QMC for Finance beyond Black-Scholes

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Abstract

QMC Methods are used to approximate integrals of high dimensionality. However, if the problem under consideration is of unbounded dimensionality, it is not obvious if one can apply QMC methods at all. In this text, we introduce a hybrid approach combining QMC and MC methods and apply it to a finance problem of unbounded dimensionality. We find that this hybrid approach improves on approaches only relying on MC methods.

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1 Introduction

QMC methods are techniques for numerical integration. The pricing of financial derivatives in Gaussian models results in integration problems to which QMC methods are routinely applied, e.g. [1]. However, the shortcomings of Gaussian models are well-known, and from a finance point of view, the usage of jump-diffusion models such as the Kou model, e.g. [4], might be more appropriate. Yet, the pricing of financial derivatives in the Kou model can result in problems of unbounded dimensionality, and it is not obvious if QMC methods can be applied at all.

Having recalled properties of QMC point sets and the Kou model and defined lookback options in Section 2, we show in Section 3.1 why the pricing of lookback options in the Kou model can result in an integration problem of unbounded dimensionality. In Section 3.2, we recall stratification, a variance reduction technique, and use it to construct a hybrid approach combining QMC and MC methods in Section 4. In Section 5, we present numerical results showing that the hybrid approach outperforms both the stratification and the standard Monte Carlo approach.

2 QMC Point Sets and the Finance Problem

In this section, we firstly recall some basic properties of QMC point sets and then introduce the finance problem.

2.1 Properties of QMC Point Sets

QMC methods are equal-weight integration formulas to estimate high-dimensional integrals over the unit cube. In formulating the integration problem we often need to invert probability distributions and we hence recall the well-known generalised inverse function, e.g. [3], p. 55: If $Y \sim F$, then the generalised inverse of F, $F^{-1}(u)$, is given by

$$F^{-1}(u) = \inf\left\{z \in \mathbb{R} : F(z) \geq u\right\}, \quad u \in [0,1].$$

We point out that all generalised inverse functions needed for this paper are available in standard computer packages such as MATLAB.

For purposes of this paper, we will make use of digital nets, e.g. [5], and randomize these using a digital b-ary shift, e.g. [5], to compute standard errors. Assuming that the mixed partial derivatives of the integrand satisfy a Lipschitz condition, one can obtain a variance bound for the QMC estimator of $\mathcal{O}(n^{-2}(\ln(n))^d)$, e.g. [5], where d denotes the dimensionality of the problem. We remark that this smoothness condition is not satisfied by the functions dealt with in this paper, hence it is important to be able to investigate standard errors numerically. Lastly, instead of digital nets, other QMC point sets such as lattice rules, e.g. [5], could have been used.

2.2 The Finance Problem

Firstly, we recall properties of the Kou model, e.g. [4]. We assume that we deal with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we introduce the stochastic processes $(Z_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ in Definition 2.1. The following density function will be used in Definition 2.1 :

$$f(x) = p\eta^+ \exp(-\eta^+ x)\mathbb{I}_{x\geq 0} + (1-p)\eta^- \exp(\eta^- x)\mathbb{I}_{x<0}, \quad \forall x \in \mathbb{R} \tag{1}$$

where $0 \leq p \leq 1$, $\eta^+ > 1$, $\eta^- > 0$. The cumulative distribution corresponding to density f, denoted by F, is of course invertible and we denote its inverse by $F^{-1}(u)$, $u \in [0,1]$. The inverse can be computed explicitly as

$$F^{-1}(u) = \frac{\ln(\frac{u}{1-p})}{\eta^-}\mathbb{I}_{u\leq 1-p} + \frac{\ln(\frac{1-u}{p})}{-\eta^+}\mathbb{I}_{u>1-p}, u \in [0,1].$$

We also set $\mu = r - \frac{1}{2}\sigma^2 - \lambda \int_{\mathbb{R}}(\exp(x)-1)f(x)dx$, for $\sigma \in \mathbb{R}^+$, $r \in \mathbb{R}^+$.

