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Mortar finite elements for a heat transfer problem on sliding meshes

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Abstract. We consider a heat transfer problem with sliding bodies, where heat is generated on the interface due to friction. Neglecting the mechanical part, we assume that the pressure on the contact interface is a known function. Using mortar techniques with Lagrange multipliers, we show existence and uniqueness of the solution in the continuous setting. Moreover, two different mortar formulations are analyzed, and optimal a priori estimates are provided. Numerical results illustrate the flexibility of the approach.

Key words. Mortar finite elements, Lagrange multiplier, saddle point problem, domain decomposition, interface problem.

Mathematics Subject Classification (1991): 65N30, 65N55

1. Introduction

We consider here the problem of two bodies sliding on each other and generating heat due to contact friction (see also [10,13,14,16, 19,23]). We denote by $\Omega_k \subset \mathbf{R}^d$, $d \in \{2,3\}$, the two disjoint regions occupied by the bodies during the sliding process, and by $\partial\Omega_k$ the

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corresponding piecewise smooth boundaries for $k = 1, 2$. The surface of the bodies is decomposed into two parts: the first Γ^c, $\overline{\Gamma^c} := \partial\Omega_1 \cap \partial\Omega_2$, representing the common contact area of the two bodies, the second, Γ_k^s is given by

$$\Gamma_k^s := \partial\Omega_k \backslash \overline{\Gamma^c}, \quad k = 1, 2.$$

In order to investigate the heat conduction during the sliding process, we introduce the variable $u_k(t, x)$ denoting the temperature distribution of the corresponding body depending on the time $t \in [0, T]$, where $T > 0$ is a prescribed time-level, and the space variable $x \in \Omega_k$. Neglecting the mechanical part, we consider the heat equation given by

$$\rho_k c_k \frac{\partial u_k}{\partial t}(t, x) - \nabla \cdot (\alpha_k \nabla u_k(t, x)) = f_k(t, x) \quad \text{in} \quad (0, T) \times \Omega_k, \quad (1)$$

for $k = 1, 2$. A complete thermo-mechanical model can be found in [13,23]. Here ρ_k, c_k and α_k represent the density, the specific heat and the thermal conductivity respectively, and $f_k \in L^2(Q_{k,T})$ is the heat source within Ω_k. Moreover, the temperature distribution has to satisfy the following initial and boundary conditions: for $k = 1, 2$

$$\begin{aligned} u_k(0, x) &= u_k^0(x) && \text{in } \Omega_k, \ \text{ for } t = 0, && (2)\\ -\alpha_k \nabla u_k(t, x) \cdot n_k &= q_k^{fr}(t, x) + q_k^{ex}(t, x) && \text{on } \Gamma_T^c := (0, T) \times \Gamma^c, && (3)\\ -\alpha_k \nabla u_k(t, x) \cdot n_k &= q_k^{co}(t, x) && \text{on } \Gamma_{k,T}^s := (0, T) \times \Gamma_k^s, && (4) \end{aligned}$$

where $u_k^0 \in L^2(\Omega_k)$ is the initial temperature, n_k is the outward unit normal vector of $\partial\Omega_k$ and q_k^{fr}, q_k^{ex} and q_k^{co} are given by

$$\begin{aligned} q_1^{fr}(t, x) &= -c_D \beta v(t, x) p(t, x) && \text{on} \quad \Gamma_T^c, && (5)\\ q_2^{fr}(t, x) &= -c_D (1 - \beta) v(t, x) p(t, x) && \text{on} \quad \Gamma_T^c, && (6) \end{aligned}$$

$$\begin{aligned} q_1^{ex}(t, x) &= \hat{\alpha}(u_1(t, x) - u_2(t, x)) && \text{on} \quad \Gamma_T^c, && (7)\\ q_2^{ex}(t, x) &= \hat{\alpha}(u_2(t, x) - u_1(t, x)) && \text{on} \quad \Gamma_T^c, && (8) \end{aligned}$$

and

$$\begin{aligned} q_1^{co}(t, x) &= \sigma_1(u_1(t, x) - u_\infty) && \text{on} \quad \Gamma_{1,T}^s, && (9)\\ q_2^{co}(t, x) &= \sigma_2(u_2(t, x) - u_\infty) && \text{on} \quad \Gamma_{2,T}^s. && (10) \end{aligned}$$

For $k = 1, 2$, (3) represents the transmission conditions on the interface Γ_T^c given in terms of the heat fluxes $\alpha_1 \nabla u_1(t, x) \cdot n_1$ and $\alpha_2 \nabla u_2(t, x) \cdot n_2$ flowing out of the first and second body, respectively.

