, \quad j = & \quad 0, 1, 2, \cdots, N \\ t_n = nk, \quad n = & \quad 0, 1, 2, \cdots \end{aligned}$$

For simplicity we take a uniform mesh, with h and k constant, although most of the methods discussed can be extended to variable meshes. Sumardi

(2007) have given the schema:

$$\begin{aligned} U_j^{n+1} &= \left(I + kA + \frac{k^2A^2}{2} - \frac{k^2C^2}{h^2}\right)U_j^n \\ &+ \left(\frac{kC}{2h} + \frac{k^2(AC+CA)}{4h} + \frac{k^2C^2}{2h^2}\right)U_{j-1}^n \\ &+ \left(-\frac{kC}{2h} - \frac{k^2(AC+CA)}{4h} + \frac{k^2C^2}{2h^2}\right)U_{j+1}^n \end{aligned} \tag{8}$$

This valid for all the regular points, i.e. those for which a discontinuous point do not lie the interval of the 3-points stencil. Now consider the two irregular point J and $J+1$, with

$$x_J < \beta < x_{J+1}$$

i.e., β is the interface location in between of points x_J and x_{J+1}.

We will assume that β is not exactly green point, but lies strictly between points x_J and x_{J+1}. Then the Taylor series expansion is still valid, the solution is smooth in both x and t at every grid point, every those near the interface. The problem comes in approximating U_x and U_{xx} by difference formulas that uses values of U lying both sides of the interface. However, by taking into account the jumps in U and its derivatives at β, it is possible to possible to find linear combinations of grid values that give accurate approximation to the derivatives at grid points.

Rather than deriving expressions for U_x and U_{xx} separately, it is easiest to define three points schema of the form

$$\begin{aligned} U_J^{n+1} &= U_J^n + \frac{k}{h}\left(\Gamma_{J,1}U_{J-1}^n + \Gamma_{J,2}U_J^n + \Gamma_{J,3}U_{J+1}^n\right) \\ U_{J+1}^{n+1} &= U_{J+1}^n + \frac{k}{h}\left(\Gamma_{J+1,1}U_{J+2}^n + \Gamma_{J+1,2}U_{J+1}^n + \Gamma_{J+1,3}U_J^n\right) \end{aligned} \tag{9}$$

Here U is a vector variable and Γ's are 2×2 matrices, we want to choose the Γ's to get second order accuracy. Sumardi(2007) got at J from a system of matrix equation:

$$\begin{aligned} \beta_{11}\Gamma_{J,1} + \beta_{12}\Gamma_{J,2}\beta_{13}D_1\Gamma_{J,3} &= hA + \frac{1}{2}hkA^2 \\ \beta_{21}\Gamma_{J,1} + \beta_{22}\Gamma_{J,2}\beta_{23}D_1\Gamma_{J,3} &= (x_J-\beta)A - C^- - \frac{k}{2}(AC^- + C^-A) \\ \beta_{31}\Gamma_{J,1} + \beta_{32}\Gamma_{J,2}\beta_{33}D_1\Gamma_{J,3} &= -2C^-\frac{(x_J-\beta)}{h} + \frac{k}{h}(C^-)^2 \end{aligned}$$

where

$$\beta_{ij} = \left(\frac{x_{J+(j-2)}-\beta}{h}\right)^{i-1} \quad i,j = 1,2,3 \tag{10}$$

and

$$D_j = \begin{pmatrix} \frac{c_1^-}{c_1^+} & 0 \\ 0 & -\frac{c_2^-}{c_2^+} \end{pmatrix}^j \quad j = 1,2,3 \tag{11}$$

for jump conditions $[CU] = 0$,

$$D_j = \begin{pmatrix} \frac{c_1^-}{c_1^+} & 0 \\ 0 & -\frac{c_2^-}{c_2^+} \end{pmatrix}^{j-1} \quad j = 1,2,3 \tag{12}$$

for jump conditions $[U] = 0$.

