Physical Combinatorics:

Fermionic Paths and Virasoro Algebra

Giovanni Feverati

Paul A. Pearce

Mathematics & Statistics

University of Melbourne

Part I: (PAP)

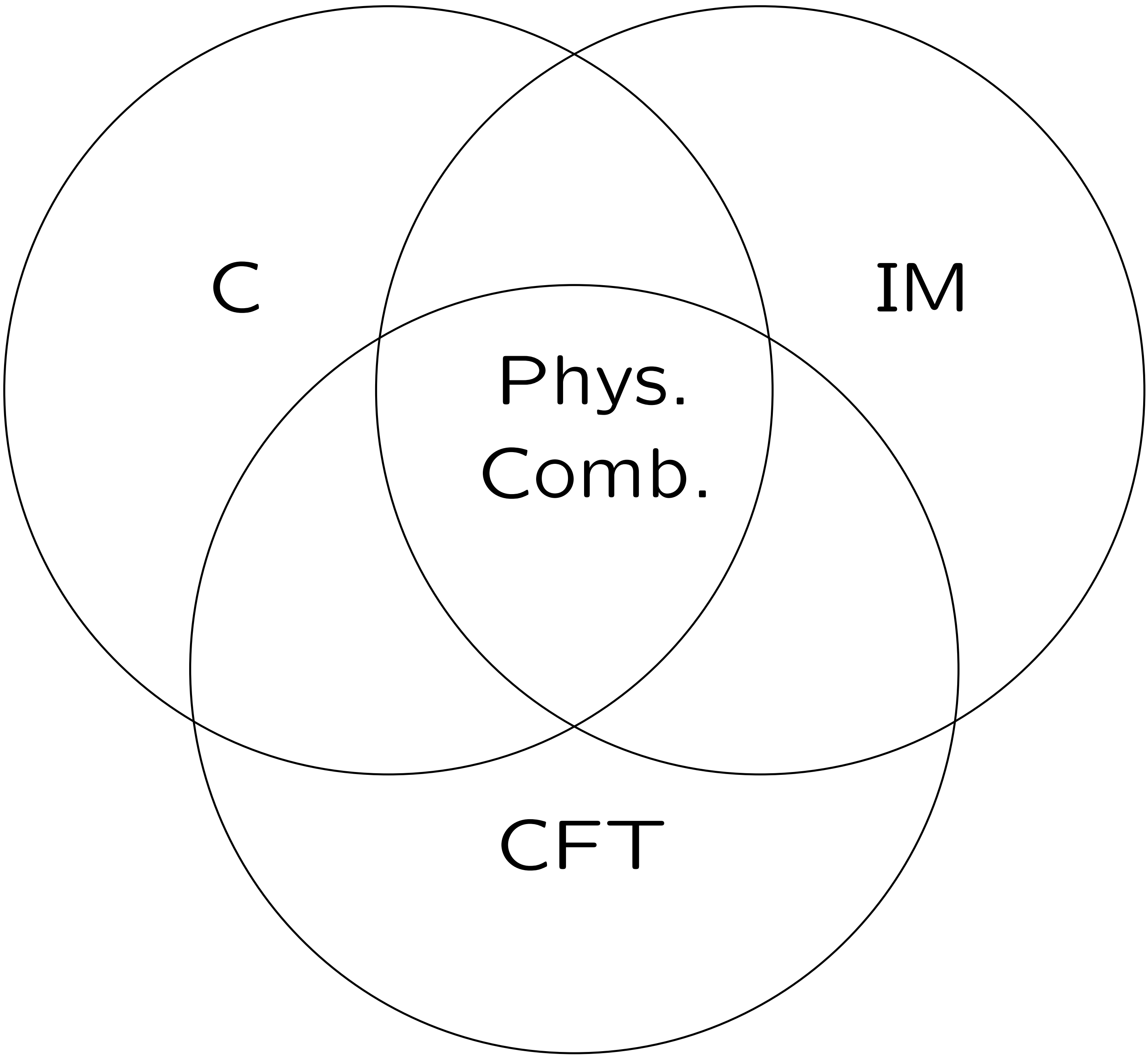
CFT: Virasoro algebra and states; Integrable RSOS models: classification of eigenstates of double row transfer matrices; Fermionic algebra: fermionic paths, finite Verma modules, finitized characters; Bijection.

Part II: (GF)

Virasoro algorithm: L_1 basis, Virasoro matrices; Primary field algorithm: matrices for critical Ising, tricritical Ising, Yang-Lee and 3-state Potts.

Physical Combinatorics

$$\text{C} \cap \text{IM} \cap \text{CFT} = \{\text{Physical Combinatorics}\}$$



Conformal Field Theory

Minimal Models: $\mathcal{M}(p', p)$ (Unitary if $p = p' + 1$)

$$
\begin{aligned}
\mathcal{M}(3,4) &= \text{critical Ising model } (c = 1/2) \\
\mathcal{M}(4,5) &= \text{tricritical Ising model } (c = 7/10) \\
\mathcal{M}(2,5) &= \text{Yang-Lee theory } (c = -22/5)
\end{aligned}
$$