In particular, each heat flux $\alpha_k \nabla u_k(t,x) \cdot n_k$ consists of two parts: the first, $q_k^{fr}(t,x)$ (see (5)-(6)) arises from the frictional dissipation and involves the frictional constant c_D (from experiment, $c_D \leq 1$), and the parameter $\beta := \frac{\alpha_1}{\alpha_1+\alpha_2}$. Here $v(t,x)$ and $p(t,x)$ are the relative velocity of the bodies and the contact pressure, respectively. The second part, $q_k^{ex}(t,x)$ (see (7)-(8)) represents the heat exchange arising from the different temperatures of the bodies at the contact interface. The heat transfer parameter $\hat{\alpha} = \hat{\alpha}(p)$ depends on the pressure p, and we assume here to satisfy $\hat{\alpha}(p) \geq 0$. Finally, for $k = 1, 2$, (4) represents boundary conditions of Robin type given in terms of the convective heat flux $q_k^{co}(t,x)$ (see (9)-(10)), where $\sigma_1 > 0$ and $\sigma_2 > 0$ are coefficients of the surface heat transfer and u_∞ is the ambient temperature of the surrounding medium.

We present here an approach based on mortar techniques and Lagrange multipliers, which well suits to treat discontinuous solutions and for allowing non-matching grids across the subdomain boundaries.

The structure of the rest of the work is as follows: Section 2 is devoted to the formulation of the continuous problem and to the presentation of relevant results on well-posedness of the time-independent problem. In Section 3, two discrete formulations are obtained and error estimates are given for two different approaches. Finally in Section 4, we show some numerical experiments.

2. The saddle point formulation

We start by introducing the functional spaces we will work with. We denote by $X := H^1(\Omega_1) \times H^1(\Omega_2)$ the unconstrained product space endowed with the broken norm

$$\|w\|_X^2 := \|w\|_{1,\Omega_1}^2 + \|w\|_{1,\Omega_2}^2,$$

and by $M_2 := H^{\frac{1}{2}}(\Gamma^c)$ and $M_1 := (H^{\frac{1}{2}}(\Gamma^c))'$ the trace space on the common interface and the corresponding dual space, respectively. To simplify the readability of the forthcoming formulas, we use the nonstandard notation $H^{-\frac{1}{2}}(\Gamma^c) := (H^{\frac{1}{2}}(\Gamma^c))'$ for the dual space. From now on, we skip the space and time variables of our unknowns, where there is no ambiguity.

We introduce the heat flux on Γ_T^c from the second body, $\lambda(t,x) := \alpha_2 \nabla u_2(t,x) \cdot n_2 \in L^2(0,T; M_1)$, as a Lagrange multiplier which, according to (6) and (8), can be written as

$$\lambda = c_D(1-\beta)vp + \hat{\alpha}[u], \tag{11}$$

where $[u] := u_1 - u_2$ denotes the jump of the temperature across Γ^c_T. The mortar formulation is now obtained by considering the weak form of (1) and (11). We remark that, in contrast to the mortar approach for the Laplace operator, where the constraint reads $[u] = 0$, we cannot use $H^{-\frac{1}{2}}(\Gamma^c)$ as test space for the constrained equation (11). Here we have to work with $H^{\frac{1}{2}}(\Gamma^c)$ itself. Denoting by $< \cdot, \cdot >_{\Gamma^c}$ the duality pairing between $H^{\frac{1}{2}}(\Gamma^c)$ and $H^{-\frac{1}{2}}(\Gamma^c)$ on the interface, we can write, for any $\varphi \in H^{\frac{1}{2}}(\Gamma^c)$

$$< \lambda(t), \varphi >_{\Gamma^c} = \int_{\Gamma^c} c_D(1-\beta)v(t,s)p(t,s)\varphi(s)ds + \int_{\Gamma^c} \hat{\alpha}(p)[u(t,s)]\varphi(s)ds,$$

for $t \in [0,T]$. The weak formulation of our partial differential equation can be written as the following generalized saddle point problem: find $(u(t), \lambda(t)) \in L^2(0,T;X) \times L^2(0,T;M_1)$ such that

