MORTAR FINITE ELEMENTS FOR COUPLING COMPRESSIBLE AND NEARLY INCOMPRESSIBLE MATERIALS IN ELASTICITY

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Abstract. We consider the coupling of compressible and nearly incompressible materials within the framework of mortar methods. Taking into account the locking effect, we use a suitable discretization for the nearly incompressible material and work with a standard conforming discretization elsewhere. The coupling of different discretization schemes in different subdomains are handled by flexible mortar techniques. A priori error analysis is carried out for the coupled problem, and several numerical examples are presented. Using dual Lagrange multipliers, the Lagrange multipliers can easily be eliminated by local static condensation.

Key words. Mortar finite elements, Lagrange multipliers, dual space, non-matching triangulations, mixed formulations, saddle point problems
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1. Introduction. Often coupled problems with completely different material properties in different subdomains occur in solid mechanics. To get optimal a priori estimates, a proper discretization scheme should be used in each subdomain. Here, we consider coupling of compressible and nearly incompressible linear elastic materials with mortar techniques. The boundary value problem of elasticity involves a critical Lamé parameter λ. For nearly incompressible materials the Lamé parameter λ is very large, and it is well-known that working with low order finite elements with displacement based formulation suffers from so-called locking effect yielding a poor convergence, see [Bra01, BF91, BS92a]. Various approaches have been proposed to overcome this difficulty. Among these are to apply higher-order finite elements with a standard displacement formulation. For example, in [SV85], it is shown that working with the h-version finite elements of order higher than three on a class of triangular meshes completely avoid locking. On the other hand, in [BS92a], it has been shown that the h-version can never be fully free of locking in rectangular meshes no matter how higher-order finite elements are used in the sense that optimal orders of convergence are not obtained. The other approach is related to working with mixed methods. The linear elasticity problem can be formulated as a mixed formulation in many different ways, see [BF91, Bra01, Bra96, Wie00, AW02, ABD84]. The general approach in these mixed formulations is to introduce extra variables leading to a problem of saddle point type with a penalty term. The essential point is to prove that the method is robust for the limiting problem, which is the Stokes problem. Methods associated with nonconforming finite elements have also been analyzed leading to the uniform convergence in the nearly incompressible case, see [Fal91, BS92b, LLS03, Bre93]. The central point in these approaches is to construct an interpolation operator at each element which preserves zero divergence. We point out that many different methods like the reduced integration, the enhanced assumed strain and the mixed enhanced strain can be analyzed within the framework of mixed formulation, see [BF91, Bra98, BCR04, SR90, KT00a, KT00b, LRW06]. All these approaches have in common that the finite element approximation is robust for nearly incompressible materials.

In order to avoid the problem of locking-effect, we consider suitable discretization schemes for nearly incompressible materials. Introducing the pressure as an additional unknown for the nearly incompressible case, we arrive at the problem of coupling a saddle point problem with

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a positive definite one. Working exclusively with non-matching triangulations, we use mortar techniques to realize the coupling of different discretization schemes.

This paper is organized as follows. In the next section, we describe the boundary value problem of linear elasticity and introduce a new formulation of the boundary value problem in the continuous setting suitable for coupling a nearly incompressible material with a compressible material. In Section 3, we show the stability of the scheme and prove optimal a priori estimates. Finally in Section 4, we present some numerical results illustrating the performance of our approach.

2. The problem of linear elasticity in the mortar framework. We consider a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, which is decomposed into two non-overlapping subdomains Ω_1 and Ω_2 with the common interior interface Γ, $\bar{\Gamma} = \partial\Omega_1 \cap \partial\Omega_2$. For simplicity, we restrict ourselves to the case of two subdomains. However, the approach can easily be generalized to more than two subdomains.

We assume that the subdomains Ω_1 and Ω_2 are occupied with different isotropic linear elastic materials. Furthermore, the material in Ω_1 is supposed to be nearly incompressible, whereas Ω_2 is occupied with a compressible material. We consider the following linear elasticity problem of finding the displacement field $\mathbf{u}$ in Ω such that

$$\begin{array}{rcll} -\mathrm{div}\,(\mathcal{C}_1\boldsymbol{\varepsilon}(\mathbf{u})) & = & \boldsymbol{f}_1 & \text{in} \quad \Omega_1, \\ -\mathrm{div}\,(\mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u})) & = & \boldsymbol{f}_2 & \text{in} \quad \Omega_2 \end{array} \tag{2.1}$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$. Here, $\mathcal{C}_1$ and $\mathcal{C}_2$ are constant and symmetric fourth-order elasticity tensors corresponding to different materials in Ω_1 and Ω_2, respectively. Denoting the identity tensor by $\mathbf{1}$, their actions on the strain tensor are defined as

$$\mathcal{C}_1\boldsymbol{\varepsilon}(\mathbf{u}) = \lambda_1(\mathrm{div}\,\mathbf{u})\mathbf{1} + 2\mu_1\,\boldsymbol{\varepsilon}(\mathbf{u}), \text{ and } \mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u}) = \lambda_2(\mathrm{div}\,\mathbf{u})\mathbf{1} + 2\mu_2\,\boldsymbol{\varepsilon}(\mathbf{u}).$$

Moreover, the plane strain is assumed in the two-dimensional case. We define the global Hooke tensor $\mathcal{C}$ which takes the value $\mathcal{C}_1$ on Ω_1 and $\mathcal{C}_2$ on Ω_2, and set $\mathbf{u}_1 := \mathbf{u}_{|\Omega_1}$ and $\mathbf{u}_2 := \mathbf{u}_{|\Omega_2}$. We assume that $\boldsymbol{f}_i \in (L^2(\Omega_i))^d, i = 1, 2$. The interface conditions on Γ are given by

$$\begin{aligned} [\mathbf{u}] &:= \mathbf{u}_1 - \mathbf{u}_2 = 0 \quad \text{on} \quad \Gamma, \\ [\mathbf{u}]_n &:= (\mathcal{C}_1\boldsymbol{\varepsilon}(\mathbf{u}_1))\mathbf{n} - (\mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u}_2))\mathbf{n} = 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{2.2}$$

where $\mathbf{n}$ is the outer normal to Γ from Ω_1.

In order to write the variational formulation of the linear elasticity problem (2.1), we introduce $\mathbf{H}^1(\Omega_k) := (H^1(\Omega_k))^d$ for $k = 1, 2$ and define the unconstrained product space

$$\mathbf{X} := \prod_{k=1}^{2} \{\mathbf{v} \in \mathbf{H}^1(\Omega_k) |\ \mathbf{v}_{|\partial\Omega\cap\partial\Omega_k} = 0\}.$$

The interpolation space $\mathbf{H}_{00}^{1/2}(\Gamma)$ is defined by $\mathbf{H}_{00}^{1/2}(\Gamma) := \left(H_{00}^1(\Gamma)\right)^d$, and its dual space will be denoted by $\mathbf{H}^{-1/2}(\Gamma)$. The weak matching condition on the interface is imposed by introducing the vector-valued Lagrange multiplier space $\mathbf{M} := \mathbf{H}^{-1/2}(\Gamma)$ on the interface Γ. Here, we consider the positive definite variational problem on the constrained finite element space which is given by means of the global Lagrange multiplier space $\mathbf{M}$