Similarly Γ at the point x_{J+1}, we obtain the linear equations

$$\begin{aligned}
\hat{\beta}_{11}\Gamma_{J+1,1} + \hat{\beta}_{12}\Gamma_{J+1,2}\hat{\beta}_{13}D_1\Gamma_{J+1,3} &= hA + \frac{1}{2}hkA^2 \\
\hat{\beta}_{21}\Gamma_{J+1,1} + \hat{\beta}_{22}\Gamma_{J+1,2}\hat{\beta}_{23}D_1\Gamma_{J+1,3} &= (x_{J+1} - \beta)A - C^+ - \frac{k}{2}(AC^+ + C^+A) \\
\hat{\beta}_{31}\Gamma_{J+1,1} + \hat{\beta}_{32}\Gamma_{J+1,2}\hat{\beta}_{33}D_1\Gamma_{J+1,3} &= -2C^+\frac{(x_{J+1} - \beta)}{h} + \frac{k}{h}(C^+)^2
\end{aligned}$$

where

$$\hat{\beta}_{ij} = \left(\frac{x_{J+1(j-2)} - \beta}{h}\right)^{i-1} \quad i,j = 1,2,3 \tag{13}$$

and D_j are computed as (11) and (12).

3 Numerical Result

Here we will show two examples. Figure 1 gives the numerical solution to the two channels dissipation model with jump $[U] = 0$(left) and $[CU] = 0$ (right) at $\beta = \frac{5}{9}$, at $\alpha = 1$, at $t = 0$,$t = 0.15$,$t = 0.30$ and $t = 0.40$. The initial data are three humps given by

$$\begin{aligned}
u(x,0) &= \begin{cases} \frac{1}{2}(1 + \cos(\frac{(x-0.28)\pi}{.08}) & 0.04 \le x \le 0.52 \\ 0 & \text{otherwise} \end{cases} \\
v(x,0) &= 0.5
\end{aligned}$$

Figure 1 gives the solutions on grid size $h = \frac{1}{200}$ and $k = 0.001$. After reaching the interface, the wave speed changes accordingly on the other sides. For the case of the jump conditions $[U] = 0$, the solutions of $u(x,t)$ is continuous. The magnitude of $u(x,t)$ is not changed by the change of speed at the