$$\begin{aligned} &\rho_k c_k \tfrac{d}{dt}(u(t),w) + a(u(t),w) + b_1(w,\lambda(t)) = (f(t),w),\ w \in X, \\ &b_2(u(t),\varphi) - c(\lambda(t),\varphi) \qquad\qquad\qquad = (g(t),\varphi),\ \ \varphi \in M_2, \end{aligned} \tag{12}$$

where the bilinear forms $a : X \times X \longrightarrow \mathbb{R}$, $b_1 : X \times M_1 \longrightarrow \mathbb{R}$, $b_2 : X \times M_2 \longrightarrow \mathbb{R}$ and $c : M_1 \times M_2 \longrightarrow \mathbb{R}$ are defined as follows

$$\begin{aligned} a(u,w) &:= \sum_{k=1}^{2} \Big[\int_{\Omega_k} \Big(\alpha_k \nabla u_k(x) \cdot \nabla w_k(x)\Big) dx + \int_{\Gamma^s_k} \sigma_k u_k(t,x) w_k(x) ds \Big], \\ b_1(w,\lambda) &:= < [w], \lambda >_{\Gamma^c}, \\ b_2(u,\varphi) &:= \int_{\Gamma^c} \hat{\alpha}[u]\varphi ds, \\ c(\lambda,\varphi) &:= < \lambda, \varphi >_{\Gamma^c}, \end{aligned}$$

and the right hand sides $f(t) : X \longrightarrow \mathbb{R}$ and $g(t) : M_2 \longrightarrow \mathbb{R}$ as

$$\begin{aligned} (f(t),w) &:= \int_{\Omega} f(t,x)w(x)dx + \int_{\Gamma^c} c_D v(t,s)p(t,s)w(s)_{|\Omega_1} ds \\ &\quad + \sum_{k=1}^{2} \int_{\Gamma^s_k} \sigma_k u_\infty w(s) ds, \\ (g(t),\varphi) &:= \int_{\Gamma^c} c_D(1-\beta)v(t,s)p(t,s)\varphi ds. \end{aligned}$$

We denote by $\|f(t)\|_{X'}$ and $\|g(t)\|_{-\frac{1}{2},\Gamma^c}$ the dual norms of the linear operators $f(t)$ and $g(t)$, defined by

$$\|f(t)\|_{X'} := \sup_{w \in X \setminus \{0\}} \frac{|(f(t),w)|}{\|w\|_X}, \qquad \|g(t)\|_{-\frac{1}{2},\Gamma^c} := \sup_{\varphi \in M_2 \setminus \{0\}} \frac{|(g(t),\varphi)|}{\|\varphi\|_{\frac{1}{2},\Gamma^c}}.$$

The mathematical analysis of generalized saddle point problems has become a subject of different papers. We address the reader to [3, 7, 17] for a deeper study on this subject. We remark that, due to the non-symmetry of the bilinear form $c(\cdot,\cdot)$, existence and uniqueness of (12) is not a straightforward consequence of the above mentioned results. In order to prove the well-posedness of such a problem, we therefore start by analyzing the continuous setting for the time-independent case.

2.1. The time-independent problem

We aim here at proving existence and uniqueness of Problem (12) in the stationary limit, that is when the temperature does not depend on t. In this case the problem reads: find $(u,\lambda) \in X \times M_1$ such that

$$\begin{aligned} a(u,w) + b_1(w,\lambda) &= (f,w), \qquad & w \in X, \\ b_2(u,\varphi) - c(\lambda,\varphi) &= (g,\varphi), \qquad & \varphi \in M_2. \end{aligned} \tag{13}$$

In the following, we will use the notation $A \lesssim B$ (resp. $A \gtrsim B$) to signify that the quantity A is bounded from above (resp. below) by $C{\cdot}B$, where C is a constant generally depending on the coefficients α_k, σ_k, $\hat{\alpha}(p)$ and the aspect ratio of the subdomains, but not depending, for the discrete setting, on the mesh size. Assuming $\sigma_k \geq \sigma_0 > 0$ for $k = 1, 2$, it is easy to show that the bilinear form $a(\cdot,\cdot)$ is coercive on X. Moreover, we recall that the following inf-sup condition for the bilinear form $b_1(\cdot,\cdot)$ holds [12]: there exists $\beta_1 > 0$ such that

$$\forall q \in M_1, \qquad \sup_{w \in X} \frac{b_1(w,q)}{\|w\|_X} \geq \beta_1 \|q\|_{-\frac{1}{2},\Gamma^c}. \tag{14}$$