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} |\ \int_\Gamma [\mathbf{v}] \cdot \boldsymbol{\psi}\, d\boldsymbol{\sigma} = 0,\ \boldsymbol{\psi} \in \mathbf{M}\}. \tag{2.3}$$

Then, the variational problem of linear elasticity in the mortar formulation can be written as: given $l \in (L^2(\Omega))^d$ find $\mathbf{u} \in \mathbf{V}$ such that

$$a(\mathbf{u},\mathbf{v}) = l(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}, \tag{2.4}$$

where the bilinear form $a(\cdot,\cdot)$ and the linear form $l(\cdot)$ are defined by

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) :&= \int_{\Omega_1} \mathcal{C}_1 \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, dx + \int_{\Omega_2} \mathcal{C}_2 \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, dx, \quad \text{and} \\ l(\mathbf{v}) :&= \int_{\Omega_1} \boldsymbol{f}_1 \cdot \mathbf{v}\, dx + \int_{\Omega_2} \boldsymbol{f}_2 \cdot \mathbf{v}\, dx, \end{aligned}$$

respectively. Taking into account the definition of $\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})$, we can write the variational formulation (2.4) as

$$\sum_{i=1}^{2} 2\mu_i \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, dx + \lambda_i \int_{\Omega_i} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx = \sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{f}_i \cdot \mathbf{v}\, dx. \tag{2.5}$$

From the assumption on $\mathcal{C}$, we find that $a(\cdot,\cdot)$ is symmetric, continuous and $\mathbf{V}$-elliptic, and hence the problem (2.5) has a unique solution $\mathbf{u} \in \mathbf{V}$. Since the material occupying Ω_1 is supposed to be nearly incompressible λ_1 is very large, and hence the divergence of the exact solution $\operatorname{div} \mathbf{u}_1$ is very small. This constraint for the low order approximation based on displacement approach leads to the locking. In the next paragraph, we will relax this constraint by introducing an additional variable for the pressure.

There are many efficient numerical approaches to handle a nearly incompressible material, see [SR90, BF91, Bra96, LRW06]. In general, they are more complex than the standard displacement formulation. Our goal is to combine the standard formulation with a suitable scheme for a nearly incompressible material without losing the simplicity and optimality of the approach. For that purpose, we want to get a variational formulation which is uniformly well-posed in terms of λ_1. Now we introduce an additional unknown variable $p := \lambda_1 \operatorname{div} \mathbf{u}$ in Ω_1 leading to a mixed formulation. Then the variational problem (2.5) is given by: find $(\mathbf{u}, p) \in \mathbf{V} \times L^2(\Omega_1)$ such that

$$\begin{array}{lclcllcl} \tilde{a}(\mathbf{u},\mathbf{v}) & + & \tilde{b}(\mathbf{v},p) & = & l(\mathbf{v}), & \mathbf{v} & \in & \mathbf{V}, \\ \tilde{b}(\mathbf{u},q) & - & \frac{1}{\lambda_1}\tilde{c}(p,q) & = & 0, & q & \in & L^2(\Omega_1), \end{array} \tag{2.6}$$

where

$$\begin{aligned} \tilde{a}(\mathbf{u},\mathbf{v}) &:= \sum_{i=1}^{2} 2\mu_i \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, dx + \lambda_2 \int_{\Omega_2} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx, \\ \tilde{b}(\mathbf{v},q) &:= \int_{\Omega_1} \operatorname{div} \mathbf{v}\, q\, dx \quad \text{and} \quad \tilde{c}(p,q) := \int_{\Omega_1} p\, q\, dx. \end{aligned}$$

As usual, for $v \in H^s(\Omega)$ ($H^0(\Omega) \equiv L^2(\Omega)$), $s \in \mathbb{R}$, $\|v\|_{s,\Omega}$ denotes the standard norm in $H^s(\Omega)$, and we use the same notation for norms on $\mathbf{H}^s(\Omega)$ and $H^s(\Omega)$, whereas a broken norm is used on $\mathbf{X}$ defined as

$$\|\mathbf{v}\|_1 := \|\mathbf{v}\|_{1,\Omega_1} + \|\mathbf{v}\|_{1,\Omega_2}.$$

We remark that in contrast to the setting of the Stokes problem with homogeneous boundary condition, where $p \in L^2_0(\Omega_1)$, here, the pressure $p \in L^2(\Omega_1)$. The essential points for the existence and the uniqueness of the solution of a saddle point problem are ellipticity, continuity and a suitable inf-sup condition. Furthermore, for the saddle point problem with penalty, it is necessary that the bilinear form $\tilde{c}(,\cdot,)$ should be positive semi-definite and bounded, see [Bra96]. The bilinear form $\tilde{a}(\cdot,\cdot)$ is symmetric, continuous and $\mathbf{V}$-elliptic uniformly with respect to λ_1. It is also clear that the bilinear form $\tilde{c}(\cdot,\cdot)$ is continuous, symmetric and positive definite. The continuity of $\tilde{b}(\cdot,\cdot)$ follows from its definition.

Lemma 2.1. *The bilinear form $\tilde{b}(\cdot,\cdot)$ on $\mathbf{V}\times L^2(\Omega_1)$ satisfies an inf-sup condition uniformly with respect to λ_1.*

Proof. The proof is based on applying the argument due to Boland and Nicolaides [BN83]. Given $q\in L^2(\Omega_1)$, we split $q=q_0+q_c$, where $\int_{\Omega_1} q_0\,dx=0$ and q_c is a constant such that $\int_{\Omega_1} q\,dx=\int_{\Omega_1} q_c\,dx=|\Omega_1|q_c$. Thus $\|q\|^2_{0,\Omega_1}=\|q_0\|^2_{0,\Omega_1}+\|q_c\|^2_{0,\Omega_1}$. Since $q_0\in L^2_0(\Omega_1)$, there exists a $\mathbf{v}_0\in\mathbf{H}^1_0(\Omega_1)$ with $\|\mathbf{v}_0\|_{1,\Omega_1}\le C\|q_0\|_{0,\Omega_1}$ such that

$$\|q_0\|^2_{0,\Omega_1}=\tilde{b}(\mathbf{v}_0,q_0),\quad \text{see [GR86, Corollary 2.4].}$$

Hence $\|q\|^2_{0,\Omega_1}=\tilde{b}(\mathbf{v}_0,q_0)+q_c^2|\Omega_1|$. Now, we define a piecewise constant function $\tilde{f}$ in Ω with

$$\tilde{f}(x):=\begin{cases} q_c & \text{if } x\in\Omega_1,\\ -\frac{q_c|\Omega_1|}{|\Omega_2|} & \text{if } x\in\Omega_2\end{cases}$$

so that $\tilde{f}\in L^2_0(\Omega)$, and hence the divergence equation

$$\nabla\cdot\mathbf{w}=\tilde{f}\quad\text{in}\quad\Omega \tag{2.7}$$

has a solution $\mathbf{v}_c\in\mathbf{H}^1_0(\Omega)$ with $\|\mathbf{v}_c\|_1\le C\|\tilde{f}\|_0$, see [ASV88, Gal97]. Thus

$$\|q\|^2_{0,\Omega_1}=\tilde{b}(\mathbf{v}_0,q_0)+q_c^2|\Omega_1|=\tilde{b}(\mathbf{v}_0,q_0)+\int_{\Omega_1}\nabla\cdot\mathbf{v}_c q_c\,dx=\tilde{b}(\mathbf{v}_0,q_0)+\tilde{b}(\mathbf{v}_c,q_c).$$