Definition 2.1 *Let* $(N_t)_{t\geq 0}$ *be a Poisson process with intensity* $\lambda > 0$ *and jump times* $(\tau_i)_{i=1}^{N_T}$, $(W_t)_{t\geq 0}$ *a Brownian motion and* $(Y_k)_{k\geq 1}$ *independent, identically distributed random variables with distribution* F. *Then*

$$Z_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k \tag{2}$$

and, for $S_0 \in \mathbb{R}^+$,

$$S_t = S_0 \exp(Z_t). \tag{3}$$

Of course $(S_t)_{t\geq 0}$ is the stock price process. We can now focus on the financial derivative relevant for this paper. We deal with a particular lookback option, a continuously monitored lookback put option with floating strike price, from now on referred to as lookback option. Its pay-off is given by

$$\max(B, \max_{0\leq t\leq T} S_t) - S_T$$

where $B \in \mathbb{R}^+$, $B > S_0$, and its price by

$$\mathbb{E}[\exp(-rT)(\max(B, \max_{0\leq t\leq T} S_t) - S_T)] = \mathbb{E}[\exp(-rT)\max(B, \max_{0\leq t\leq T} S_t)] - S_0.$$

So the integration problem is reduced to approximating

$$\mathbb{E}[\exp(-rT)\max(B, \max_{0\leq t\leq T} S_t)] \tag{4}$$

and we denote

$$V = \exp(-rT)\max(B, \max_{0\leq t\leq T} S_t). \tag{5}$$

We will now show how to estimate prices of lookback options, assuming that the stock price is given by (3).

3 MC Approaches to Pricing Lookback Options in the Kou Model

In Section 3.1, we estimate (4) using a Monte Carlo approach and in Section 3.2, we modify this approach using stratification. Besides being a variance reduction technique, the benefit of stratification is that the resulting problem formulation is amenable to the hybrid approach combining QMC and MC methods presented in section 4.

3.1 Naive MC Approach

In this section, we show how to price a lookback option in the Kou model using a Monte Carlo approach. This seems new, most likely due to the fact that closed form solutions for prices of lookback options in the Kou model are available in terms of special functions: [4]. We note however, that the methodology presented in this section is not limited to the Kou model: Instead of assuming that jumps follow (1), any distribution that can be simulated can be used. We make the convention that $\tau_0 = 0$ and $\tau_{N_T+1} = T$.
We recall the definition of V, (5), and denote independent, identically distributed copies of V by V_i, $i = 1, \ldots, n_{MC}$, $n_{MC} \in \mathbb{Z}^+$, so n_{MC} corresponds to the number of function evaluations for the naive MC approach. It is obvious that

$$I_{MC} = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} V_i \tag{6}$$

is an unbiased estimator of $\mathbb{E}[V]$ and an unbiased estimator of its variance is given by

$$\sigma^2_{MC} = \frac{1}{n_{MC}(n_{MC}-1)} \sum_{i=1}^{n_{MC}} (V_i - I_{MC})^2. \tag{7}$$

However, the simulation of V_i, $i = 1, \ldots, n_{MC}$, is not immediately obvious and is explained below. The following notation is introduced:

$$M_l = \max_{\tau_{l-1} \leq t < \tau_l} S_t, \quad l = 1, \ldots, N_T + 1$$

and we define the sigma-algebra

$$\mathcal{F}^* = \sigma((N_t)_{t \in [0,T]}, (Y_l)_{l=1}^{N_T}, (W_{\tau_l})_{l=1}^{N_T}, W_T).$$

Of course we can express $\mathbb{E}[V]$ as follows:

$$\mathbb{E}[V] = \mathbb{E}[\mathbb{E}[\exp(-rT) \max(B, \max_{l=1,\ldots,N_T+1} M_l) \,|\, \mathcal{F}^*]]. \tag{8}$$

Equality (8) suggests that one can simulate V by simulating $(M_l)_{l=1}^{N_T+1}$. To be able to generate the M_l, we need to know their distributions and be able to invert these. This is addressed in the next lemma, parts ii) and iii), where we set $S_{\tau_l^-} = S_{l^-}$ and $S_{\tau_{l-1}} = S_{l-1}$, $l = 1, \ldots, N_T + 1$.