We note that, choosing $\varphi = [w] \in M_2$ in (12) and using the definition of $b_1(\cdot,\cdot)$ and $c(\cdot,\cdot)$, we get

$$\begin{aligned} a(u,w) + b_1(w,\lambda) + b_2(u,[w]) - c(\lambda,[w]) &= a(u,w) + b_2(u,[w]) \\ &= f(w) + g([w]). \end{aligned} \tag{15}$$

As a result, Problem (12) can be rewritten: find $u \in X$ such that

$$\tilde{a}(u,w) = \tilde{f}(w), \qquad w \in X, \tag{16}$$

where

$$\tilde{a}(u,w) := a(u,w) + b_2(u,[w]) \qquad \text{and} \qquad \tilde{f}(w) := f(w) + g([w]).$$

The following result holds:

Theorem 1. *Problem (12) admits a unique solution $(u, \lambda) \in X \times M_1$. Moreover, denoting by γ the coercivity constant of $a(\cdot,\cdot)$, the following stability estimates hold:*

$$\begin{aligned} \|u\|_X &\leq C_1\Big(\|f\|_{X'} + \|g\|_{-\frac{1}{2},\Gamma^c}\Big), \\ \|\lambda\|_{-\frac{1}{2},\Gamma^c} &\leq C_2\|f\|_{X'} + C_3\|g\|_{-\frac{1}{2},\Gamma^c}, \end{aligned} \tag{17}$$

where

$$C_1 = 1/\gamma, \qquad C_2 = \frac{1}{\beta_1}\Big(1 + \|a\|/\gamma\Big), \qquad C_3 = \frac{1}{\beta_1}\|a\|/\gamma.$$

Proof. We start by considering (16). The continuity and coercivity of the bilinear form $\tilde{a}(\cdot,\cdot)$ follow directly from the properties of $a(\cdot,\cdot)$ and the fact that $b_2(\cdot,\cdot) \geq 0$. Furthermore, $\tilde{f}(\cdot)$ is also continuous. Therefore, Problem (16) admits a unique solution $u \in X$. Once u is given, existence and uniqueness of $\lambda \in M_1$ is retrieved from the second line of (12). In order to prove (17), we start by choosing $\varphi = [u]$ in the first line of (12) and find

$$a(u,u) \leq a(u,u) + b_2(u,[u]) = f(u) + g([u]),$$

and thus

$$\|u\|_X^2 \leq \frac{1}{\gamma}a(u,u) \leq \Big(\|f\|_{X'} + \|g\|_{-\frac{1}{2},\Gamma^c}\Big)\|u\|_X. \tag{18}$$

Moreover, according to (14) we can write

$$\begin{aligned} \beta_1\|\lambda\|_{-\frac{1}{2},\Gamma^c} &\leq \sup_{w\in X\setminus\{0\}} \frac{b_1(w,\lambda)}{\|w\|_X} = \sup_{w\in X\setminus\{0\}} \frac{f(w) - a(u,w)}{\|w\|_X} \\ &\leq \|f\|_{X'} + \|a\|\|u\|_X, \end{aligned}$$

where $\|a\|$ denotes the continuity constant of the bilinear form $a(\cdot,\cdot)$. This yields, together with (18),

$$\begin{aligned} \|\lambda\|_{-\frac{1}{2},\Gamma^c} &\leq \frac{1}{\beta_1}\|f\|_{X'} + \frac{\|a\|}{\beta_1\gamma}\Big(\|f\|_{X'} + \|g\|_{-\frac{1}{2},\Gamma^c}\Big) \\ &= \frac{1}{\beta_1}\Big(1 + \frac{\|a\|}{\gamma}\Big)\|f\|_{X'} + \frac{\|a\|}{\beta_1\gamma}\|g\|_{-\frac{1}{2},\Gamma^c}. \end{aligned}$$

It is clear that the solution of the time independent problem (13) is a solution of (16). Reversely, it is not difficult to verify that the couple (u, λ), with u solution of (16) and λ retrieved from the second line of equation (13), is a solution of equation (15).