Since $\mathbf{v}_0\in\mathbf{H}^1_0(\Omega_1)$, we can extend $\mathbf{v}_0$ trivially on Ω by defining $\tilde{\mathbf{v}}_0:=\mathbf{v}_0$ in Ω_1 and $\tilde{\mathbf{v}}_0:=0$ in Ω_2, and find that $\tilde{\mathbf{v}}_0\in\mathbf{H}^1_0(\Omega)$. Hence $\tilde{b}(\mathbf{v}_0,q_0)=\tilde{b}(\tilde{\mathbf{v}}_0,q_0)$. On the other hand,

$$\tilde{b}(\tilde{\mathbf{v}}_0+\mathbf{v}_c,q_0+q_c)=\tilde{b}(\tilde{\mathbf{v}}_0,q_0)+\tilde{b}(\mathbf{v}_c,q_c)+\tilde{b}(\tilde{\mathbf{v}}_0,q_c)+\tilde{b}(\mathbf{v}_c,q_0).$$

Noting that $\tilde{b}(\tilde{\mathbf{v}}_0,q_c)=0$, and $\tilde{b}(\mathbf{v}_c,q_0)=0$, we get

$$\|q\|^2_{0,\Omega_1}=\tilde{b}(\tilde{\mathbf{v}}_0+\mathbf{v}_c,q_0+q_c).$$

Finally, taking into account that $\tilde{\mathbf{v}}_0\in\mathbf{H}^1_0(\Omega)$ we get $\mathbf{v}_c+\tilde{\mathbf{v}}_0=:\mathbf{v}\in\mathbf{H}^1_0(\Omega)$ with $\|\mathbf{v}\|_1\le\|\mathbf{v}_c+\tilde{\mathbf{v}}_0\|_1\le C(\|\tilde{f}\|_0+\|q_0\|_{0,\Omega_1})$, which completes the proof. □

An immediate consequence of the previous lemma is the following theorem.

Theorem 2.2. *The problem 2.6 has a unique solution and there exists a constant C independent of λ_1 such that*

$$\|\mathbf{u}\|_1+\|p\|_{0,\Omega_1}\le C\|l\|_0.$$

3. Mortar discretizations and a priori estimates. In this section, we briefly review mortar finite elements and prove optimal a priori estimates for the discretization errors. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be independent shape regular triangulations on Ω_1 and Ω_2 with mesh-sizes bounded by h_1 and h_2, respectively. We define the unconstrained discrete finite element space for the displacement $\mathbf{X}_h := \mathbf{X}_1 \times \mathbf{X}_2$, where $\mathbf{X}_k := X_k^d$, X_k being the conforming finite element space of order $p_k > 1$ in Ω_k. We recall that no interface condition is imposed on $\mathbf{X}_h$, and the elements in $\mathbf{X}_h$ do not have to satisfy a continuity condition at the interface. The pressure space $L^2(\Omega_1)$ is discretized by some finite elements and will be denoted by $R_h \subset L^2(\Omega_1)$. The efficiency and optimality of the mortar method depends on the choice of a discrete Lagrange multiplier space, which should satisfy assumptions stated in [Lam06, Assumptions 2–4]. Without loss of generality, the Lagrange multiplier space is based on a "$d-1$"-dimensional mesh $\mathcal{T}_{\Gamma_2}$ inherited from $\mathcal{T}_2$, and its basis functions are defined locally having the same support as finite element basis functions associated with the interior nodes of the slave side.

We observe that since the normal has jumps if Γ has corners although $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega_1)$, $\boldsymbol{\varepsilon}(\mathbf{u})\mathbf{n}$ is, in general, not an element in $\mathbf{H}^{s-1/2}(\Gamma)$ when $s > \frac{1}{2}$. Therefore, we decompose Γ into a finite number of subsets γ_i, $1 \le i \le N$, such that each γ_i entirely lies in a "$d-1$"-dimensional hyperplane, and

$$\bar{\Gamma} = \bigcup_{i=1}^{N} \bar{\gamma}_i,$$

where $\gamma_k \cap \gamma_l = \emptyset$, and $\bar{\gamma}_k \cup \bar{\gamma}_l$ does not entirely lie in a "$d-1$"-dimensional hyperplane, $1 \le k \ne l \le N$. Denoting the discrete Lagrange multiplier spaces on γ_i by M_i, $1 \le i \le N$, we define $\mathbf{M}_i := M_i^d$, and our global discrete Lagrange multiplier space is then given as the product space

$$\mathbf{M}_h := \prod_{i=1}^{N} \mathbf{M}_i.$$

The finite element nodes in $\partial\gamma_i$ on the slave side, $1 \le i \le N$, are the crosspoints and they do not carry any degree of freedom for the Lagrange multipliers. We assume that $\mathbf{W}_i^m$ and $\mathbf{W}_i^s$ are the trace spaces of $\mathbf{X}_1$ and $\mathbf{X}_2$ restricted to γ_i, respectively, satisfying homogeneous boundary conditions on $\partial\gamma_i$, and we set

$$\mathbf{W}_h^m := \prod_{i=1}^{N} \mathbf{W}_i^m, \ \mathbf{W}_h^s := \prod_{i=1}^{N} \mathbf{W}_i^s.$$

As in the continuous setting, we consider the positive-definite variational problem on the constrained finite element space $\mathbf{V}_h$ which is given by means of the discrete global Lagrange multiplier space $\mathbf{M}_h$

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{X}_h \mid b(\mathbf{v}_h, \boldsymbol{\psi}_h) = 0, \ \boldsymbol{\psi}_h \in \mathbf{M}_h\}, \tag{3.1}$$

where $b(\mathbf{v}_h, \boldsymbol{\psi}_h) := \sum_{i=1}^{N} \int_{\gamma_i} [\mathbf{v}_h] \cdot \boldsymbol{\psi}_h \, d\boldsymbol{\sigma}$. We remark that the elements of the space $\mathbf{V}_h$ satisfy a weak continuity condition on the skeleton Γ in terms of the discrete Lagrange multiplier space $\mathbf{M}_h$. However, $\mathbf{V}_h$ is, in general, not a subspace of $\mathbf{H}_0^1(\Omega)$. Replacing the space $\mathbf{V} \times L^2(\Omega_1)$ by our discrete space $\mathbf{V}_h \times R_h$ in (2.6), we obtain our discrete variational problem: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times R_h$ such that

$$\begin{array}{rcll} \tilde{a}(\mathbf{u}_h, \mathbf{v}) + \tilde{b}(\mathbf{v}, p_h) & = & l(\mathbf{v}), & \mathbf{v} \in \mathbf{V}_h, \\ \tilde{b}(\mathbf{u}_h, q) - \frac{1}{\lambda_1}\tilde{c}(p_h, q) & = & 0, & q \in R_h. \end{array} \tag{3.2}$$

To establish a priori estimates for the discretization errors, we consider the saddle point formulation (3.2) of the elasticity problem and apply the theory of mixed finite elements. The continuity of the bilinear form $\tilde{a}(\cdot,\cdot)$ on $\mathbf{V}_h \times \mathbf{V}_h$, of $\tilde{b}(\cdot,\cdot)$ on $\mathbf{V}_h \times R_h$ and of $\tilde{c}(\cdot,\cdot)$ on $R_h \times R_h$ is straightforward. Moreover, the continuity constants are independent of λ_1. Furthermore, we need the ellipticity of the bilinear form $\tilde{a}(\cdot,\cdot)$ on $\mathbf{V}_h \times \mathbf{V}_h$, and a uniform inf-sup condition for the bilinear form $\tilde{b}(\cdot,\cdot)$ on $\mathbf{V}_h \times R_h$.