Central charges: (Belavin-Polyakov-Zamolodchikov 1984)

$$c = 1 - \frac{6\,(p-p')^2}{p\,p'}, \quad p' < p, \quad p, p' \text{ coprime}$$

Conformal weights: $r = 1, 2, \ldots, p-1; \quad s = 1, 2, \ldots, p'-1$

$$h = h_{r,s} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'}$$

Virasoro characters (bosonic):

$$
\begin{aligned}
\chi_h(q) &= \frac{q^{-c/24+h}}{(q)_\infty} \sum_{k=-\infty}^{\infty} \left(q^{k(kpp'+rp-sp')} - q^{(kp+s)(kp'+r)}\right) \\
&= q^{-c/24} \sum_{\ell \in \mathbb{N}+h} d_\ell\, q^\ell, \qquad d_\ell \in \mathbb{N}
\end{aligned}
$$

$$(q)_n = \prod_{k=1}^{n} (1 - q^k)$$

Primary fields: $\phi(z) = \phi^h(z)$

$\mathbb{Z}_k$ Parafermions: (Zamolodchikov-Fateev 1985)

$$\mathbb{Z}_3 = \text{3-state Potts } (c = 4/5)$$

Central charges:

$$c = \frac{2(k-1)}{k+2}$$

Virasoro Algebra and States

Virasoro Algebra:

$$\text{Vir} = \langle L_n, n \in \mathbb{Z} \rangle, \qquad L_{-n} = L_n^T$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\,\delta_{n,-m}$$

Energy-momentum tensor:

$$T(z) = \sum_{n \in \mathbb{Z}} L_n \, z^{-n-2}$$

Primary States:

Vacuum $|0\rangle$ and primary states $|h\rangle$:

$$L_0|h\rangle = h|h\rangle; \qquad L_n|0\rangle = 0, \quad n \geq -1; \qquad L_n|h\rangle = 0, \quad n > 0$$

Primary states $\leftrightarrow$ primary fields:

$$\lim_{z \to 0} \phi^h(z)|0\rangle = |h\rangle$$

Hilbert Space: $\mathcal{H} = \oplus_h \mathcal{V}_h$

Verma module:

$$\mathcal{V}_h = \oplus_\ell \mathcal{V}_{h,\ell}, \qquad \ell = \sum_{i=1}^{j} n_j + h = \text{level}$$

$$L_{-n_j} L_{-n_{j-1}} \ldots L_{-n_1}|h\rangle, \qquad n_j \geq n_{j-1} \geq \cdots \geq n_1 \geq 1$$

Virasoro character:

$$\chi_h(q) = \text{Tr}_{\mathcal{V}_h} q^{L_0 - c/24} = q^{-c/24} \sum_{\ell \in \mathbb{N}+h} d_\ell \, q^\ell$$

$$d_\ell = d_\ell^h = \dim \mathcal{V}_{h,\ell} = \text{degeneracy}, \qquad q = \text{modular parameter}$$

Verma Module

1	$\|0\rangle$			
0	$-$			
q^2	$L_{-2}\|0\rangle$			
q^3	$L_{-3}\|0\rangle$			
$2q^4$	$L_{-4}\|0\rangle$	$L^2_{-2}\|0\rangle$		
$2q^5$	$L_{-5}\|0\rangle$	$L_{-3}L_{-2}\|0\rangle$		
$4q^6$	$L_{-6}\|0\rangle$	$L_{-4}L_{-2}\|0\rangle$	$L^2_{-3}\|0\rangle$	$L^3_{-2}\|0\rangle$
$4q^7$	$L_{-7}\|0\rangle$	$L_{-5}L_{-2}\|0\rangle$	$L_{-4}L_{-3}\|0\rangle$	$L_{-3}L^2_{-2}\|0\rangle$
$7q^8$	$L_{-8}\|0\rangle$	$L_{-6}L_{-2}\|0\rangle$	$L_{-5}L_{-3}\|0\rangle$	$L^2_{-4}\|0\rangle$
	$L_{-4}L^2_{-2}\|0\rangle$	$L^2_{-3}L_{-2}\|0\rangle$	$L^4_{-2}\|0\rangle$	
$8q^9$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$12q^{10}$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Generic Verma module $\mathcal{V}_0$ of A_∞ in the vacuum $h = 0$ sector.

$$q^{c/24}\chi_0(q) = \prod_{n=2}^{\infty}(1-q^n)^{-1}$$
$$= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 \cdots$$

- Virasoro states are not orthonormal — Gram matrix of inner products.

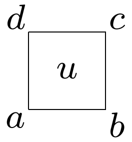
- Virasoro states are not linearly independent — null vectors.

- Basis of linearly independent Virasoro states not known even for the Ising model.

Integrable RSOS Models

Critical A_L RSOS Face Weights: (ABF84/FB85)

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) =$$



$$= \frac{\sin(\lambda-u)}{\sin\lambda}\,\delta_{a,c} + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_a S_c}{S_b S_d}}\,\delta_{b,d}$$

$$a,b,c,d \in A_L, \qquad \lambda = \frac{(p-p')\pi}{p}, \qquad S_a = \sin a\lambda, \qquad p = L+1$$

Continuum Scaling Limit:

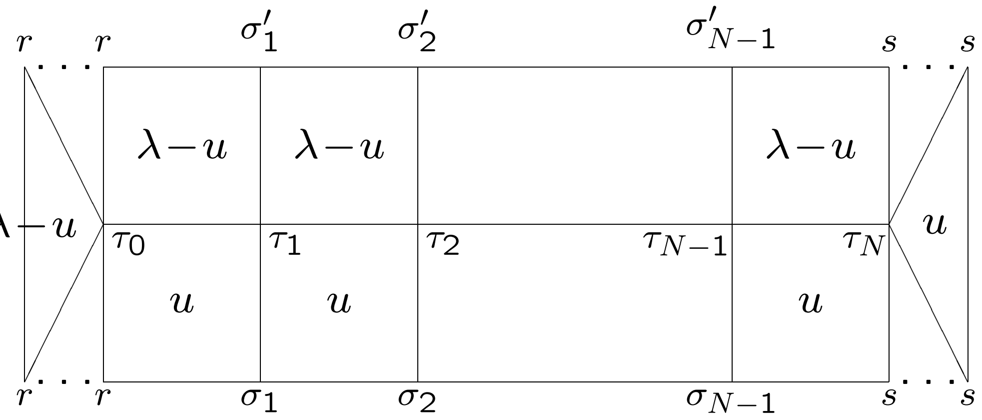
$$\mathcal{M}(p',p): \quad 0 < u < \lambda$$

$$\mathbb{Z}_{L-1}: \quad \lambda - \pi/2 < u < 0, \quad p' = L$$

Double Row Transfer Matrices $\boldsymbol{D}(u)$: (BPO96)

$$\text{YBE} + \text{BYBE} \;\Rightarrow\; [\boldsymbol{D}(u), \boldsymbol{D}(v)] = 0$$

$$\boldsymbol{D}(u)_{\sigma,\sigma'} = \sum_{\tau_0,\ldots,\tau_N}$$



• Integrable boundary conditions exist for each (r,s) conformal boundary condition. (BP01)

• Orthonormal basis of eigenstates independent of u.

$$Z(q) = \operatorname{Tr}\frac{\boldsymbol{D}(u)}{D_{max}(u)} = \sum_E q^E = q^{\frac{c}{24}}\,\chi^{(N)}_{r,s}(q)$$

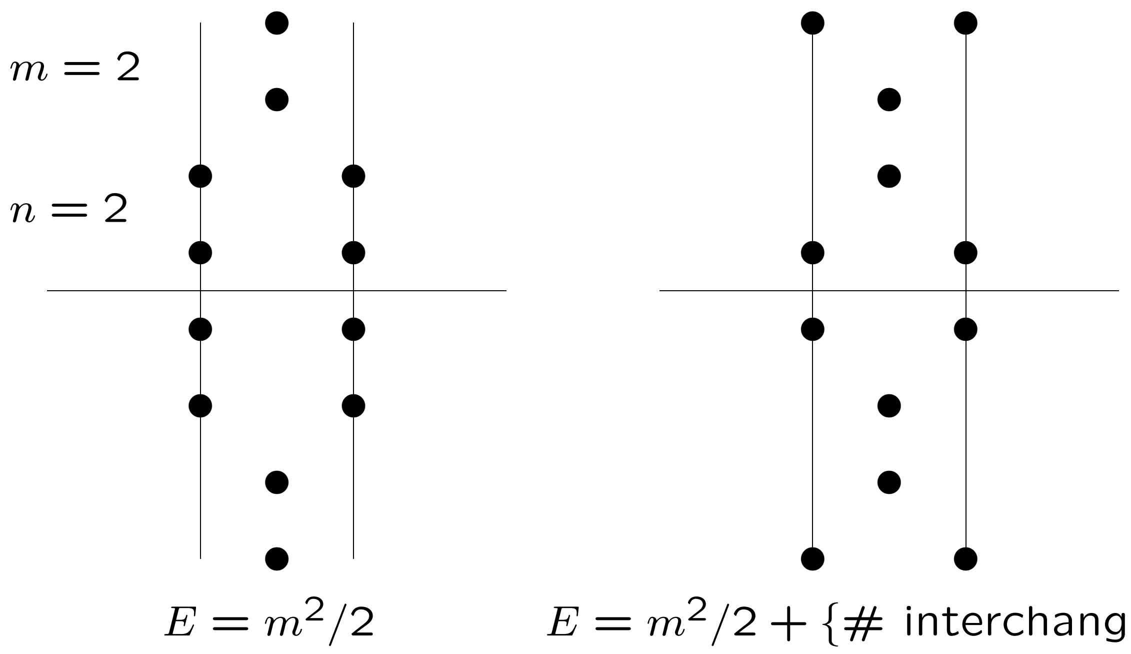
$$q = \exp\!\left(-\frac{2\pi\sin(L+1)u}{N}\right) = \text{modular parameter}$$

Critical Ising Model/Free Fermion $(h = 0)$

(m, n) System: $m, N \in 2\mathbb{N}$ (OPW96)

Analyticity strip: $-\frac{\lambda}{2} < \mathrm{Re}(u) < \frac{3\lambda}{2}$, $\lambda = \pi/4$

$$m + n = \frac{N}{2}, \qquad m = \#\ \text{1-strings}, \qquad n = \#\ \text{2-strings}$$



Finitized Fermionic Character:

$$\chi_0^{(N)}(q) = q^{-c/24} \sum_E q^E = q^{-c/24} \sum_{m \in 2\mathbb{N}} q^{m^2/2} \begin{bmatrix} N/2 \\ m \end{bmatrix}_q$$

$$\to q^{-c/24} \sum_{m \in 2\mathbb{N}} \frac{q^{m^2/2}}{(q)_m} = \chi_0(q), \quad \text{as } N \to \infty$$

q-binomials:

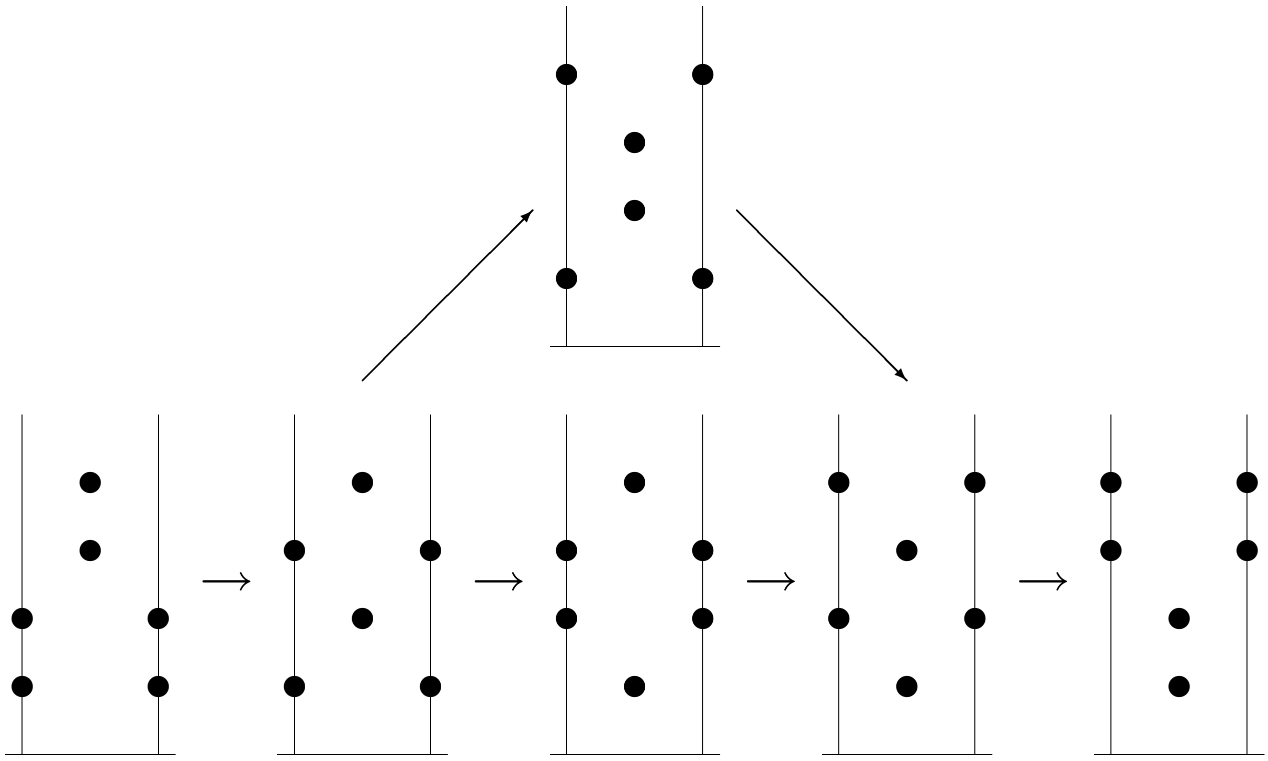
$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{cases} \frac{(q)_{m+n}}{(q)_m (q)_n}, & m, n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

q-Binomials

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{cases} \dfrac{(q)_{m+n}}{(q)_m\,(q)_n}, & m,n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Example:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$



$$1 \quad + \quad q \quad + \quad 2q^2 \quad + \quad q^3 \quad + \quad q^4$$

Tricritical Ising Model $(h = 0)$

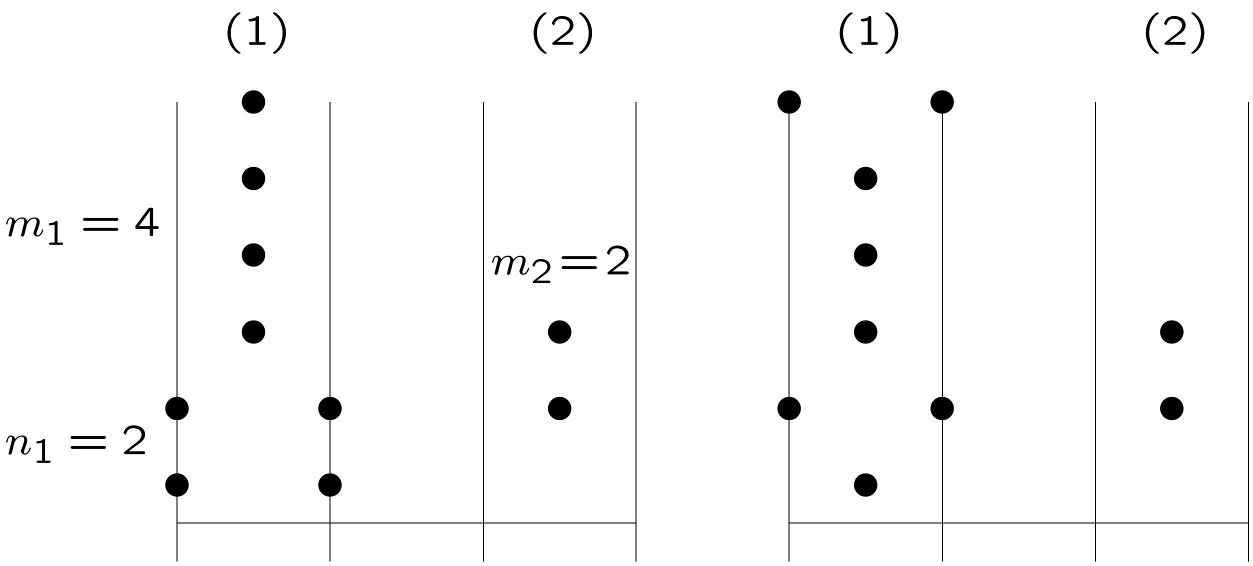
(m,n) System: $m_1, m_2, N \in 2\mathbb{N}$ (OPW97)