3. The discrete setting and error estimates

In this section, we introduce the finite element approximation spaces $X_h \subset X$, $M_h \subset M_1$ and $W_h \subset M_2$, and we consider two different discrete mortar formulations of Problem (1) which basically differ on the evaluation of the bilinear form $b_2(\cdot,\cdot)$ with respect to the second argument. Moreover, optimal a priori error estimates are shown for both approaches.
Let us introduce two independent shape regular meshes $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ on Ω_1 and Ω_2, with mesh-sizes bounded by h_1 and h_2, respectively. In the forthcoming, to fix the ideas, the elements K of the meshes will be either triangles ($d = 2$) or tetrahedra ($d = 3$), but it is not difficult to see that the analysis contains all the ingredients also for quadrilaterals ($d = 2$) or hexahedra ($d = 3$). We assume that the meshes are quasi-uniform and that $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$. Without loss of generality, we choose the side of Γ^c associated with Ω_2 as slave side and the one associated with Ω_1 as master side. Moreover we assume that Γ^c is the complete $(d-1)$-dimensional union of faces of the elements on the slave side, inheriting its mesh from $\mathcal{T}_{h_2}$. We denote by $\mathcal{T}_{\Gamma^c}$ the mesh on Γ^c with mesh-size bounded by h_2 whose elements are boundary edges ($d = 2$) or faces ($d = 3$) of $\mathcal{T}_{h_2}$. The unconstrained discrete finite element space is denoted by

$$X_h := \prod_{k=1}^{2} X_h^k,$$

where $X_h^k = \{w_h \in C^0(\Omega_k) \mid w_{h|K} \in \mathbb{P}^1(K)\ ,\ K \in \mathcal{T}_{h_k}\}$ stands for the space of linear conforming finite elements in the subdomain Ω_k associated to the mesh $\mathcal{T}_{h_k}$. We note that no interface condition is imposed on X_h, and the elements in X_h do not have to satisfy a continuity condition at the interface. Let W_h be the trace space of finite element basis functions from the slave side, i.e., of X_h^2, restricted to Γ^c. The crucial point is that the Lagrange multiplier space M_h should be defined in a suitable way so that the bilinear form $b_1(\cdot,\cdot)$ satisfies a suitable inf-sup condition. Here, we use a dual Lagrange multiplier space such that $n_s := \dim M_h = \dim W_h$ and the following biorthogonality property is satisfied (for details see [21]): denoting by $\{\mu_i\}_{1\le i\le n_s}$ and $\{\varphi_i\}_{1\le i\le n_s}$ the basis function of M_h and W_h, respectively, it holds

$$\int_{\Gamma^c} \mu_i \varphi_j ds = \delta_{ij} \int_{\Gamma^c} \varphi_j ds, \qquad 1 \le i, j \le n_s. \tag{19}$$

3.1. First approach

With the choices of the discrete spaces introduced above, we can consider the following Petrov–Galerkin mortar formulation: find $(u_h, \lambda_h) \in X_h \times M_h$ such that

$$a(u_h, w_h) + b_1(w_h, \lambda_h) = f(w_h), \qquad w_h \in X_h, \tag{20}$$
$$b_2(u_h, \varphi_h) - c(\lambda_h, \varphi_h) = g(\varphi_h), \qquad \varphi_h \in W_h. \tag{21}$$

We point out that $M_h \subset (H^{\frac{1}{2}}(\Gamma^c))'$ but $M_h \not\subset H^{\frac{1}{2}}(\Gamma^c)$, whereas $W_h \subset H^{\frac{1}{2}}(\Gamma^c)$. In order to analyze the approximation error in this case, we recall well-known results. We denote by π_h the mortar projection operator $\pi_h : L^2(\Gamma^c) \longrightarrow W_h$ defined by

$$\int_{\Gamma^c} \pi_h w \mu_h \, ds = \int_{\Gamma^c} w \mu_h \, ds, \qquad \mu_h \in M_h. \tag{22}$$

It has been proved [12] that π_h is L^2- and H^1-stable. Moreover, by space interpolation, the $H^{\frac{1}{2}}$-stability holds

$$\|\pi_h w\|_{\frac{1}{2},\Gamma^c} \lesssim \|w\|_{\frac{1}{2},\Gamma^c}, \qquad w \in H^{\frac{1}{2}}(\Gamma^c). \tag{23}$$

We point out that, in contrast with standard mortar approaches, we work with the dual norm of $H^{\frac{1}{2}}(\Gamma^c)$ and not with the dual norm of $H_{00}^{\frac{1}{2}}(\Gamma^c)$. The validity of the uniform discrete inf-sup condition [12]

$$\|\mu_h\|_{-\frac{1}{2},\Gamma^c} \leq C \sup_{w_h \in X_h \setminus \{0\}} \frac{b_1(w_h, \mu_h)}{\|w_h\|_X} \tag{24}$$

yields the following result [6,12]:

Theorem 2. *The discrete variational problem (20)-(21) has a unique solution $(u_h, \lambda_h) \in X_h \times M_h$ and the following a priori bounds hold*

$$\|u - u_h\|_X \lesssim \inf_{w_h \in X_h} \|u - w_h\|_X + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_{-\frac{1}{2},\Gamma^c}, \tag{25}$$
$$\|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma^c} \lesssim \inf_{w_h \in X_h} \|u - w_h\|_X + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_{-\frac{1}{2},\Gamma^c}. \tag{26}$$

Using the best approximation properties of X_h and M_h, we obtain optimal a priori estimates for the solution u and for the Lagrange multiplier λ. We refer to [12] for a proof of the following result.