3.1. Uniform inf-sup condition and ellipticity. The following two assumptions will be crucial to prove the inf-sup condition in the discrete setting and are supposed to hold in the following.

Assumption 3.1.

3.1(i) *For a constant $q_c \in \mathbb{R}$, there exist functions $\mathbf{v}_h^s \in \mathbf{W}_i^s, \mathbf{v}_h^m \in \mathbf{W}_i^m$ for some $i \in \{1, \cdots, N\}$ with $\|\mathbf{v}_h^s\|_{\mathbf{H}_{00}^{1/2}(\gamma_i)} \le C|q_c|, \|\mathbf{v}_h^m\|_{\mathbf{H}_{00}^{1/2}(\gamma_i)} \le C|q_c|$ so that*

$$\int_{\gamma_i} \mathbf{v}_h^s \cdot \mathbf{n}\, d\boldsymbol{\sigma} = q_c, \text{ and} \int_{\gamma_i} (\mathbf{v}_h^s - \mathbf{v}_h^m) \cdot \boldsymbol{\psi}\, d\boldsymbol{\sigma} = 0, \ \boldsymbol{\psi} \in \mathbf{M}_i.$$

3.1(ii) *For any $q \in R_h \cap L_0^2(\Omega_1)$, there exists a constant $C > 0$ independent of the meshsize such that*

$$\sup_{\mathbf{v}_h \in \mathbf{X}_1 \cap \mathbf{H}_0^1(\Omega_1)} \frac{\tilde{b}(\mathbf{v}_h, q)}{\|\mathbf{v}_h\|_{1,\Omega_1}} \ge C\|q\|_{0,\Omega_1}.$$

Assumption 3.1 (i) is readily met if the triangulation is fine enough and the discrete Lagrange multiplier space satisfies the stability assumption [Lam06, Assumption 2], and Assumption 3.1 (ii) tells that the spaces $\mathbf{X}_1$ and R_h should be chosen carefully so that they form a stable pair for the Stokes problem. The following lemma provides a necessary tool to prove inf-sup condition.

Lemma 3.2. *For a constant $q_c \in \mathbb{R}$, there exists a $\mathbf{v}_h \in \mathbf{V}_h$ with $\|\mathbf{v}_h\|_1 \le C|q_c|$ such that $\int_{\Omega_1} \nabla \cdot \mathbf{v}_h\, dx = q_c$.*

Proof. Because of Assumption 3.1 (i), we can choose a function $\mathbf{v}_h^s \in \mathbf{W}_i^s$ with

$$\|\mathbf{v}_h^s\|_{\mathbf{H}_{00}^{1/2}(\gamma_i)} \le C|q_c| \text{ such that } \int_{\gamma_i} \mathbf{v}_h^s \cdot \mathbf{n}\, d\boldsymbol{\sigma} = q_c,$$

and define a function $\mathbf{v}_h^m \in \mathbf{W}_i^m$ with $\|\mathbf{v}_h^m\|_{\mathbf{H}_{00}^{1/2}(\gamma_i)} \le C|q_c|$ so that $\int_{\gamma_i} (\mathbf{v}_h^s - \mathbf{v}_h^m) \cdot \boldsymbol{\psi}\, d\boldsymbol{\sigma} = 0,\ \boldsymbol{\psi} \in \mathbf{M}_i$. Since $\mathbf{v}_h^s, \mathbf{v}_h^m \in \mathbf{H}_{00}^{1/2}(\gamma_i)$ both $\mathbf{v}_h^s$ and $\mathbf{v}_h^m$ can trivially be extended to functions in $\mathbf{W}_h^s$ and $\mathbf{W}_h^m$, respectively, still denoted by $\mathbf{v}_h^s$ and $\mathbf{v}_h^m$. Using the discrete harmonic extension, we obtain functions $\mathbf{w}_h^m \in \mathbf{X}_1$ and $\mathbf{w}_h^s \in \mathbf{X}_2$ so that $\mathbf{w}_{h|\Gamma}^s = \mathbf{v}_h^s$ and $\mathbf{w}_{h\ |\Gamma}^m = \mathbf{v}_h^m$. Defining a function $\mathbf{v}_h \in \mathbf{X}_h$ with $\mathbf{v}_{h|\Omega_1} = \mathbf{w}_h^m$, and $\mathbf{v}_{h|\Omega_2} = \mathbf{w}_h^s$, we find that $\mathbf{v}_h \in \mathbf{V}_h$, and from the well known property of harmonic extension we have $\|\mathbf{v}_h\|_1 \le C|q_c|$. Finally, the result follows from

$$\int_{\Omega_1} \nabla \cdot \mathbf{v}_h\, dx = \int_{\Gamma} \mathbf{v}_h^s \cdot \mathbf{n}\, d\boldsymbol{\sigma} = \int_{\gamma_i} \mathbf{v}_h^s \cdot \mathbf{n}\, d\boldsymbol{\sigma} = q_c.$$

□

THEOREM 3.3. *For any $q_h \in R_h$, there exists a constant C independent of λ_1 and the meshsize such that*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\tilde{b}(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq C \|q_h\|_{0,\Omega_1}.$$

Proof. As in the continuous case, we resort to the argument due to Boland and Nicolaides [BN83] to prove the inf-sup condition. We take $q_h \in R_h$ and split $q_h = q_{0h} + q_{ch}$, where $\int_{\Omega_1} q_{0h}\, dx = 0$ and q_{ch} is the L^2-projection of q_h onto $\mathbb{R}$ such that $\int_{\Omega_1} q_h\, dx = \int_{\Omega_1} q_{ch}\, dx$. Since $q_{0h} \in R_h \cap L_0^2(\Omega_1)$, from Assumption 3.1 (ii), we get a $\mathbf{v}_{0h} \in \mathbf{X}_1 \cap \mathbf{H}_0^1(\Omega_1)$ with $\|\mathbf{v}_{0h}\|_{1,\Omega_1} \leq C\|q_{0h}\|_{0,\Omega_1}$ so that $\|q_{0h}\|^2_{0,\Omega_1} = \tilde{b}(\mathbf{v}_{0h}, q_{0h})$. Hence

$$\|q_h\|^2_{0,\Omega_1} = \tilde{b}(\mathbf{v}_{0h}, q_{0h}) + q_{ch}^2 |\Omega_1|. \tag{3.3}$$

From Lemma 3.2, we get a $\mathbf{v}_{ch} \in \mathbf{V}_h$ such that $\int_{\Omega_1} \nabla \cdot \mathbf{v}_{ch}\, q_{ch}\, dx = q_{ch}^2|\Omega_1|$. Using this in (3.3), we get

$$\|q_h\|^2_{0,\Omega_1} = \tilde{b}(\mathbf{v}_{0h}, q_{0h}) + \tilde{b}(\mathbf{v}_{ch}, q_{ch})$$