Analyticity strips: $\lambda = \pi/5$

$$(1) \quad -\frac{\lambda}{2} < \mathrm{Re}(u) < \frac{3\lambda}{2}, \qquad (2) \quad 2\lambda < \mathrm{Re}(u) < 4\lambda$$

$$\begin{aligned} m_i &= \{\text{number of 1-strings in strip } i = 1,2\} \\ n_i &= \{\text{number of 2-strings in strip } i = 1,2\} \end{aligned}$$

$$m_1 + n_1 = \frac{N + m_2}{2}, \qquad m_2 + n_2 = \frac{m_1}{2}$$



$$E = (m_1^2 - m_1 m_2 + m_2^2)/2 \qquad E = (m_1^2 - m_1 m_2 + m_2^2)/2 + \{\#\ \text{interchanges}\}$$

Finitized Fermionic Character:

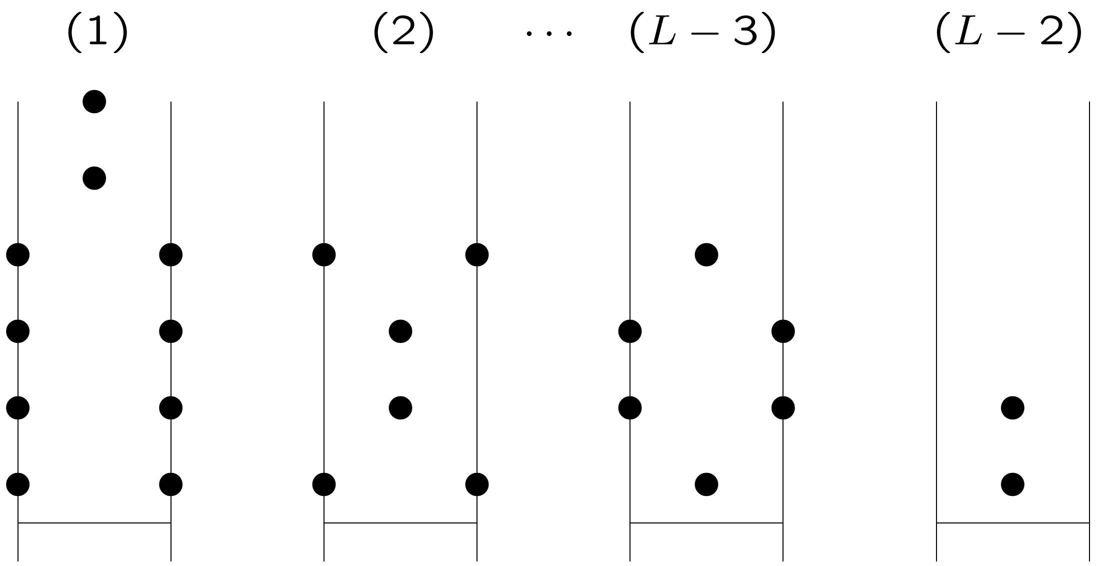
$$\begin{aligned} \chi_0^{(N)}(q) &= q^{-c/24} \sum_{m_1, m_2 \in 2\mathbb{N}} q^{(m_1^2 - m_1 m_2 + m_2^2)/2} \begin{bmatrix} (N+m_2)/2 \\ m_1 \end{bmatrix}_q \begin{bmatrix} m_1/2 \\ m_2 \end{bmatrix}_q \\ &\to q^{-c/24} \sum_{m_1, m_2 \in 2\mathbb{N}} \frac{q^{(m_1^2 - m_1 m_2 + m_2^2)/2}}{(q)_{m_1}} \begin{bmatrix} m_1/2 \\ m_2 \end{bmatrix}_q = \chi_0(q) \end{aligned}$$

A_L Unitary Minimal Model $(h = 0)$

$(\boldsymbol{m},\boldsymbol{n})$ System: $\quad m_i, N \in 2\mathbb{N} \quad$ (Melzer-Berkovich 1994)

$$\boldsymbol{m}+\boldsymbol{n} = \frac{1}{2}(N\boldsymbol{e}_1 + A\boldsymbol{m}), \qquad A = \{A_{L-2} \text{ adjacency matrix}\}$$

$$m_i + n_i = \frac{1}{2}(N\delta_{i,1} + m_{i-1} + m_{i+1}), \qquad i = 1, 2, \ldots, L-2$$



$$E = \tfrac{1}{4}\boldsymbol{m}C\boldsymbol{m} + \{\#\text{ interchanges}\}$$

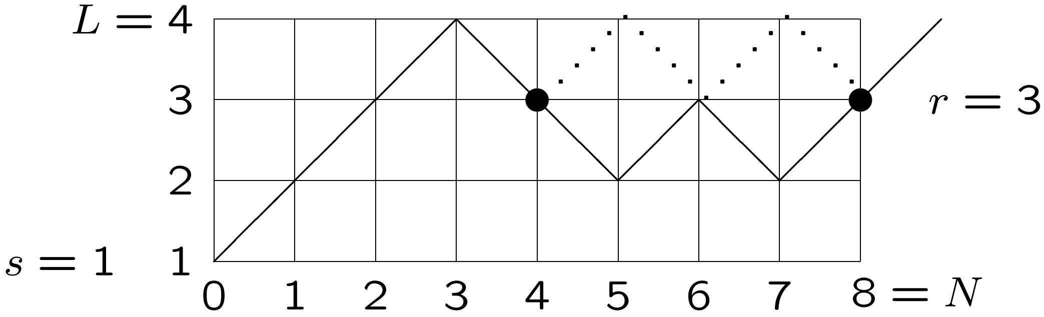
$$C = 2\boldsymbol{I} - A = \{\text{Cartan matrix}\}$$