Corollary 1. *Assume that* $u \in \prod_{k=1}^{2} H^{r_k+1}(\Omega_k)$ *with* $r_1 \geq 0$ *and* $r_2 > \frac{1}{2}$*. Then, we have the following a priori estimate for the discretization error*

$$\|u - u_h\|_X \lesssim (h_1^{2s_1} \|u\|_{s_1+1,\Omega_1}^2 + h_2^{2s_2} \|u\|_{s_2+1,\Omega_2}^2)^{\frac{1}{2}}, \tag{27}$$

$$\|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma^c} \lesssim (h_1^{2s_1} \|u\|_{s_1+1,\Omega_1}^2 + h_2^{2s_2} \|u\|_{s_2+1,\Omega_2}^2)^{\frac{1}{2}}, \tag{28}$$

where $s_k := min(1, r_k)$, $k = 1, 2$*. If* $0 \leq r_2 \leq \frac{1}{2}$ *and* $\lambda \in H^{r_2-\frac{1}{2}}(\Gamma^c)$*, then we have*

$$\begin{aligned}\|u - u_h\|_X \lesssim (h_1^{2s_1} \|u\|_{s_1+1,\Omega_1}^2 + h_2^{2s_2} \|u\|_{s_2+1,\Omega_2}^2 \\ + h_2^{2r_2} \|\lambda\|_{r_2-\frac{1}{2},\Gamma^c}^2)^{\frac{1}{2}}.\end{aligned} \tag{29}$$

$$\begin{aligned}\|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma^c} \lesssim (h_1^{2s_1} \|u\|_{s_1+1,\Omega_1}^2 + h_2^{2s_2} \|u\|_{s_2+1,\Omega_2}^2 \\ + h_2^{2r_2} \|\lambda\|_{r_2-\frac{1}{2},\Gamma^c}^2)^{\frac{1}{2}}.\end{aligned} \tag{30}$$

Remark 1. Using the same notation for the finite element solution and its vector representation, we can write the algebraic formulation of the variational equations (20)-(21) as

$$\begin{pmatrix} A & B_1^T \\ B_2 & -C \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} f_h \\ g_h \end{pmatrix}, \tag{31}$$

where the matrices A, B_1, B_2 and C correspond to the bilinear forms $a(\cdot, \cdot)$, $b_1(\cdot, \cdot)$, $b_2(\cdot, \cdot)$ and $c(\cdot, \cdot)$, respectively, and f_h and g_h represent the discrete forms of the linear forms $f(\cdot)$ and $g(\cdot)$, respectively. We note that, as a consequence of the biorthogonality between the nodal basis functions of the trace space W_h and the Lagrange multiplier M_h, the mass matrix on the slave side of the matrix B_1 is reduced to a diagonal one as well as the matrix C. However, due to the choice of the test space in the second equation of equation (12), we cannot take advantage of the biorthogonality property for the mass matrix on the slave side of the matrix B_2 as well. As a result, the Lagrange multiplier can be locally eliminated but no local static condensation for the degrees of freedom on the slave side of the interface can be carried out. Thus we cannot apply the multigrid schemes introduced in [21, 22] as iterative solvers. In the next subsection, we provide a second approach where both sets of variables can be locally eliminated.