Lemma 3.1 *The following properties hold:*

i) Conditional on $\mathcal{F}^*$, the $(M_l)_{l=1}^{N_T+1}$ are independent.

ii) The distribution of M_l, conditional on $\mathcal{F}^*$, F_{M_l}, is given by

$$F_{M_l}(b) = \mathbb{P}^*\left(M_l \leq b \,|\, \mathcal{F}^*\right) = (1-\exp(-\frac{2\ln(\frac{S_{l^-}}{b})\ln(\frac{S_{l-1}}{b})}{(\tau_l - \tau_{l-1})\sigma^2}))\mathbb{I}_{b \geq \max(S_{l^-}, S_{l-1})}.$$

iii) The inverse of F_{M_l}, $F_{M_l}^{-1}$ is given by

$$\ln(F_{M_l}^{-1}(u)) = \frac{\ln(S_{l^-}S_{l-1}) + \sqrt{(\ln(\frac{S_{l^-}}{S_{l-1}}))^2 - 2(\tau_l - \tau_{l-1})\sigma^2 \ln(1-u)}}{2}.$$

Proof. i) and ii) are given in [2], p.177, iii) follows from ii). □

Next, we recall that if Y follows the Beta distribution with parameters a, b, $Y \sim Beta(a, b)$, then its density is given by

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 \leq x \leq 1 \tag{9}$$

and its inverse is denoted by $F_{Beta}^{-1}(u, a, b)$. Regarding notation, if X is normal with mean μ and variance σ^2, we write $X \sim N(\mu, \sigma^2)$, and the inverse of its distribution is denoted by $F_N^{-1}(u, \mu, \sigma^2)$. If Y is Poisson with rate λ, we write $Y \sim P(\lambda)$, and the inverse of its distribution is denoted by $F_P^{-1}(u, \lambda)$. Finally we state the following lemma:

Lemma 3.2 *Conditional on* $N_T = n$, $\tau_l = a$, $\tau_m = b$, $0 \leq a < b \leq T$ *and* $0 \leq l < m \leq n+1$, *the law of* $(\tau_{l+o})_{o=1}^{m-l-1}$ *is given by*

$$\frac{\tau_{l+o} - \tau_l}{\tau_m - \tau_l} \sim Beta(o, m-l-o), o = 1, \ldots, m-l-1.$$

Proof. The proof follows from [6], Theorem 5.2 and Example 2.38. □

We are now in a position to state the algorithm showing how to obtain one realisation of V, which is needed for the $(V_i)_{i=1}^{n_{MC}}$ in (6).

Algorithm 3.1 *The naive MC approach to simulating* V*:*

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1. Simulate $N_T \sim P(\lambda T)$
2. for $l = 1 : N_T$
    • Simulate $\frac{\tau_l - \tau_{l-1}}{T - \tau_{l-1}} \sim Beta(1, N_T - (l-1))$
    • Simulate $Y_l \sim F$
    • Simulate $W_{\tau_l} - W_{\tau_{l-1}} \sim N(0, \tau_l - \tau_{l-1})$
    • Simulate $M_l \sim F_{M_l}$
3. end
4. Simulate $W_T - W_{\tau_{N_T}} \sim N(0, T - \tau_{N_T})$
5. Simulate $M_{N_T+1}$
6. Set $V = \exp(-rT) \max(B, M_1, \ldots, M_{N_T+1})$
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Remark 3.1 We remark that in Algorithm 3.1, we only deal with Beta variates with parameters 1 and b, $b \geq 1$. Using (9), we conclude that the distribution of such variates is given by $F(y) = 1 - (1-y)^b$ and its inverse is given by $F^{-1}(u) = 1 - (1-u)^{\frac{1}{b}}$. This should be exploited when implementing Algorithm 3.1.

The following remark shows that for purposes of QMC, Algorithm 3.1 is not useful.

Remark 3.2 We note that the formulation of the problem provided by Algorithm 3.1 is not suitable for QMC methods. This is because N_T is not bounded resulting in a problem whose dimension is very high, in principle infinite. Furthermore, dealing with a QMC point set, a particular dimension should correspond to a particular random variable, else the favourable uniformity properties of the QMC point set are not preserved. It is clear that Algorithm 3.1 does not ensure this.