Finitized Fermionic Characters:

$$\begin{aligned}\chi_0^{(N)}(q) &= q^{-c/24} \sum_{(m,n)} q^{\frac{1}{4}\boldsymbol{m}C\boldsymbol{m}} \prod_{i=1}^{L-2} \begin{bmatrix} m_i+n_i \\ m_i \end{bmatrix}_q \\ &\to q^{-c/24} \sum_{m_i \in 2\mathbb{N}} \frac{q^{\frac{1}{4}\boldsymbol{m}C\boldsymbol{m}}}{(q)_{m_1}} \prod_{i=2}^{L-2} \begin{bmatrix} m_i+n_i \\ m_i \end{bmatrix}_q = \chi_0(q)\end{aligned}$$

Fermionic Paths

RSOS Paths in (r,s) Sector: $\sigma_h = |h\rangle, \quad h = h_{r,s}$



(r,s) : $\sigma_0 = s, \qquad \sigma_N = r, \quad \sigma_{N+1} = r+1$

1-d Configurational Sums: (CTMs: Baxter80, ABF84)

$$H(\sigma_{j-1},\sigma_j,\sigma_{j+1}) = \tfrac{1}{2}|\sigma_{j-1}-\sigma_{j+1}| = \begin{cases} 0, & \sigma_{j+1}=\sigma_{j-1} \\ 1, & \sigma_{j+1}-\sigma_{j-1}=\pm 2 \end{cases}$$

$$E(\sigma) = \tfrac{1}{2}\sum_{j=1}^{N} j\, H(\sigma_{j-1},\sigma_j,\sigma_{j+1})$$

$$\chi_h^{(N)}(q) = q^{-c/24+h}\sum_{\{\sigma\}} q^{E(\sigma)-E(\sigma_h)} \to \chi_h(q) \qquad \text{as } N\to\infty$$

Fermion Operators: $b_{-\frac{j}{2}} = b^{\dagger}_{\frac{j}{2}}$

$b_{-\frac{j}{2}} = \begin{cases} \text{creates straight segment at } j \\ \text{annihilates corner at } j \end{cases}$

$b_{\frac{j}{2}} = \begin{cases} \text{creates corner at } j \\ \text{annihilates straight segment at } j \end{cases}$

$b_{-\frac{j}{2}} b_{\frac{j}{2}} =$ number operator

Fermionic State: $|\sigma\rangle = b_{-4}\, b_{-2}|h\rangle$

Fermionic Algebra

$$\mathcal{F} = \oplus_h \mathcal{F}_h, \qquad \mathcal{F}_h = \langle b_j = b_j^h : \ j \in \mathbb{Z}/2, \ j \neq 0 \rangle$$

$$\textbf{CAR}: \qquad \{b_j, b_k\} = \delta_{j,-k}, \qquad b_{-j} = b_j^\dagger = b_j^T$$

Fermionic Matrices

The three 8×8 fermionic matrices for $N = 3$ are

$$b_{-\frac{1}{2}} = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix} = b_{\frac{1}{2}}^T$$

$$b_{-1} = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&-1&0&0&0&0&0&0 \end{pmatrix} = b_1^T$$

$$b_{-\frac{3}{2}} = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0 \end{pmatrix} = b_{\frac{3}{2}}^T$$

The algebra $\mathcal{F}^{(3)}$ has dimension 8. For the vacuum sector $(r,s) = (1,1)$, only two states are physical:

$$\begin{aligned} |0\rangle &= (1,0,0,0,0,0,0,0) \\ b_{-\frac{3}{2}} b_{-\frac{1}{2}} |0\rangle &= (0,1,0,0,0,0,0,0) \end{aligned}$$

Fermionic Verma Module

Generic Verma module $\mathcal{V}_0$ of A_∞ in the vacuum $h = 0$ sector:

1	$\|0\rangle$			
0	$-$			
q^2	$b_{-\frac{3}{2}}b_{-\frac{1}{2}}\|0\rangle$			
q^3	$b_{-\frac{5}{2}}b_{-\frac{1}{2}}\|0\rangle$			
$2q^4$	$b_{-\frac{7}{2}}b_{-\frac{1}{2}}\|0\rangle$	$b_{-\frac{5}{2}}b_{-\frac{3}{2}}\|0\rangle$		
$2q^5$	$b_{-\frac{9}{2}}b_{-\frac{1}{2}}\|0\rangle$	$b_{-\frac{7}{2}}b_{-\frac{3}{2}}\|0\rangle$		
$4q^6$	$b_{-\frac{11}{2}}b_{-\frac{1}{2}}\|0\rangle$	$b_{-\frac{9}{2}}b_{-\frac{3}{2}}\|0\rangle$	$b_{-\frac{7}{2}}b_{-\frac{5}{2}}\|0\rangle$	$b_{-\frac{5}{2}}b_{-2}b_{-1}b_{-\frac{1}{2}}\|0\rangle$
$4q^7$	$b_{-\frac{13}{2}}b_{-\frac{1}{2}}\|0\rangle$	$b_{-\frac{11}{2}}b_{-\frac{3}{2}}\|0\rangle$	$b_{-\frac{9}{2}}b_{-\frac{5}{2}}\|0\rangle$	$b_{-\frac{7}{2}}b_{-2}b_{-1}b_{-\frac{1}{2}}\|0\rangle$
$7q^8$	$b_{-\frac{15}{2}}b_{-\frac{1}{2}}\|0\rangle$	$b_{-\frac{13}{2}}b_{-\frac{3}{2}}\|0\rangle$	$b_{-\frac{11}{2}}b_{-\frac{5}{2}}\|0\rangle$	$b_{-\frac{9}{2}}b_{-\frac{7}{2}}\|0\rangle$
	$b_{-\frac{7}{2}}b_{-\frac{5}{2}}b_{-\frac{3}{2}}b_{-\frac{1}{2}}\|0\rangle$	$b_{-\frac{9}{2}}b_{-2}b_{-1}b_{-\frac{1}{2}}\|0\rangle$	$b_{-\frac{7}{2}}b_{-3}b_{-1}b_{-\frac{1}{2}}\|0\rangle$	
$8q^9$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Virasoro character:

$$q^{c/24}\chi_0(q) = \prod_{n=2}^{\infty}(1-q^n)^{-1}$$
$$= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + \cdots$$

- The fermionic states are orthonormal.

- For finite L some states are unphysical (null) — these are automatically incorporated in the basis.

Hamiltonian and Characters

Fermionic Hamiltonian: (r,s) sector

$$L_0 = \mathcal{P}_h\Big(\sum_{j=1}^{\infty} \frac{j}{2}\, b_{-\frac{j}{2}} b_{\frac{j}{2}} - E(\sigma_h) + h\Big)\mathcal{P}_h, \qquad h = h_{r,s}$$

$\mathcal{P}_h$ is projector onto physical states:

$$\mathcal{P}_h = \sum_{|\sigma\rangle} |\sigma\rangle\langle\sigma|$$

Matrix elements of physical operator:

$$\phi = \mathcal{P}_h\, \phi\, \mathcal{P}_h = \sum_{|\sigma\rangle,|\sigma'\rangle} \langle\sigma|\phi|\sigma'\rangle\ |\sigma\rangle\langle\sigma'|$$

$$|0\rangle\langle 0| = \prod_{j=1}^{\infty} (1 - b_{-\frac{j}{2}} b_{\frac{j}{2}}), \quad \text{etc}$$

L_0 Eigenstates and Characters:

$$L_0|\sigma\rangle = L_0\, b_{-\frac{j_1}{2}} b_{-\frac{j_2}{2}} \ldots b_{-\frac{j_n}{2}}|h\rangle = \Big(E(\sigma) - E(\sigma_h) + h\Big) b_{-\frac{j_1}{2}} b_{-\frac{j_2}{2}} \ldots b_{-\frac{j_n}{2}}|h\rangle$$

$$E(\sigma) = \tfrac{1}{2} \sum_{j=1}^{N} j\, H(\sigma_{j-1}, \sigma_j, \sigma_{j+1}) = \sum_{k=1}^{n} \frac{j_k}{2}$$

$$j_k = \{\text{positions of straight segments}\}$$

Virasoro character:

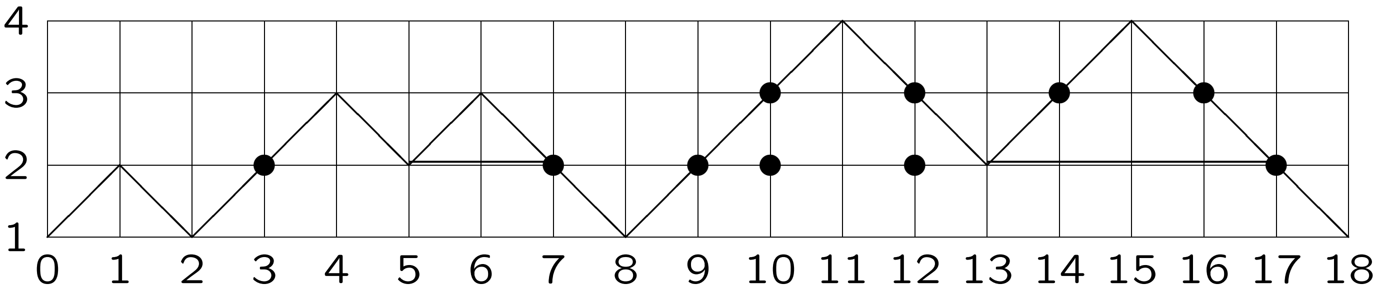
$$\chi_h(q) = q^{-c/24}\, \mathrm{Tr}_{\mathcal{V}_h} q^{L_0}$$

Finitized character: $(b_{\frac{j}{2}} = 0 \text{ for } |j| > N)$

$$\chi_h^{(N)}(q) = q^{-c/24}\, \mathrm{Tr}_{\mathcal{V}_h^{(N)}}\, q^{L_0}$$

Fermion Particles

RSOS path = complex of overlapping particles: $(A_4, N = 18)$



Particles: (Warnaar 1995)

- $L - 1$ types of particles:

$$n_i = \{\# \text{ particles of type } i = 1, 2, 3, \ldots, L-1\}$$

- Particle type i: peak i units above the baseline, width $2i$.
- Particles of type 1, 2, 1, 3, 2 at $j = 1, 4, 6, 11, 15$.
- Particle content: $n_1 = 2, \quad n_2 = 2, \quad n_3 = 1$
- n_{L-1} determined by: $\sum_{i=1}^{L-1} 2i\, n_i = N$

Dual Particles:

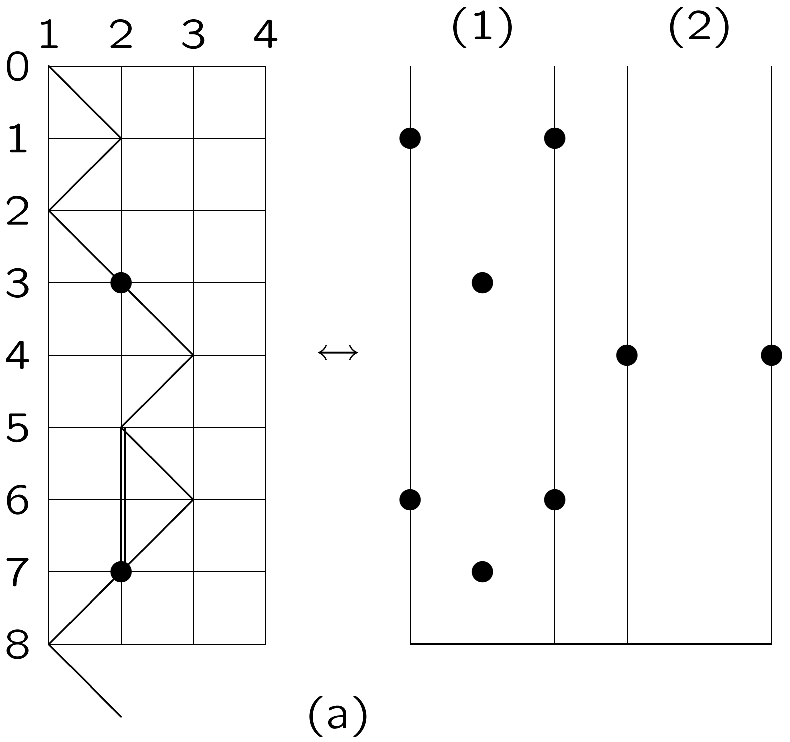
- $L - 2$ types of dual particles:
- Type i: straight segment i units above the baseline.

$$p_i = \{\# \text{ dual particles of type } i = 1, 2, \ldots, L-2\}$$
$$m_i = \sum_{i' \geq i} p_{i'}$$

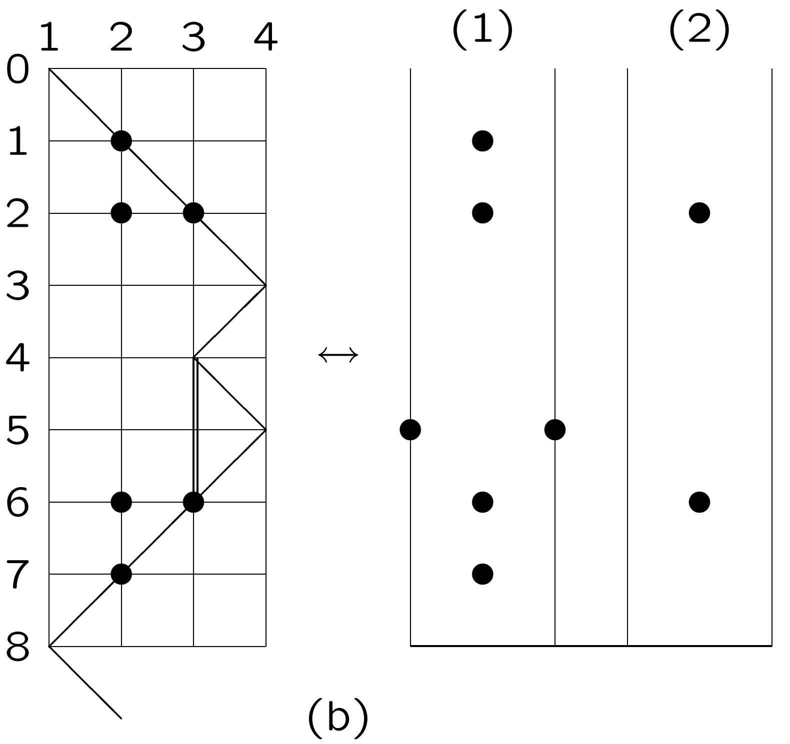
- Dual particles of type 1, 1, 1, 2, 2, 1, 1, 1
  at $j =$ 3, 7, 9, 10, 12, 14, 16, 17.
- Dual particle content:

$$p_1 = 6, \qquad p_2 = 2$$
$$m_1 = 8, \qquad m_2 = 2$$

Energy-Preserving Bijection
RSOS Paths ↔ Strings



(a)



(b)

String (particle) content:

(a) $m_1 = 2$, $n_1 = 2$, $m_2 = 0$, $n_2 = 1$, $n_3 = 0$

(b) $m_1 = 4$, $n_1 = 1$, $m_2 = 2$, $n_2 = 0$, $n_3 = 1$

Summary (Part 1)

• Eigenstates of double row transfer matrices (RTMs) are labelled by RSOS (fermionic) paths, that is, there exists an energy-preserving bijection. This is general for A_L models in all (r, s) sectors.

• Verma modules of orthonormal states are constructed by the action of a fermionic algebra on primary states. The physical states are in 1-1 correspondence with allowed RSOS paths.

• Physical quantities (operators, fields) are obtained by projecting onto the physical space of fermion states (generalizing the free fermion Ising case).

• A consistent finitization is obtained by setting fermion modes $b_{\frac{j}{2}} = 0$ for $|j| > N$.

• Spectrum generating functions of double RTMs are finitized fermionic Virasoro characters given by 1-d RSOS configuration sums with q the modular parameter.