3.2. Second approach

We consider the case that the heat transfer parameter $\hat{\alpha}$ is constant and strictly positive, so that the bilinear form $b_2(\cdot,\cdot)$ has the form $b_2(u,\varphi) = \hat{\alpha}\int_{\Gamma^c}[u]\varphi ds$. The idea is to introduce here a discrete approach that allows us to statically condense out the degree of freedom on the slave side of the interface in such a way that the global linear system is symmetric and positive definite and special multigrid techniques , such as the one proposed in [22], can be applied. To this aim, we introduce an isomorphism $I_h : M_h \to W_h$ defined as $I_h\mu_h = \sum_{i=1}^{n_s}\alpha_i\varphi_i$ for each $\mu_h \in M_h$ with $\mu_h = \sum_{i=1}^{n_s}\alpha_i\mu_i$. We remark that I_h is well-defined since $\dim M_h = \dim W_h$, and it is L^2-bounded with L^2-bounded inverse. Using this isomorphism, we introduce an alternative approach to the Petrov–Galerkin problem, which is given as follows: find $(\tilde{u}_h, \tilde{\lambda}_h) \in X_h \times M_h$ such that

$$a(\tilde{u}_h, w_h) + b_1(w_h, \tilde{\lambda}_h) = f(w_h), \qquad w_h \in X_h, \tag{32}$$

$$b_2(\tilde{u}_h, \mu_h) - c(\tilde{\lambda}_h, I_h\mu_h) = g(I_h\mu_h), \qquad \mu_h \in M_h. \tag{33}$$

We remark that Problem (32)-(33) differs from (20)-(21) by simply replacing the evaluation of the bilinear form $b_2(\cdot,\cdot)$ with respect to the second argument. The algebraic form of the variational equations (32)-(33) can also be written as in (31). Under the assumption that $\hat{\alpha}$ is constant, it is not difficult to see that, after a proper reordering of the unknowns, B_1 and B_2 can be written as

$$B_1 = \begin{pmatrix} 0 \ M \ D \end{pmatrix}^T \quad \text{and} \quad B_2 = \hat{\alpha} B_1^T,$$

where the entries of the mass matrices M and D are given by $(m)_{ij} := \int_{\Gamma^c}\varphi_j^m\,\mu_i\,d\sigma$ and $(d)_{ij} := \int_{\Gamma^c}\varphi_j^s\,\mu_i\,d\sigma$, respectively. Here, μ_i denotes the basis functions of M_h and φ_j^m and φ_j^s stand for the nodal basis functions in X_h associated with the nodes on the master and on the slave side of the interface Γ^c, respectively. We note that the matrix D is diagonal due to the use of a dual Lagrange multiplier space and that $C = D$. Scaling the equation (33) with $\hat{\alpha}^{-1}$, we arrive at a symmetric saddle point system

$$\begin{pmatrix} A & B_1^T \\ B_1 & -\frac{1}{\hat{\alpha}}D \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} f_h \\ \frac{1}{\hat{\alpha}} g_h \end{pmatrix}. \tag{34}$$

Hence, working with this second approach, static condensation of the Lagrange multiplier can be done as in the first approach [12], but also the degrees of freedom of the slave side of Γ^c can be eliminated. As a consequence, a multigrid approach for a positive definite system on nonconforming spaces can be applied (see [22] for the mortar setting).

Theorem 3. *Problem (32)-(33) admits a unique solution $(\tilde{u}_h, \tilde{\lambda}_h) \in X_h \times M_h$ and the following error estimates hold:*

$$\begin{aligned}\|u-\tilde{u}_h\|_X \lesssim & \inf_{w_h\in X_h}\|u-w_h\|_X + \inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_{-\frac{1}{2},\Gamma^c} \\ & +\|I_h^{-1}\pi_h[u]-[u]\|_{-\frac{1}{2},\Gamma^c}, \qquad (35)\\ \|\lambda-\tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma^c} \lesssim & \inf_{w_h\in X_h}\|u-w_h\|_X + \inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_{-\frac{1}{2},\Gamma^c} \\ & +\|I_h^{-1}\pi_h[u]-[u]\|_{-\frac{1}{2},\Gamma^c}. \qquad (36)\end{aligned}$$

Proof. According to the definition of π_h and to (33), it follows that

$$\begin{aligned}b_1(w_h,\tilde{\lambda}_h) &= c(\tilde{\lambda}_h, I_h\left(I_h^{-1}\pi_h[w_h]\right)) \\ &= b_2(\tilde{u}_h, I_h^{-1}\pi_h[w_h]) - g(I_h\left(I_h^{-1}\pi_h[w_h]\right)). \qquad (37)\end{aligned}$$