The rest of the proof follows exactly as in continuous setting. □

REMARK 3.4. *Working with bilinear or trilinear finite elements and piecewise constant pressure (Q_1P_0) in the subdomain with the nearly incompressible material, it is well known that the uniform inf-sup condition does not hold, and one can observe some spurious pressure modes. Since Assumption 3.1 (ii) does not hold, the theoretical analysis does not cover this case. However, as analyzed in [GR86] for a problem posed in a single domain with homogeneous Dirichlet boundary condition, the spurious pressure modes do not substantially affect the displacement. Furthermore, through the numerical results we will show that the Q_1P_0 formulation can be successfully used in a subdomain with nearly incompressible material.*

Now we turn our attention to the ellipticity of the bilinear form $\tilde{a}(\cdot,\cdot)$ on the space $\mathbf{V}_h$. If $\partial\Omega_k \cap \partial\Omega$ has a non-zero measure for $k = 1, 2$, we can apply Korn's and Poincare's inequalities to each subdomain and obtain the desired results

$$\tilde{a}(\mathbf{v},\mathbf{v}) = \sum_{k=1}^{2} \tilde{a}_k(\mathbf{v},\mathbf{v}) \geq C \sum_{k=1}^{2} \|\mathbf{v}\|^2_{1,\Omega_k} = C\|\mathbf{v}\|_1^2, \quad \mathbf{v} \in \mathbf{X}_h,$$

where $\tilde{a}_k(\cdot,\cdot)$ stands for the restriction of $\tilde{a}(\cdot,\cdot)$ to the subdomain Ω_k. Thus $\tilde{a}(\cdot,\cdot)$ is elliptic on $\mathbf{X}_h \times \mathbf{X}_h$. Unfortunately, there are many interesting situations where we cannot satisfy this assumption. However, it is sufficient to have ellipticity of $\tilde{a}(\cdot,\cdot)$ in $\mathbf{V}_h \times \mathbf{V}_h$ for the problem (3.2) to be uniquely solvable. Since the bilinear form $\tilde{a}(\cdot,\cdot)$ does not involve λ_1 the ellipticity can been shown exactly as in [Woh01, Bre04, HT04] uniformly with respect to λ_1. It is shown in [Bre04, HT04] that the ellipticity constant is independent of the number and the size of different subdomains of the decomposition.

REMARK 3.5. *Using the Stokes equation in the subdomain Ω_1 instead of equation of elasticity we arrive at the Stokes flow coupled with a linear elastic body. The coupled problem can be written as: given $l \in L^2(\Omega)$ find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times R_h$ such that*

$$\begin{aligned}
\int_{\Omega_2} \mathcal{C}_2 \boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h)\, dx \quad + \mu_1 \int_{\Omega_1} \nabla \mathbf{u}_h : \nabla \mathbf{v}_h\, dx + \int_{\Omega_1} div\, \mathbf{v}_h\, p_h\, dx &= \quad l(\mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h \\
\int_{\Omega_1} div\, \mathbf{u}_h\, q_h\, dx &= \quad 0, \quad q_h \in R_h,
\end{aligned}$$

where $\mathbf{u}_h$ restricted to the subdomain Ω_2 represents the displacement, $\mathbf{u}_h$ restricted to the subdomain Ω_1 represents the velocity, and μ_1 is the kinematic viscosity for the incompressible fluid. The mathematical analysis of mortar finite elements for the Stokes problem can be found in [Ben00, Ben04]. The mortar finite element method for mixed elasticity problems is analyzed in [BCS03].

3.2. A priori estimates. The immediate consequence of the above discussion is the well-posedness of the discrete problem (3.2). From the theory of saddle point problem, see, e.g., [BF91], we have

LEMMA 3.6. *The discrete problem* (3.2) *has exactly one solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times R_h$ *which is uniformly stable with respect to the data* $\boldsymbol{f}_i, i = 1, 2$, *and there exists a constant* C *independent of Lamé parameter* λ_1 *such that*

$$\|\mathbf{u}_h\|_1 + \|p_h\|_{0,\Omega_1} \leq C\|\boldsymbol{f}\|_0.$$

The convergence theory is provided by an abstract result about the approximation of saddle point problems by nonconforming methods, see [DM87, Ben00, BCS03].

LEMMA 3.7. *Assume that* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *be the solutions of problems* (2.6) *and* (3.2), *respectively. Then, we have the following error estimate uniform with respect to* λ_1*:*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_{0,\Omega_1} \\ &\leq C\left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_1 + \inf_{q_h \in R_h} \|p - q_h\|_{0,\Omega_1} + \sup_{\mathbf{v}_h \in \mathbf{V}_h \setminus \{0\}} \frac{|\tilde{a}(\mathbf{u}, \mathbf{v}_h) + \tilde{b}(\mathbf{v}_h, p) - l(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_1}\right). \end{aligned} \tag{3.4}$$

We note that the first two terms in the right hand side of (3.4) denote the best approximation error and the last one is the consistency error. In the following, any integral over Γ is to be understood as a duality pairing between $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathbf{H}^{\frac{1}{2}}(\Gamma)$.

LEMMA 3.8. *The following identity holds for the consistency error in Lemma 3.7*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h \setminus \{0\}} \frac{|\tilde{a}(\mathbf{u}, \mathbf{v}_h) + \tilde{b}(\mathbf{v}_h, p) - l(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_1} = \sup_{\mathbf{v}_h \in \mathbf{V}_h \setminus \{0\}} \frac{|\int_\Gamma \mathcal{C}_2 \boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n} \cdot [\mathbf{v}_h]\, d\boldsymbol{\sigma}|}{\|\mathbf{v}_h\|_1}.$$

Proof.

$$\begin{aligned} \tilde{a}(\mathbf{u}, \mathbf{v}_h) + \tilde{b}(\mathbf{v}_h, p) - l(\mathbf{v}_h) &= \sum_{k=1}^{2} \int_{\Omega_k} \mathcal{C}_k \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}_h)\, dx \\ &+ \int_{\Omega_1} (p - \lambda_1 \nabla \cdot \mathbf{u}) \nabla \cdot \mathbf{v}_h\, dx - l(\mathbf{v}_h) \\ &= \int_\Gamma \mathcal{C}_2 \boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n} \cdot [\mathbf{v}_h]\, d\boldsymbol{\sigma}, \end{aligned}$$

where in the last step we have used the second equation of (2.2), and $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$. □

The a priori error estimate is obtained by combining the approximation of the saddle point problem in our nonconforming situation with the best approximation property of $\mathbf{V}_h$, R_h and $\mathbf{M}_h$.

THEOREM 3.9. *Assume that* $\mathbf{u} \in \Pi_{k=1}^{2}\mathbf{H}^{r_k+1}(\Omega_k)$, $p \in H^{r_1}(\Omega_1)$, *and* $\boldsymbol{\chi} := \mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n} \in \Pi_{i=1}^{N}\mathbf{H}^{r_2-\frac{1}{2}}(\gamma_i)$ *with* $r_k > \frac{1}{2}$, $k = 1, 2$. *Moreover, assume that*

$$\inf_{q_h \in R_h} \|q - q_h\|_{0,\Omega_1} \leq C h_1^{p_1} \|q\|_{p_1,\Omega_1}, \quad q \in H^{p_1}(\Omega_1).$$

Then the following a priori error estimate holds for the discretization error

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_{0,\Omega_1} \leq C \left(\sum_{k=1}^{2} h_k^{t_k} \|\mathbf{u}\|_{t_k+1,\Omega_k} + h_1^{t_1} \|p\|_{t_1,\Omega_1} \right),$$

where $t_k := min(r_k, p_k)$, $k = 1, 2$.