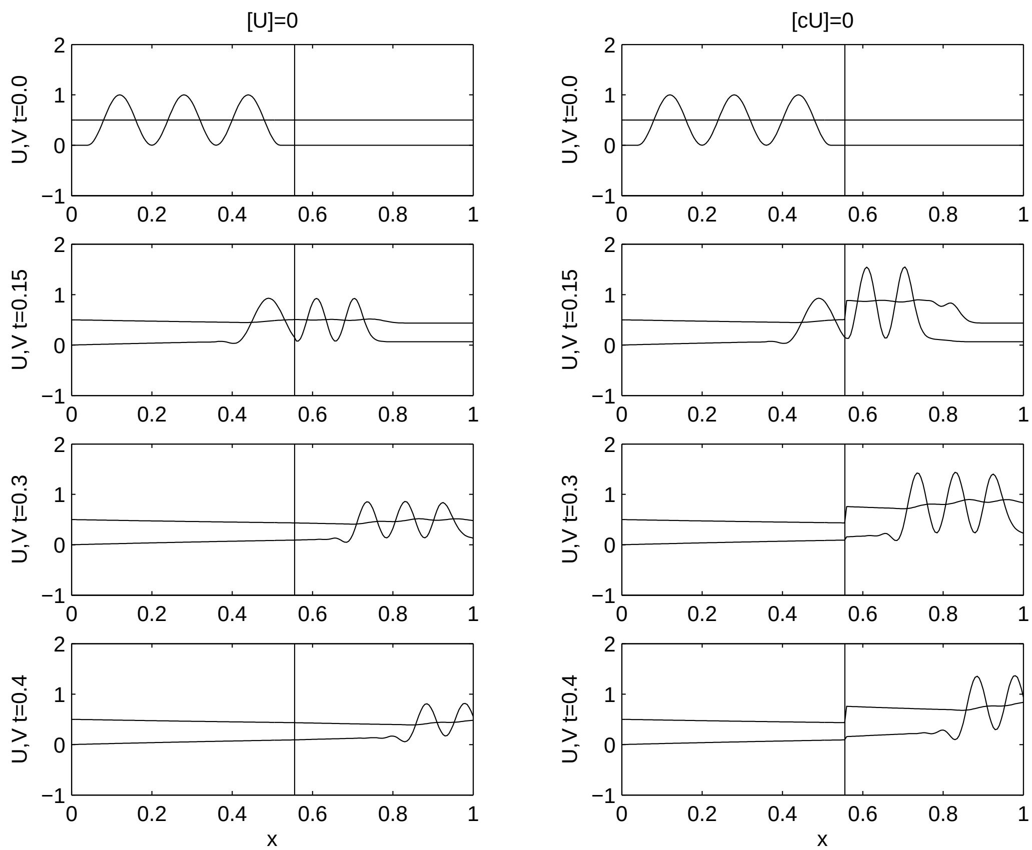

Figure 1: Numerical solution the two channels dissipation model with jump [U]=0(left) and [CU]=0(right) at $\beta = 5/9$, at $\alpha = 1$, at $t = 0, 0.1, 0.15, 0.3$ and $t = 0.4$

interface, but the wave length of $u(x,t)$ is changed by the change of speed at in the interface. The magnitude is changed by the term of dissipative $u(x,t)$ and $v(x,t)$. The solution of $v(x,t)$ is also continuous and occurs three humps from the term of in the second equation of two channels dissipation model.

For the case of the jump conditions $[CU]=0$, the solution of $u(x,t)$ is not continuous at the interface. The magnitude of $u(x,t)$ is changed by the change of speed at the interface, and the wave length is also changed by the change of speed at the interface. The magnitude is also changed by the term of dissipative. The solution of $v(x,t)$ is not continuous at the interface and occurs three humps from the term of $u(x,t)$ in the second equation of two channels dissipation model.

4 Theory of the Convergence and the Stability

In this section, we consider Cauchy problem in the space $[0,L] \subset R$ and time t ranges on an interval, typically $[0,T]$. A constant-coefficient first-order system is determined by matrix C and B given in $M_N(R)$, where is the size of the system. Then the Cauchy problem consists in finding solution $U(x,t)$ of initial value problem:

$$\begin{aligned} \frac{\partial U(x,t)}{\partial t} &= C\frac{\partial U(x,t)}{\partial x} + BU(x,t) \\ U(x,0) = U_0(x), &\qquad x \in [0,L] \times [0,T] \end{aligned} \tag{14}$$

The next definition gives the meaning of a solution of the problem (14).

Definition 4.1 *A function $U : [0,L] \times [0,T] \longrightarrow R^N$ is a **solution** of the initial value problem (14) if $U(x,t)$ is a continuously differentiable and when substitute into (14) reduces (16) into identity on domain $[0,L] \times [0,T]$.*

Definition 4.2 *For on complex M-dimensional vectors $u = (u_0, u_1, u_2, \cdots, u_M) \in R^{M+1}$ and $v = (v_0, v_1, v_2, \cdots, v_M) \in R^{M+1}$, we define **inner product** and **norm**:*

$$< u, v >= \frac{1}{M}\sum_{j=0}^{M} u_j \bar{v}_j, \qquad \|u\| = \sqrt{< u, u >} \tag{15}$$

*where the overbar denotes the complex conjugate. Let function f and g defined on $D \subset R$, we define **inner product** and **norm**:*

$$< f, g >= \frac{1}{M}\int_D f(x)g(x)dx, \qquad \|f\| = \sqrt{< f, f >} \tag{16}$$

and operator T on Hilbert Space , we defined norm:

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} \tag{17}$$

Definition 4.3 *The initial value problem (14) is* ***well-posed*** *if for$U_0 \in C^1([0,L])$, there is a unique solution $U \in \{C^1([0,L]\times[0,T])\}^N$ and the solution depends continuously on the initial value : There exists a constant $c_0 > 0$ such that $U(x,t)$ and $\bar{U}(x,t)$ are the solutions for the initial values $U_0, \bar{U}_0 \in C^1([0,L])$, then*

$$\sup_{0\leq t\leq T} \|U(x,t) - \bar{U}(x,t)\| \leq c_0\|U_0 - \bar{U}_0\|. \tag{18}$$

Proposition 4.1 *Assume the problem (14) is well-posed and the solution is denoted as*

$$U(x,t) = S(t)U_0, \qquad U_0 \in \{C^1(R)\}^N \tag{19}$$

Then the solution operator $S(t)$ for any t is a linear operator.

