POSSIBLE HONOURS PROJECTS, 2002

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1. Nonlinear heat equation.

The heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \ x \in \mathbb{R}$$

is well understood. Given reasonable initial data $u(0, x)$, the solution exists for all time and is very well-behaved. If a nonlinear term is added,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^p,$$

then much more interesting things can happen. For example, for some values of p, the solution may 'blow up' (ie, become infinite) at a finite time. The way in which this happens can be analysed, and the asymptotics of the solution near the blow up time specified quite precisely.

This project would involve learning about the linear and nonlinear heat equations, understanding the literature about blowup, and analysing some interesting examples in detail — possibly involving new forms of blow up that have yet been fully described.

Reference: Herrero, M. A.; Velzquez, J. J. L. Some results on blow up for semilinear parabolic problems. Degenerate diffusions (Minneapolis, MN, 1991), 105–125, IMA Vol. Math. Appl., 47, Springer, New York, 1993.

2. Eigenfunctions and concentration phenomena.

Suppose a domain Ω in $\mathbb{R}^2$ with smooth boundary is given. A *Dirichlet* or *Neumann* eigenfunction on Ω is a function on $\overline{\Omega}$, the closure of Ω, that satisfies the equation $-\Delta u = \lambda u$ inside the domain and either the Dirichlet boundary condition $u = 0$, or the Neumann boundary condition, $\partial_\nu u = 0$, at the boundary. Here $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian, and ν is the unit normal vector at the boundary. It turns out that there is a discrete set of eigenfunctions λ, which can be arranged in a sequence $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to \infty$ for which there is a nontrivial solution to the eigenfunction equation, and corresponding eigenfunctions u_1, u_2, ... These eigenfunctions have some pleasant properties; for example, they can be chosen to form on orthonormal basis of the Hilbert space $L^2(\Omega)$.

More generally, one can look at curved domains, such as a subdomain of the sphere.

This project is about understanding the relationship between the boundary and interior behaviour of eigenfunctions. For example, if the L^2 norm of the u_j is normalised to 1, how big can u_j be at the boundary? (This only makes sense for the Neumann boundary condition, since for Dirichlet the eigenfunction vanishes at the boundary by definition.) More precisely, are there bounds

$$C^{-1} \leq \|u|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C \ ?$$

There are some interesting concentration phenomena that can happen, for example when there are periodic geodesics inside Ω (think of a great circle on the sphere as an example).

This project would involve analysing concentration phenomena in some interesting special cases, such as subdomains of the sphere, and trying to prove some positive results under the assumption that there are no periodic geodesics. The general idea is to understand how eigenfunctions with large λ behave, and to link the behaviour to geometric properties of the domain.

Reference: A. Hassell and T. Tao, 'Upper and lower bounds for normal derivatives of Dirichlet eigenfunctions', preprint, 2001.