Proof. The best approximation property of $\mathbf{V}_h$ is quite standard and can be found, e.g., in [BMP93, BMP94]. Hence using Lemma 3.7 it is sufficient to consider the consistency error. The definition of space $\mathbf{V}_h$, the best approximation property of $\mathbf{M}_h$ and the trace theorem yield for $\boldsymbol{\psi}_h \in \mathbf{M}_h$

$$\begin{aligned}
\int_\Gamma \boldsymbol{\chi} \cdot [\mathbf{v}_h]\, d\boldsymbol{\sigma} &= \int_\Gamma (\mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n} - \boldsymbol{\psi}_h) \cdot [\mathbf{v}_h]\, d\boldsymbol{\sigma} \\
&\leq \sum_{i=1}^{N} \inf_{\boldsymbol{\psi}_h \in \mathbf{M}_i} \|\mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n} - \boldsymbol{\psi}_h\|_{(\mathbf{H}^{1/2}(\gamma_i))'} \|[\mathbf{v}_h]\|_{1/2,\gamma_i} \\
&\leq C h_2^{t_2} \sum_{i=1}^{N} \|\mathcal{C}_2\boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n}\|_{t_2-1/2,\gamma_i} \|\mathbf{v}_h\|_1 \\
&\leq C h_2^{t_2} \|\mathbf{u}\|_{t_2+1,\Omega_2} \|\mathbf{v}_h\|_1.
\end{aligned}$$

□

The assumption of Theorem 3.9 requires a strong assumption on the regularity of the solution $\mathbf{u} \in \Pi_{k=1}^{2}\mathbf{H}^{r_k+1}(\Omega_k)$ with $r_k > \frac{1}{2}$, $k = 1, 2$. In the following, invoking a result about the regularity of the co-normal derivative on Lipschitz domain [Cos88], we prove an optimal estimate under a weaker regularity assumption such that $0 < r_k < \frac{1}{2}$. We note that we have to exclude the case $r_k = \frac{1}{2}$ as the result in [Cos88] does not cover this case.

THEOREM 3.10. *Assume that* $\mathbf{u} \in \Pi_{k=1}^{2}\mathbf{H}^{r_k+1}(\Omega_k)$, $p \in H^{r_1}(\Omega_1)$ *with* $0 < r_k < \frac{1}{2}$, $k = 1, 2$. *If*

$$\inf_{q_h \in R_h} \|q - q_h\|_{0,\Omega_1} \leq C h_1^{r_1} \|q\|_{r_1,\Omega_1}, \quad q \in H^{r_1}(\Omega_1),$$

the following a priori error estimate holds for the discretization error

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_{0,\Omega_1} \leq C \left(\sum_{k=1}^{2} h_k^{r_k} \|\mathbf{u}\|_{r_k+1,\Omega_k} + h_1^{r_1} \|p\|_{r_1,\Omega_1} + h_2^{r_2} \|\boldsymbol{f}_2\|_{0,\Omega_2} \right).$$

Proof. As $\boldsymbol{f}_2 \in (L^2(\Omega_2))^d$ and $\mathbf{u}_{|\Omega_2} \in \mathbf{H}^{r_2}(\Omega_2)$, Lemma 4.3 of [Cos88] yields $\boldsymbol{\chi}_{|\Gamma} \in \mathbf{H}^{r_2-\frac{1}{2}}(\Gamma)$ with

$$\|\boldsymbol{\chi}\|_{r_2-\frac{1}{2},\Gamma} \leq C\left(\|\mathbf{u}\|_{1+r_2,\Omega_2} + \|\boldsymbol{f}_2\|_{0,\Omega_2}\right).$$

Let $\boldsymbol{\psi}_h \in \mathbf{M}_h$. Proceeding exactly as in Theorem 3.9 and using the previous result, we obtain

$$\begin{aligned}\int_\Gamma \boldsymbol{\chi} \cdot [\mathbf{v}_h]\, d\boldsymbol{\sigma} &\le C h_2^{r_2} \sum_{i=1}^N \|\mathcal{C}_2 \boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n}\|_{r_2-1/2,\gamma_i} \|\mathbf{v}_h\|_1 \\ &\le C h_2^{r_2} \left(\|\mathbf{u}\|_{1+r_2,\Omega_2} + \|\boldsymbol{f}_2\|_{0,\Omega_2}\right) \|\mathbf{v}_h\|_1.\end{aligned}$$

We note that the inegral $\int_\Gamma \boldsymbol{\chi}\cdot[\mathbf{v}_h]\, d\boldsymbol{\sigma}$ is to be understood as a duality pairing between $\mathbf{H}^{r_2-\frac{1}{2}}(\Gamma)$ and $\mathbf{H}^{\frac{1}{2}-r_2}(\Gamma)$. ☐

Here we have assumed $\boldsymbol{f}_2 \in (L^2(\Omega_2))^d$ to use the result of [Cos88]. For a loading function $\boldsymbol{f}_2$ with low regularity and for the case with some $r_k = \frac{1}{2}$, results similar to those of [Cos88] are obtained in [Hu08].

Remark 3.11. *If Ω_1 is on the slave side of the interface Γ, then we have to estimate the term*

$$\sum_{i=1}^N \inf_{\boldsymbol{\psi}_h \in \mathbf{M}_i} \|\mathcal{C}_1 \boldsymbol{\varepsilon}(\mathbf{u}_1)\mathbf{n} - \boldsymbol{\psi}_h\|_{(\mathbf{H}^{1/2}(\gamma_i))'},$$

where now the Lagrange multiplier spaces $\mathbf{M}_i$ are defined on γ_i, $1 \le i \le N$, with the mesh inherited from $\mathcal{T}_1$. In this case, assuming $r_2, r_1 > \frac{1}{2}$, we can use the second equation of (2.2) *to obtain*

$$\begin{aligned}\int_\Gamma \boldsymbol{\chi} \cdot [\mathbf{v}_h]\, d\boldsymbol{\sigma} &\le \sum_{i=1}^N \inf_{\boldsymbol{\psi}_h \in \mathbf{M}_i} \|\mathcal{C}_1 \boldsymbol{\varepsilon}(\mathbf{u}_1)\mathbf{n} - \boldsymbol{\psi}_h\|_{(\mathbf{H}^{1/2}(\gamma_i))'} \|[\mathbf{v}_h]\|_{1/2,\gamma_i} \\ &= \sum_{i=1}^N \inf_{\boldsymbol{\psi}_h \in \mathbf{M}_i} \|\mathcal{C}_2 \boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n} - \boldsymbol{\psi}_h\|_{(\mathbf{H}^{1/2}(\gamma_i))'} \|[\mathbf{v}_h]\|_{1/2,\gamma_i} \\ &\le C h_1^{t_1} \sum_{i=1}^N \|\mathcal{C}_2 \boldsymbol{\varepsilon}(\mathbf{u}_2)\mathbf{n}\|_{t_1-1/2,\gamma_i} \|\mathbf{v}_h\|_1 \\ &\le C h_1^{t_1} \|\mathbf{u}\|_{t_1+1,\Omega_2} \|\mathbf{v}_h\|_1.\end{aligned}$$

4. Numerical results. In this section, we investigate the computational performance of our approach through some numerical examples. In particular, we compare the results from the standard approach and mortar approach for different test examples. In the following, Q_1 or Q_2 denotes that standard bilinear or quadratic serendipity elements are used in the whole domain Ω, whereas Q_1-Q_1P_0, Q_2-Q_2P_0 or Q_2-Q_2P_1 denotes that Q_1P_0, Q_2P_0 or Q_2P_1 formulation is used in subdomains with a nearly incompressible material ($\nu \to 0.5$) in combination with the standard Q_1 or Q_2 formulation in subdomains with smaller ν. We note that the mathematical theory presented in the previous sections do not cover the case of Q_1P_0 discretizations as these discretizations do not satisfy a local inf-sup condition.