Now we introduce a finite difference method defined by one-parameter family of uniformly bounded linear operators on grid function set $R^{M+1} = \{(u_0, u_1, u_2, \cdots, u_M) | u_j \in R, j = 0, 1, 2, \cdots, M\}$:

$$C(\Delta t) : R^{M+1} \to R^{M+1}, \qquad 0 < \Delta t \leq \Delta t_0. \tag{20}$$

Definition 4.4 *The* ***finite difference method*** *is the approximate exact solution $U(j\Delta x, n\Delta t) = S(j\Delta x, n\Delta t)U_0(j\Delta x)$ of the problem (14) that is defined by*

$$\begin{aligned}
C_j(\Delta t)U_j^n &= \sum_{s\in I_j} C_{js}U_{j+s}^n \\
U^{n+1} &= C(\Delta t)U^n = (C_0(\Delta t)U_0^n, C_1U_1^n, \cdots, C_M(\Delta t)U_M^n) \\
C(\Delta t)U^n &= C(\Delta t)^n U^0 \qquad j = 0, 1, 2, \cdots, M \\
U^n &= (U_0^n, U_1^n, U_2^n, \cdots, U_M^n) \qquad n = 0, 1, 2, 3, \cdots \\
U_j^n &= \begin{pmatrix} u_{j1}^n \\ u_{j2}^n \\ \vdots \\ u_{jN}^n \end{pmatrix} \in R^N
\end{aligned} \tag{21}$$

Definition 4.5 *The finite difference method (21) is* ***consistency*** *for the corresponding solution U(x,t) of the initial value problem (14) if*

$$\lim_{(\Delta t,\Delta x)\to(0,0)} \|\frac{1}{\Delta t}(C(\Delta t)U(x,t) - U(x,t+\Delta t))\| = 0 \tag{22}$$

Definition 4.6 *The finite difference method (21) is* ***convergent*** *if for any fixed* $t \in [0, T]$*, any* $u_0 \in V$*, we have*

$$\lim_{\Delta t_i \to 0} \|C(\Delta t_i)^n U_0 - S(t)U_0\| = 0 \tag{23}$$

where n_i *is sequence of integer and* Δt_i *a sequence of step time sizes such that*

$$\lim_{i \to \infty} n_i \Delta t_i = t.$$

Definition 4.7 *The finite difference method (21) is* ***stable*** *if the operator*

$$\{C(\Delta t)^n | 0 < \Delta t \leq \Delta t_0, n\Delta t \leq T\} \tag{24}$$

are uniformly bounded; i.e., there exists a constant $M_0 > 0$ *such that*

$$\|C(\Delta t)^n\| \leq M_0 \qquad \forall n : n\Delta t \leq T \qquad \forall \Delta t \leq \Delta t_0. \tag{25}$$

We now give the main result theorem of convergence analysis:

Theorem 4.1 (Lax Equivalence theorem) *Suppose the initial value problem (14) is well-posed. For a consistency finite difference method (21), stability is equivalent to convergence.*

Theorem 4.2 *Assuming that* M *is even, we define the vectors*

$$w^{(k)} = (w_0^{(k)}, w_1^{(k)}, \cdots, w_M^{(k)}),$$
$$k = -\frac{M}{2}, \cdots, 0, 1, \cdots, \frac{M}{2}$$

as the values of continuous exponential evaluated at discrete points:

$$w_j^{(k)} = e^{\frac{2\pi jk\Delta x}{L}} \tag{26}$$

Then

1. *The vector* $w^{(k)}$ *form a complete orthonormal basis with respect to the inner product (15)*
2. $w_{j+s}^{(k)} = e^{is\theta} w_j^{(k)}, \qquad \theta = \frac{2\pi\pi k\Delta x}{L}$

Let U some vector that defined

$$\begin{aligned} U &= (U_0, U_1, U_2, \cdots, U_M) \\ U_j &= \begin{pmatrix} u_{j1} \\ u_{j2} \\ \vdots \\ u_{jN} \end{pmatrix} \in R^N \end{aligned}$$

