RIEMANN MAPPING THEOREM IN $\mathbb{R}^3$

D H Hamilton

The Riemann Mapping Theorem, a special case of Hilbert's XXII problem, is equivalent to mapping a planar domain to the unit disk by a quasiconformal (QC) mapping; see Ahlfors and Bers 1960. These QC mappings generalize to higher dimensions (where conformal mappings are trivial). The main problem was to characterize the QC images of the unit ball, i.e., QC balls. In his reviews Ahlfors emphasized the related question of characterizing the images of the unit sphere under QC mappings of space, i.e., quasispheres. In their plenary talks at the ICM, both Ahlfors and Gehring reiterated these questions.

Our solution comes from reflection: a sense-reversing idempotent homeomorphism of $\widehat{\mathbb{R}}^3$ onto itself. Any reflection F of the sphere $\widehat{\mathbb{R}}^3$ has a set $\mathbf{T}$ of fixed points forming a topological sphere. Bing showed the existence of a wild reflection, i.e. the complementary domains are not simply connected. He asked for an explicit example which we construct (and show a picture of). On the other hand smooth reflections are tame so Sullivan, Heinonen, Semmes etc asked if there are wild QC reflections. Nevertheless, we prove:

The complementary domains of a QC reflection are simply connected.

A uniform sphere is a topological sphere whose blowups only have topolological spheres as limits. So we have

$\mathbf{T}$ *is the fixed set of a QC reflection iff it is a uniform sphere.*

We show how this solves Ahflors' problem of characterising quasispheres. The QC Riemann Mapping Theorem is a one-sided version of the result for quasispheres. We discuss the general ideas, as well as illustrating with pictures and applications.