In all our examples, we work with non-matching triangulations and employ dual Lagrange multiplier spaces introduced in [Woh01] to realize the weak matching condition. Construction of dual Lagrange multiplier spaces for higher order finite elements can be found in [Lam06]. For the pressure space, piecewise constant pressure is used for Q_1P_0 and Q_2P_0, whereas discontinuous linear pressure is used for Q_2P_1 case. Furthermore, we do not specify the measurement units, and they should be understood with proper scaling.

Example 1: Cook's membrane problem. In this example, we consider a structure occupying a region $\Omega := \mathrm{conv}\,\{(0,0),(48,44),(48,60),(0,44)\}$, where $\mathrm{conv}\,\xi$ is the convex hull of the set ξ. The left boundary of Ω is fixed and an in-plane shearing load of 100N is applied along the positive y-direction on the right boundary. Here, the domain Ω is decomposed into two subdomains Ω_1 and Ω_2 with

$$\Omega_2 := \mathrm{conv}\{(12,20.25),(36,38.75),(36,50.25),(12,38.75)\},$$

and $\Omega_1 := \Omega\backslash\bar{\Omega}_2$. The decomposition of domain Ω and the initial triangulation are given in Figure 4.1. The material parameters are taken to be $E_1 = 250$, $E_2 = 80$, $\nu_1 = 0.49999$, and

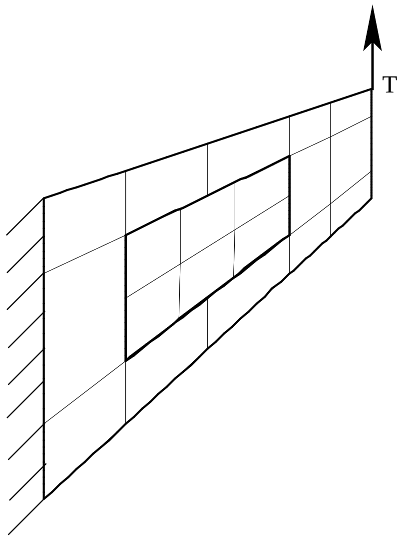

Fig. 4.1: Cook's membrane decomposed into two subdomains

$\nu_2 = 0.35$ to get a nearly incompressible response in Ω_1. We recall that Lamé parameters λ and μ are related to Young's modulus E and Poisson ratio ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)},$$

and note that $\lambda \to \infty$ corresponds to $\nu \to 0.5$. In Figure 4.2, we have shown the absolute error in the vertical tip displacement of the membrane at point T. We have used a reference solution in a fine mesh computed by using Q_2P_1 formulation in the whole domain Ω to obtain the error. We see that uniform convergence is obtained if we work with Q_1-Q_1P_0, Q_2-Q_2P_0, Q_2-Q_2P_1 or Q_2, see Figure 4.2. In this problem, we see that Q_1-Q_1P_0 and Q_2 elements work as good as Q_2-Q_2P_0 and Q_2-Q_2P_1. To show the influence of the choice of the master and the slave side, we have given the plot of the absolute error in the vertical tip displacement at the top right corner of the membrane in the left and right pictures of Figure 4.2 for different choices of master and slave sides. Comparing both of these pictures, we can see that there is not any essential difference between choosing Ω_1 or Ω_2 as the slave side. However, since the Lagrange multiplier space $\mathbf{M}_h$ is based on a coarser mesh if Ω_2 is on the slave side, we see some influence in the first step.

In a next step, we investigate the situation with the nearly incompressible material in Ω_2 so that the material parameters are $E_1 = 80$, $E_2 = 250$, $\nu_1 = 0.35$, and $\nu_2 = 0.49999$. As before

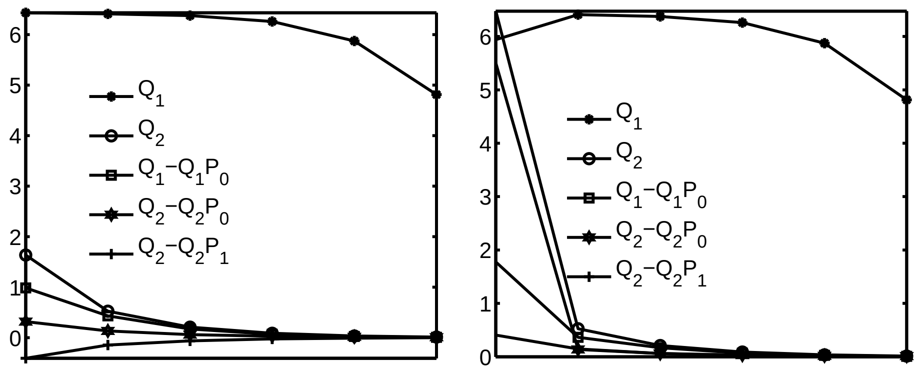

Fig. 4.2: Absolute error in the vertical tip displacement at the top right corner versus number of elements (Ω_1 master, Ω_2 slave) (left) and (Ω_2 master, Ω_1 slave) (right), Ω_1 nearly incompressible, Example 1

we also want to see the influence of the choice of the master and the slave side. The vertical tip displacement at the top right corner of the membrane for different levels of refinement are shown in the left and the right pictures of Figure 4.3 for different choices of master and slave sides. The standard approach in both subdomains leads to locking, whereas we obtain a good convergence behavior if a mixed formulation is used in Ω_2. As before, we do not see any influence of the choice of the master and slave side when we refine the mesh.

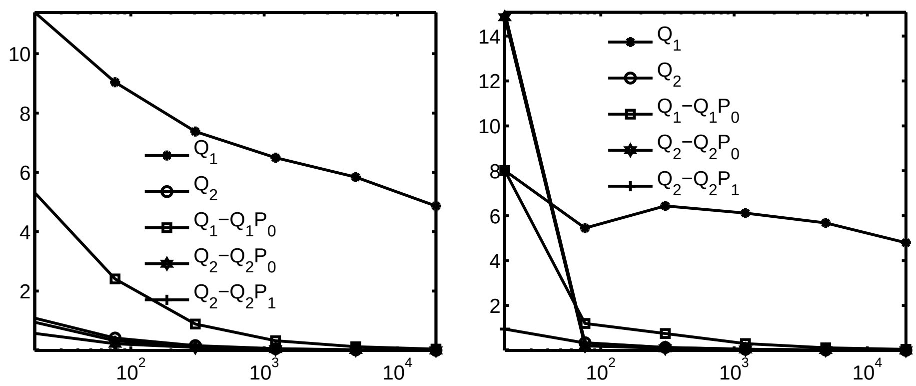

Fig. 4.3: Absolute error in the vertical tip displacement at the top right corner versus number of elements (Ω_1 master, Ω_2 slave) (left), and (Ω_2 master, Ω_1 slave) (right), Example 1, Ω_2 nearly incompressible