U can then expanded in terms of the $w^{(k)}$, so

$$U = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k w^{(k)} \tag{27}$$

where

$$B_k = \begin{pmatrix} < u_{j1}, w^{(k)} > \\ < u_{j2}, w^{(k)} > \\ \vdots \\ < u_{jN}, w^{(k)} > \end{pmatrix} \tag{28}$$

We also have

$$\|U\|^2 = \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} \|B_k\|^2. \tag{29}$$

Then evaluate the operator $C(\Delta t)U$, so we have

$$\begin{aligned} C(\Delta t)U &= C(\Delta t) \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k w^{(k)} \\ &= \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k C(\Delta t) w^{(k)} \\ &= \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k \left(C_0(\Delta t) w_0^{(k)}, C_1(\Delta t) w_1^{(k)}, \cdots, C_M(\Delta t) w_M^{(k)} \right) \\ &= \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k \left(\sum_{s \in I_0} C_{0,s} w_s^{(k)}, \sum_{s \in I_1} C_{1,s} w_{1+s}^{(k)}, \cdots, \sum_{s \in I_M} C_{M,s} w_{M+s}^{(k)} \right) \\ &= \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k \left(\sum_{s \in I_0} C_{0,s} e^{is\theta} w_0^{(k)}, \sum_{s \in I_1} C_{1,s} e^{is\theta} w_1^{(k)}, \cdots, \sum_{s \in I_M} C_{M,s} e^{is\theta} w_M^{(k)} \right) \end{aligned}$$

Here we define the coefficients $G_j(\Delta x, \Delta t, \theta) = \sum_{s\in I_j} C_{j,s}e^{is\theta}$ and called **amplification matrix at point** j.

Theorem 4.3 *The finite difference method (21) is stable if only if there exist positive constants* K, h_0, k_0 *such that*

$$\|G_j(\Delta x, \Delta t, \theta)\| \leq 1 + K\Delta t, \forall j = 0, 1, \cdots, M, \tag{30}$$

for all $0 < \Delta t < k_0, 0 < \Delta x < h_0, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Proof:

$$\begin{aligned}
\|C(\Delta t)U\|^2 &= \left\|\sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k\left(\sum_{s\in I_0} C_{0,s}e^{is\theta}w_0^{(k)}, \cdots, \sum_{s\in I_M} C_{M,s}e^{is\theta}w_M^{(k)}\right)\right\|^2 \\
&\leq \left\|\sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k\left((1+K\Delta t)w_0^{(k)}, \cdots, (1+K\Delta t)w_M^{(k)}\right)\right\|^2 \\
&= \left\|\sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k\left((1+K\Delta t)(w_0^{(k)}, \cdots, w_M^{(k)})\right)\right\|^2 \\
&= \left\|\sum_{k=-\frac{M}{2}}^{\frac{M}{2}} B_k w^{(k)}\right\|^2 (1+K\Delta t)^2 \\
&= \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} |B_k|^2(1+K\Delta t)^2 \\
&= \|U\|^2(1+K\Delta t)^2
\end{aligned}$$

Then we have $\|C(\Delta t)U\| \leq \|U\|(1+K\Delta t)$ and $\|C(\Delta t)\| \leq (1+K\Delta t)$.

$$\|C(\Delta t)^n\| \leq (1+K\Delta t)^n \leq e^{Kn\Delta t} = e^{KT} \tag{31}$$

Therefore the operator

$$\{C(\Delta t)^n | 0 < \Delta t \leq \Delta t_0, n\Delta t \leq T\} \tag{32}$$

are uniformly bounded. Hence the finite difference (21) is stable.

Here to compute norm matrix $\|G_j(\Delta x, \Delta t, \theta)\|$, based paper Lax (1961), we compute

$$\|G_j(\Delta x, \Delta t, \theta)\| = \max_{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, i=1,\cdots,N} |\lambda_{ji}(\Delta x, \Delta t, \theta)|, \tag{33}$$

where $\lambda_{ji}(\Delta x, \Delta t, \theta)$ all eigenvalues of matrix $G_j(\Delta x, \Delta t, \theta)$.