3.2 Stratification approach

Stratification is a well-known variance reduction technique, see e.g. [3], where it is described as a technique constraining the fraction of observations drawn from specific subsets (strata) of the sample space. For purposes of our problem, we fix $k^* \in \mathbb{Z}^+$ and choose the following k^*+2 strata: $\{N_T = k\}_{k=0}^{k^*}$ and $\{N_T > k^*\}$. Consequently, we rewrite $\mathbb{E}[V]$ as follows:

$$\mathbb{E}[V] = \sum_{k=0}^{k^*} \mathbb{E}[V \mid N_T = k]\mathbb{P}(N_T = k) + \mathbb{E}[V \mid N_T > k^*]\mathbb{P}(N_T > k^*). \tag{10}$$

In this section, we estimate $\mathbb{E}[V \mid N_T = k]$, $k = 0, \ldots, k^*$ and $\mathbb{E}[V \mid N_T > k^*]$ using Monte Carlo techniques; in Section 4, we estimate $\mathbb{E}[V \mid N_T = k]$, $k = 0, \ldots, k^*$ using QMC methods. We note though that using Black-Scholes arguments, we can arrive at a closed form solution for $\mathbb{E}[V \mid N_T = 0]$, denoted by I_0, hence $\mathbb{E}[V \mid N_T = 0]$ is not estimated. In addition, we set $p_k = \mathbb{P}(N_T = k)$, $k = 0, \ldots, k^*$ and $p_{k^*+1} = \mathbb{P}(N_T > k^*)$. To introduce the estimator corresponding to the stratification approach, we define the following random variables:

Definition 3.1 Let $k^* \in \mathbb{Z}^+$ and $S \in \mathcal{B}(\mathbb{R})$, then $\widetilde{V}_k$, $k = 0, \ldots, k^*+1$ are random variables with respective laws

$$\mathbb{P}(\widetilde{V}_k \in S) = \mathbb{P}(V \in S \mid N_T = k), \quad k = 0, \ldots, k^*$$

and

$$\mathbb{P}(\widetilde{V}_{k^*+1} \in S) = \mathbb{P}(V \in S \mid N_T > k^*).$$

Consequently, (10) becomes

$$\mathbb{E}[V] = \sum_{k=0}^{k^*+1} \mathbb{E}[\widetilde{V}_k] p_k. \tag{11}$$

Now define $\widetilde{V}_{ki}$ to be independent, identically distributed copies of $\widetilde{V}_k$, and we formulate the stratified estimator:

$$I(k^*)_{\text{strat}} = \sum_{k=0}^{k^*+1} I_k p_k$$

where, for $n_k \in \mathbb{Z}^+$, $k = 1, \ldots, k^*+1$, chosen below,

$$I_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \widetilde{V}_{ki}, \quad k = 1, \ldots, k^*+1. \tag{12}$$

Of course I_k is an unbiased estimator for $\mathbb{E}[\widetilde{V}_k]$, $k = 1, \ldots, k^* + 1$ and hence $I(k^*)_{\text{strat}}$ is an unbiased estimator for $\mathbb{E}[V]$. By independence of the I_k, $k = 1, \ldots, k^* + 1$, the variance of $I(k^*)_{\text{strat}}$ is given by

$$\mathrm{Var}(I(k^*)_{\text{strat}}) = \sum_{k=1}^{k^*+1} \mathrm{Var}(I_k) p_k^2$$

and an unbiased estimator of the variance is given by

$$\sigma^2(k^*)_{\text{strat}} = \sum_{k=1}^{k^*+1} \sigma_k^2 p_k^2 \tag{13}$$

where

$$\sigma_k^2 = \frac{1}{n_k(n_k - 1)} \sum_{i=1}^{n_k} (I_k - \widetilde{V}_{ki})^2. \tag{14}$$

In order to compute $I(k^*)_{\text{strat}}$, we need to know how to simulate $\widetilde{V}_k$, $k = 1, \ldots, k^* + 1$. This follows from minor modifications to Algorithm 3.1: To simulate $\widetilde{V}_{k^*+1}$, we replace step 1) by "Simulate N_T conditional on $N_T > k$" and leave the rest of Algorithm 3.1 unchanged, to simulate $\widetilde{V}_k$, $k = 1, \ldots, k^*$, we omit step 1), fix $N_T = k$ and proceeed with Algorithm 3.1. For background on simulation from conditional distributions, see e.g. [3], p. 57.
We now comment on the choice of n_k, $k = 1, \ldots, k^* + 1$ in (12): For simplicity, we make use of *proportional allocation*, e.g. [3], p.210, i.e. we set $n_k = \lfloor \mathbb{P}(N_T = k) n_{strat} \rfloor$, $k = 1, \ldots, k^*$ and $n_{k^*+1} = n_{strat} - \sum_{k=1}^{k^*} n_k$, so n_{strat} denotes the total number of function evaluations to be used for estimation. It also means, recalling our introductory statement, that we constrain the fraction of observations drawn from the set $\{N_T = k\}$ to be $\frac{n_k}{n_{strat}}$, $k = 1, \ldots, k^*$, and from the set $\{N_T > k^*\}$ to be $\frac{n_{k^*+1}}{n_{strat}}$. Though proportional allocation is a simple convention, we can still recover the following well-known results, showing that stratified sampling with proportional allocation achieves a variance reduction and that increasing k^* results in a decrease in the variance of the stratified estimator.