Example 2: Comparison of errors in the L^2- and H^1-norms. In this example, a two-dimensional region $\Omega := (-1,1) \times (-1,1)$ is decomposed into four non-overlapping subdomains defined by $\Omega_1 := (-1,0) \times (-1,0)$, $\Omega_2 := (0,1) \times (-1,0)$, $\Omega_3 := (-1,0) \times (0,1)$ and $\Omega_4 := (0,1) \times (0,1)$. The problem for this example is taken from [Bre93] with a slight modification to enforce that the jump of the flux across the interface Γ is zero. Here, the exact solution

$\mathbf{u} = (u_1, u_2)$ is

$$u_1(x,y) := \frac{\sin(2\pi y)(-1+\cos(2\pi x))(2+2\nu)}{E} + xy\sin(\pi x)\sin(\pi y)\frac{(1+\nu)(1-2\nu)}{1-\nu-2\nu^2+E\nu},$$
$$u_2(x,y) := \frac{\sin(2\pi x)(1-\cos(2\pi y))(2+2\nu)}{E} + xy\sin(\pi x)\sin(\pi y)\frac{(1+\nu)(1-2\nu)}{1-\nu-2\nu^2+E\nu},$$

where $\nu = 0.3, E = 25$ in Ω_1 and Ω_4, and $\nu = 0.49999, E = 250$ in Ω_2 and Ω_3 so that a nearly incompressible response is obtained in Ω_2 and Ω_3. In this example, the right hand side and the Dirichlet boundary conditions are computed by using the exact solution. We have given the decomposition of the domain and the initial triangulation in the left picture of Figure 4.4, and the error plot versus number of degrees of freedom for different levels of refinement for the L^2 and H^1-norms are given in the middle and the right pictures, respectively. From Figure 4.4, we can see that the optimality can be obtained by using $\mathrm{Q}_1\mathrm{P}_0$ and $\mathrm{Q}_2\mathrm{P}_1$-approaches for the nearly incompressible material, whereas the standard Q_1-approach locks. Furthermore, we can observe the sub-optimal behavior for $\mathrm{Q}_2\mathrm{P}_0$ and Q_2-discretizations.

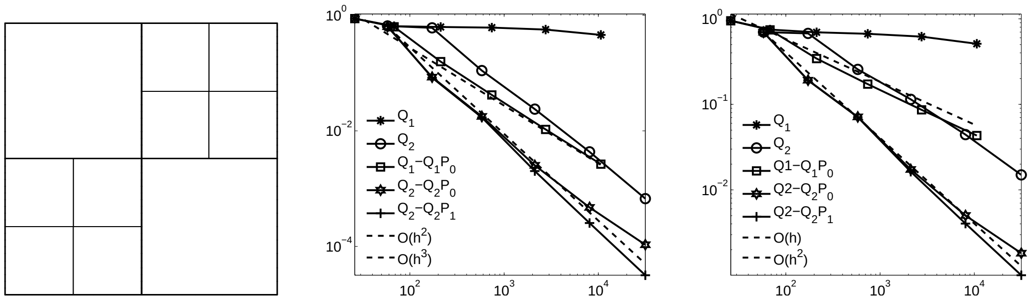

Fig. 4.4: Decomposition of the domain and initial triangulation (left), error plot versus number degrees of freedom in L^2-norm (middle) and error plot versus number of degrees of freedom in H^1-norm (right), Example 2

Example 3: Three-dimensional I-beam. In this numerical test, we consider the coupling of compressible and nearly incompressible elasticity in three-dimensional elasticity. The computational domain Ω, which is an I-beam, is decomposed into three subdomains Ω_1, Ω_2 and Ω_3 with $\Omega_1 := (0,50)\times(0,10)\times(0,2)$, $\Omega_2 := (0,50)\times(3,7)\times(2,11)$ and $\Omega_3 := (0,50)\times(0,10)\times(11,13)$. We impose zero Dirichlet boundary condition on Γ_D, where Γ_D is a part of the boundary of Ω with $x = 0$ and $x = 50$ so that the left and the right sides of each subdomain are fixed. And a constant vertical force is applied on a small part of the top boundary ($z = 13$) so that $\sigma(\mathbf{u})\mathbf{n} = \mathbf{g}_N$ on Γ_N with $\Gamma_N := \partial\Omega\backslash\Gamma_D$. The function $\mathbf{g}_N = (g_1, g_2, g_3)$ on Γ_N is given as $g_1 = g_2 = 0$, and

$$g_3 = \begin{cases} -20.35 & \text{if } 22 \le x \le 28 \text{ and } z = 13 \\ 0 & \text{otherwise} \end{cases}.$$

The material parameters are $E_1 = 250$, $\nu_1 = 0.3$, $E_2 = 300$, $\nu_2 = 0.4$, and $E_3 = 350$, $\nu_3 = 0.49999$. We have shown the setting of the problem in the left picture of Figure 4.5, and the resulting deformation of the structure is shown in the right.

In Figure 4.6, we have shown the error in the vertical displacement along the line $y = 0, z = 13$ versus x-coordinates. The error is obtained by using a reference solution computed in a fine

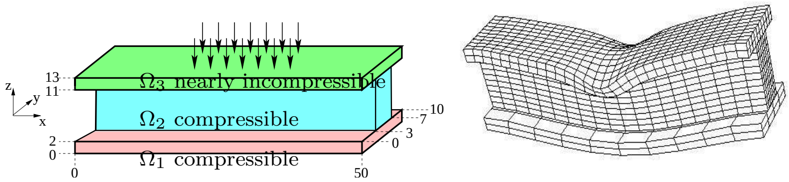

Fig. 4.5: Left: I-beam decomposed into three subdomains, Right: the distorted mesh

mesh with Q_2P_1 formulation in the whole domain Ω. As can be seen from this figure, the standard approaches show performance worse than a coupled approach due to the locking effect. In the case of coupled approach, the mixed formulation is used only in subdomain Ω_3. The vertical displacements from Q_2, Q_2-Q_2P_1 and Q_2-Q_2P_0-approaches are computed by using a one-level coarser mesh than those from Q_1-approach. We can see that numerical solutions from the coupled approach with Q_2P_0 and Q_2P_1 discretizations are almost the same.

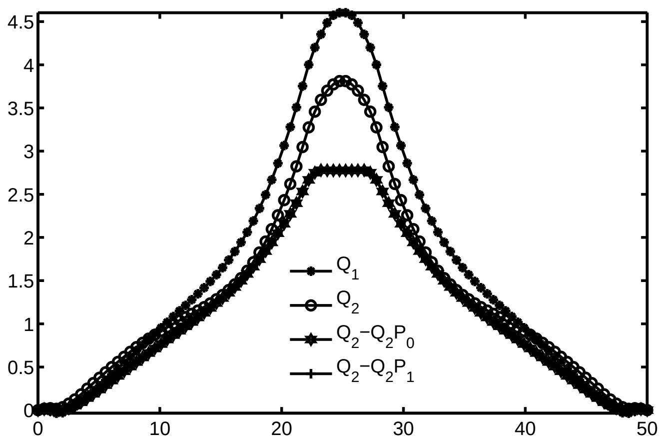

Fig. 4.6: Absolute error in the vertical displacement on the line $y = 0, z = 13$ versus x-coordinates

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