Theorem 4.4 (The Courant-Friedrichs-Levy Condition) *A necessary condition for stability of the explicit scheme (21) for the hyperbolic equation initial value problem (14):*

$$CFL = \max_{i=1,\cdots,N} |\lambda_i| \frac{\Delta t}{\Delta x} \leq 1 \tag{34}$$

where λ_i all eigenvalues matrix C.

5 The convergence of the IIM for two channels dissipation model

In this section we would prove the convergence of the immersed interfaced method for two channels dissipation model in section 2 using theory in previous section. Using Lax Equivalence Theorem and von Neumann Condition for Stability. The IIM schemas have second order accuracy in regular or irregular points, so the method is consistency. By Lax Equivalence Theorem if the method is stable then it is also convergence.

For stability the method, we only see the norm of amplification matrix scheme (8) and (9). To compute the norm of amplification matrix we use the equation (33). Using theorem 4.4 necessary condition for stability of two channels dissipation model (1-4) depend on grid size k, h and velocities:

$$c_1(x) = \begin{cases} c_1^+ & \text{if } x > \beta \\ c_1^- & \text{if } x < \beta \end{cases}, \; c_2(x) = \begin{cases} c_2^+ & \text{if } x > \beta \\ c_2^- & \text{if } x < \beta \end{cases}$$

that is

$$CFL = \max\{|c_1^+|, |c_1^-|, |c_2^+|, |c_2^-|\} \frac{k}{h} \leq 1. \tag{35}$$

For regular points we have scheme

$$\begin{aligned} U_j^{n+1} &= \left(I + kA + \frac{k^2 A^2}{2} - \frac{k^2 C^2}{h^2}\right) U_j^n \\ &+ \left(\frac{kC}{2h} + \frac{k^2(AC + CA)}{4h} + \frac{k^2 C^2}{2h^2}\right) U_{j-1}^n \\ &+ \left(-\frac{kC}{2h} - \frac{k^2(AC + CA)}{4h} + \frac{k^2 C^2}{2h^2}\right) U_{j+1}^n \end{aligned} \tag{36}$$

we have amplification matrix at regular points $j \neq J, J+1$

$$\begin{aligned}
G_j(h,k,\theta) &= \left(I + kA + \frac{k^2A^2}{2} - \frac{k^2C^2}{h^2}\right) \\
&\quad + \left(\frac{kC}{2h} + \frac{k^2(AC+CA)}{4h} + \frac{k^2C^2}{2h^2}\right) e^{-j\theta} \\
&\quad + \left(-\frac{kC}{2h} - \frac{k^2(AC+CA)}{4h} + \frac{k^2C^2}{2h^2}\right) e^{j\theta} \\
&= \left(I + kA + \frac{k^2A^2}{2} - \frac{k^2C^2}{h^2}\right) \\
&\quad - \left(\frac{kC}{h} + \frac{k^2(AC+CA)}{2h}\right) i\sin\theta + \left(\frac{k^2C^2}{h^2}\cos\theta\right) \\
&= I - \frac{kCi\sin\theta}{h} + \frac{k^2C^2(\cos\theta - 1)}{h^2} + k\left(\frac{kA^2}{2} + \frac{k(AC+CA)i\sin\theta}{h}\right) \\
&= I - \frac{kCi\sin\theta}{h} - \frac{k^2C^2 2\sin^2\frac{\theta}{2}}{h^2} + k\left(\frac{kA^2}{2} + \frac{k(AC+CA)i\sin\theta}{h}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\|G_j(h,k,\theta)\| &= \left\| I - \frac{kCi\sin\theta}{h} - \frac{k^2C^2 2\sin^2\frac{\theta}{2}}{h^2} \right\| \\
&\quad + k\left\| \frac{kA^2}{2} + \frac{k(AC+CA)i\sin\theta}{h} \right\|
\end{aligned} \tag{37}$$