Corollary 3.1 *The following properties hold:*

- *Assume that* $n_k = \mathbb{P}(N_T = k)n$ *satisfies* $n_k \in \mathbb{Z}^+$, $k = 1, \ldots, k^*$. *Then* $\mathrm{Var}(I(k^*)_{\text{strat}}) < \mathrm{Var}(I_{MC})$.
- *Let* k_1^*, $k_2^* \in \mathbb{Z}^+$, $k_2^* > k_1^*$, *then* $\mathrm{Var}(I_{\text{strat}}(k_2^*)) \leq \mathrm{Var}(I_{\text{strat}}(k_1^*))$.

Proof. [3], p.217 and p. 220. ☐

4 Hybrid Approach to Pricing Lookback Options in the Kou Model

In this section, we show how to price lookback options in the Kou model by combining QMC and MC methods, a hybrid approach. We recall Remark 3.2, stating that the naive MC approach is not suitable for QMC methods, as it results in problems of unbounded dimensionality and the allocation of variables to dimensions is not fixed. However, considering formulation (11), it is clear that the $\mathbb{E}[\widetilde{V}_k]$, $k = 1, \dots, k^*$ are of fixed dimension and that it is possible to allocate a particular dimension to a particular random variable. Consequently, we will formulate $\mathbb{E}[\widetilde{V}_k]$, $k = 1, \dots, k^*$ as integration problems and apply QMC methods to them. Of course $\mathbb{E}[\widetilde{V}_{k^*+1}]$ is still of unbounded dimension and it is not possible to allocate a particular dimension to a particular random variable, so $\mathbb{E}[\widetilde{V}_{k^*+1}]$ cannot be estimated using QMC methods but MC methods will still be used. We now state the integration problem for $k = 1$, but the same reasoning holds for $k = 2, \dots, k^*$.

$$\begin{aligned}\mathbb{E}[\widetilde{V}_1] = \mathbb{E}[\widetilde{V}_1(\tau_1, Y_1, W_{\tau_1}, M_1, W_T, M_2, 1)] = \int_{u_1=0}^{1} \dots \int_{u_6=0}^{1} \widetilde{V}_1(F_{Beta}^{-1}(u_1, 1, 1), F_J^{-1}(u_2),\\ F_N^{-1}(u_3, 0, F_{Beta}^{-1}(u_1, 1, 1)), F_{M_1}^{-1}(u_4, u_1, u_3), F_N^{-1}(u_3, 0, F_{Beta}^{-1}(u_1, 1, 1))\\ + F_N^{-1}(u_5, 0, T - F_{Beta}^{-1}(u_1, 1, 1)), F_{M_2}^{-1}(u_6, u_1, u_2, u_3, u_5))du_6 \dots du_1\end{aligned}$$

where $F_{M_1}^{-1}(u_4, u_1, u_3) = \exp(a(u_4, u_1, u_3))$ and

$$a(u_4, u_1, u_3) = \frac{\ln(S_{1^-}(u_1, u_3)S_0) + \sqrt{(\ln(\frac{S_{1^-}(u_1,u_3)}{S_0}))^2 - 2\tau_1(u_1)\sigma^2 \ln(1 - u_4)}}{2}$$

where $S_{1^-}(u_1, u_3) = S_0 \exp(\mu F_{Beta}^{-1}(u_1, 1, 1) + \sigma F_N^{-1}(u_3, 0, F_{Beta}^{-1}(u_1, 1, 1)))$ and $\tau_1(u_1) = F_{Beta}^{-1}(u_1, 1, 1)$. The formula for $F_{M_2}^{-1}(u_6, u_1, u_2, u_3, u_5)$ can be derived using similar arguments.
Given a digital net in base b, $(u_i)_{i=1}^n$, $u_i \in [0, 1]^6$, and q independent random vectors $\triangle_j$, $j = 1, \dots, q$ uniformly distributed in $[0, 1]^6$, we randomize $(u_i)_{i=1}^n$ using a digital b-ary shift, e.g. [5], where $\oplus$ means that for each dimension, we perform the digit-wise addition modulo b.