Consider the first term on the right hand of equation (37), we obtain

$$\begin{aligned}
\left\| I - \frac{kCi\sin\theta}{h} + \frac{k^2C^2 2\sin^2\frac{\theta}{2}}{h^2} \right\|^2 &= \left\| I - \frac{k^2C^2 2\sin^2\frac{\theta}{2}}{h^2} \right\|^2 \\
&\quad + \left\| \frac{kC 2\sin\frac{\theta}{2}\sin\frac{\theta}{2}}{h} \right\|
\end{aligned} \tag{38}$$

Because C is a diagonal matrix, so we have$\|I - C^2\|^2 = 1 + \|C\|^4 - 2\|C^2\|$ and $\|C^2\| = \|C\|^2$. Hence the equation (38) becomes

$$\left\| I - \frac{kCi\sin\theta}{h} + \frac{k^2C^2 2\sin^2\frac{\theta}{2}}{h^2} \right\|^2 = 1 - 4\sin^4\frac{\theta}{2}\left\|\frac{kC}{h}\right\|^2\left(1 - \left\|\frac{kC}{h}\right\|^2\right) \tag{39}$$

Using equatiaon(39), if $CFL = \left\|\frac{kC}{h}\right\| \leq 1$, then

$$\|G_j(h,k,\theta)\| \leq 1 + kK, \qquad j \neq J, J+1, \tag{40}$$

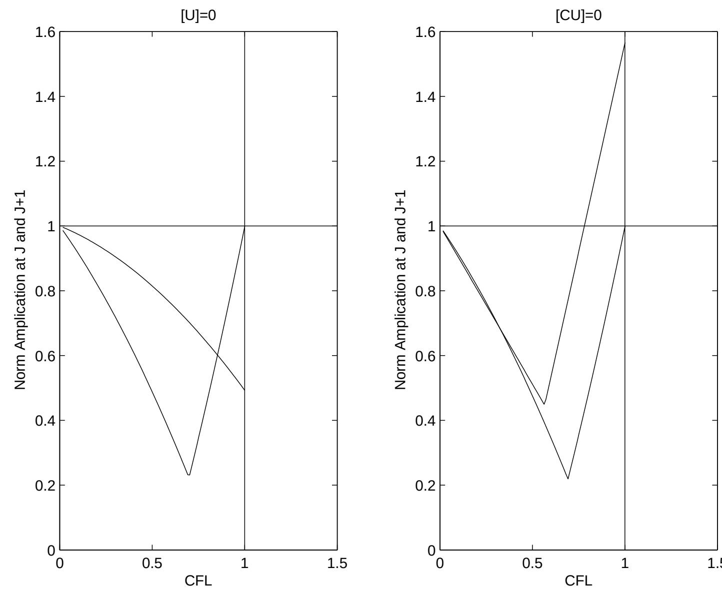

Figure 2: Graph the norm amplification matrix at irregular points.

where $K = \left\| \frac{kA^2}{2} + \frac{k(AC+CA)i\sin\theta}{h} \right\|$.

To complete proving the convergence of the immersed interface method in the section 3, we must prove

$$\|G_j(h,k,\theta)\| \leq 1 + kK, \qquad j = J, J+1, \tag{41}$$

Analytically it is difficult to prove (41), so here we prove by plotting $\|G_j(h,k,\theta)\|$ for $0 < CFL \leq 1$ at irregular point J and $J+1$ with jump condition $[U] = 0$ and $[CU] = 0$.

We see from the figure 2 that $\|G_j(h,k,\theta)\| \leq 1$ for $0 < CFL \leq 1$ at irregular points J and $J+1$ with $[U] = 0$, so the IIM is convergent for$[U] = 0$ at the interface.

In case $[CU] = 0$ we see that $\|G_J(h,k,\theta)\| \leq 1$ for $0 < CFL \leq 1$ at irregular points J, but there exist $0 < CFL \leq 1$ such that $\|G_{J+1}(h,k,\theta)\| > 1$. From the graph we see that $\|G_{J+1}(h,k,\theta)\|$ is linear with CFL that is linear with k, so $\|G_{J+1}(h,k,\theta)\|$ is also linear with k. Hence $\|G_{J+1}(h,k,\theta)\| \leq 1 + kK$,$K$ is positive gradient function $\|G_{J+1}(h,k,\theta)\|$ under k. So the IIM is also convergent for $[CU] = 0$.

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