$$\begin{aligned}\mathbb{E}[\widetilde{V}_1] \approx \frac{1}{q}\sum_{j=1}^{q} \frac{1}{n}\sum_{i=1}^{n} \widetilde{V}_1(F_{Beta}^{-1}(u_{i1} \oplus \triangle_{j1}, 1, 1), F_J^{-1}(u_{i2} \oplus \triangle_{j2}),\\ F_N^{-1}(u_{i3} \oplus \triangle_{j3}, 0, F_{Beta}^{-1}(u_{i1} \oplus \triangle_{j1}, 1, 1)), F_{M_1}^{-1}(u_{i4} \oplus \triangle_{j4}, u_{i1} \oplus \triangle_{j1}, u_{i3} \oplus \triangle_{j3}),\\ F_N^{-1}(u_{i3} \oplus \triangle_{j3}, 0, F_{Beta}^{-1}(u_{i1} \oplus \triangle_{j1}, 1, 1)) + F_N^{-1}(u_{i4} \oplus \triangle_{j4}, 0, T - F_{Beta}^{-1}(u_{i1} \oplus \triangle_{j1}, 1, 1))\\ F_{M_2}^{-1}(u_{i6} \oplus \triangle_{j6}, u_{i1} \oplus \triangle_{j1}, u_{i2} \oplus \triangle_{j2}, u_{i3} \oplus \triangle_{j3}, u_{i5} \oplus \triangle_{j5})).\end{aligned}$$

We hence get the following unbiased estimator for $\mathbb{E}[\widetilde{V}_1]$

$$\mathbb{E}[\widetilde{V}_1] \approx \frac{1}{q}\sum_{j=1}^{q}\frac{1}{n}\sum_{i=1}^{n}\widetilde{V}_1(u_i \oplus \triangle_j) = I_1.$$

Generalising this argument and using mutually independent $\triangle_j^k$, $j = 1, \ldots, q$, $k = 1, \ldots, k^*$ we obtain

$$I_k = \frac{1}{q}\sum_{j=1}^{q} I_{kj} = \frac{1}{q}\sum_{j=1}^{q}\frac{1}{n_k}\sum_{i=1}^{n_k}\widetilde{V}_k(u_i \oplus \triangle_j^k), k = 1, \ldots, k^*$$

which are unbiased estimators for $\mathbb{E}[\widetilde{V}_k]$ and we define the hybrid estimator

$$I_{HYB} = \sum_{k=0}^{k^*+1} I_k p_k.$$

As $\mathbb{E}[\widetilde{V}_{k^*+1}]$ is estimated using MC methods, I_{k^*+1} is defined as in (12). We now comment on the choice of the n_k, $k = 1, \ldots, k^*$. We fix the total number of function evaluations for the hybrid approach to be $q * n$ and again use proportional allocation to obtain $n_k = \lfloor p_k n \rfloor$, $k = 1, \ldots, k^*$. The case I_{k^*+1} is treated separately, as $\mathbb{E}[\widetilde{V}_{k^*+1}]$ is estimated via MC methods: We set n_{k^*+1}, the total number of function evaluations to be used for I_{k^*+1}, to be $n_{k^*+1} = nq - q\sum_{k=1}^{k^*} n_k$. It is easy to see that I_{HYB} is an unbiased estimator of $\mathbb{E}[V]$ and by the independence of the random variables $\triangle_j^k$, $j = 1, \ldots, q$, $k = 1, \ldots, k^*$, it follows that the variance of I_{HYP} is given by

$$\mathrm{Var}(I_{HYB}) = \sum_{k=1}^{k^*+1} \mathrm{Var}(I_k) p_k^2$$

and an unbiased estimator of $\mathrm{Var}(I_{HYB})$ is

$$\sigma_{HYB}^2 = \sum_{k=1}^{k^*}\left(\frac{1}{q(q-1)}\sum_{j=1}^{q}(I_{kj} - I_k)^2\right) p_k^2 + \sigma_{k^*+1}^2 p_{k+1}^2 \tag{15}$$

where $\sigma_{k^*+1}^2$ is given by (14) with $n_{k^*+1} = nq - q\sum_{k=1}^{k^*} n_k$. Computing I_{HYB}, we need to be able to obtain $\widetilde{V}_k(u \oplus \triangle)$, $k = 1, \ldots, k^*$, for a QMC point $u \in [0,1]^{4k+2}$ and $\triangle$ uniformly distributed in $[0,1]^{4k+2}$, which is shown in the following algorithm:

Algorithm 4.1 *Algorithm to obtain* $\widetilde{V}_k(u \oplus \triangle)$*:*

```
1. for $l = 1 : k$
    - $\tau_l = \tau_{l-1} + (T - \tau_{l-1}) F^{-1}_{Beta}(u_{4(l-1)+1} \oplus \triangle_{4(l-1)+1}, 1, k - (l-1))$
    - $Y_l = F^{-1}(u_{4(l-1)+2} \oplus \triangle_{4(l-1)+2})$
    - $W_{\tau_l} = W_{\tau_{l-1}} + F^{-1}_N(u_{4(l-1)+3} \oplus \triangle_{4(l-1)+3}, 0, \tau_l)$
    - $M_l = F^{-1}_{M_l}(u_{4(l-1)+4} \oplus \triangle_{4(l-1)+4})$
2. end
3. $W_T = W_{\tau_k} + F^{-1}_N(u_{4(k-1)+5} \oplus \triangle_{4(k-1)+5}, 0, T - \tau_k)$
4. $M_{k+1} = F^{-1}_{M_{k+1}}(u_{4(k-1)+6} \oplus \triangle_{4(k-1)+6})$
5. Set $\widetilde{V}_k(u \oplus \triangle) = \max(B, M_1, \ldots, M_{k+1})$
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For I_{k^*+1}, which is estimated using MC methods, we use the same reasoning as in Section 3.2 to simulate $\widetilde{V}_{k^*+1}$.

Remark 4.1 It is well-known, e.g. [1], that QMC methods are particularly effective if the problem under consideration has low effective dimension, e.g. [1]. Examining techniques to reduce the effective dimension of $\mathbb{E}[\widetilde{V}_k]$, $k = 1, \ldots, k^*$, is an intersting area of future research.

5 Numerical Results and Conclusion

In this section, we compare the approaches introduced in Sections 3.1, 3.2 and 4. The following set of parameters is taken from [4]. We set $B = 110$, $S_0 = 100$, $T = 1$ and $r = 0.05$, $\lambda = 3$, $\sigma = 0.2$, $p = 0.3$, $\eta^+ = 50$ and $\eta^- = 25$ in (2). For the stratification and the hybrid approach, we need to choose k^*. We choose the following two values, $k_1^* = 4$ and $k_2^* = 8$. The results presented in Table 1 were achieved as follows: For the hybrid approach (HYB), we vary n as shown in the table and set $q = 30$ in (15). To ensure the same numbers of function evaluations are used for all approaches, we set $n_{MC} = n_{strat} = q * n$ for the naive Monte Carlo approach (MC) and the stratification approach (STRAT). We choose Sobol points, a digital net in base 2, as our QMC point set. Standard errors were computed using (7), (13) and (15), results for $k_2 = 8$ are given in brackets. Table 1 shows that the hybrid approach clearly outperforms the competing ones and for this approach, we see that increasing k^* results in a decrease in standard errors. The stratification approach marginally outperforms the Monte Carlo approach. Based on Corollary 3.1, increasing k^* should result in a decrease in standard errors

	$n = 256$	$n = 1024$	$n = 4096$	$n = 16384$
MC	0.1713	0.0868	0.0433	0.0217
STRAT	0.1632 (0.1678)	0.0821 (0.0834)	0.0412 (0.0419)	0.0206 (0.0209)
HYB	0.0970 (0.0916)	0.0402 (0.0346)	0.0169 (0.0106)	0.0083 (0.0047)

Table 1: Standard errors for the MC point sets based on $q * n$ points and $k_1^* = 4$ ($k_2^* = 8$) and QMC point set based on n points, q random shifts and $k_1^* = 4$ ($k_2^* = 8$)

for the stratification approach. However, the estimates for $\mathrm{Var}(I_k)$, based on (14) are not accurate enough, one would need larger values of n_k to notice the improvement.

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