By substituting now (37) in (32), we end up with the following problem: find $\tilde{u}_h \in X_h$ such that

$$\tilde{a}_h(\tilde{u}_h, w_h) = \tilde{f}_h(w_h), \qquad w_h \in X_h, \qquad (38)$$

where the bilinear form $\tilde{a}_h(\cdot,\cdot)$ is defined as $\tilde{a}_h(\tilde{u}_h,w_h) := a(\tilde{u}_h,w_h) + b_2(\tilde{u}_h, I_h^{-1}\pi_h[w_h])$, and the right hand side is given by $\tilde{f}_h(w_h) := f(w_h) + g(\pi_h[w_h])$. We note that, according to the definition of the mortar projection (22), and to the biorthogonality property (19), the bilinear form $b_2(\cdot,\cdot)$ satisfies: for $\pi_h[v_h] = \sum_{i=1}^{n_s}\alpha_i\varphi_i$,

$$\begin{aligned}b_2(v_h, I_h^{-1}\pi_h[v_h]) \gtrsim \int_{\Gamma^c}\pi_h[v_h]I_h^{-1}\pi_h[v_h]ds &= \int_{\Gamma^c}\sum_i\alpha_i\varphi_i\sum_j\alpha_j\mu_j ds \\ &= \sum_i\alpha_i^2\int_{\Gamma^c}\varphi_i ds.\end{aligned}$$

According to the positivity of the basis functions φ_i ([21]) this yields, together with the ellipticity of the bilinear form $a(\cdot,\cdot)$, that the bilinear form $\tilde{a}_h(\cdot,\cdot)$ is positive definite. Moreover, it is uniformly coercive on $X_h \times X_h$, the coercivity constant being the coercivity constant γ of $a(\cdot,\cdot)$. Therefore, Problem (38) admits a unique solution $\tilde{u}_h$. Existence and uniqueness of $\tilde{\lambda}_h \in M_h$ easily follows from (33) by simply remarking the invertibility of the bilinear form $c(\cdot,\cdot)$. In order to derive the error estimates, we first observe that the bilinear form $\tilde{a}_h(\cdot,\cdot)$ is well defined at (u, v_h). Moreover, according to the L^2-stability of the mortar projection π_h, and recalling that I_h^{-1} is an isomorphism from W_h onto M_h with obvious equivalence of the L^2-norm, it is easy to prove that $\tilde{a}_h(\cdot,\cdot)$ verifies

$$|\tilde{a}_h(u-w_h, v_h)| \le \gamma^*\|u-w_h\|_X\|v_h\|_X, \qquad w_h, v_h \in X_h,$$

for a suitable constant γ^* independent of h. Then the following Strang inequality is obtained by a standard argument [20]:

$$\|u - \tilde{u}_h\|_X \leq \left(1 + \frac{\gamma^*}{\gamma}\right) \inf_{w_h \in X_h} \|u - w_h\|_X + \frac{1}{\gamma} \sup_{v_h \in X_h \backslash \{0\}} \frac{|\tilde{f}_h(v_h) - \tilde{a}_h(u, v_h)|}{\|v_h\|_X}. \tag{39}$$

According to the definition of $\tilde{a}_h(\cdot,\cdot)$ and $\tilde{f}_h(\cdot)$, we estimate the second term of (39) by

$$\begin{aligned}
|\tilde{f}_h(v_h) - \tilde{a}_h(u, v_h)| &= |f(v_h) + g(\pi_h[v_h]) - a(u, v_h) - b_2(u, I_h^{-1}\pi_h[v_h])| \\
&= |b_1(v_h, \lambda) - c(\lambda, \pi_h[v_h]) - b_2(u, I_h^{-1}\pi_h[v_h] - \pi_h[v_h])| \\
&\leq \int_{\Gamma^c} |\lambda([v_h] - \pi_h[v_h])|\, ds \\
&\quad + \int_{\Gamma^c} \left|[u](I_h^{-1}\pi_h[v_h] - \pi_h[v_h])\right| ds.
\end{aligned} \tag{40}$$

The definition and the $H^{\frac{1}{2}}$-stability (23) of π_h and the trace inequality yield that, for any $\mu_h \in M_h$

$$\begin{aligned}
\int_{\Gamma^c} |\lambda([v_h] - \pi_h[v_h])|\, ds &\leq \|\lambda - \mu_h\|_{-\frac{1}{2},\Gamma^c} \|[v_h] - \pi_h[v_h]\|_{\frac{1}{2},\Gamma^c} \\
&\lesssim \|\lambda - \mu_h\|_{-\frac{1}{2},\Gamma^c} \|v_h\|_X.
\end{aligned} \tag{41}$$

In order to estimate the second term of (40), we first note that the biorthogonality property of the basis functions of M_h and W_h yields the identity

$$\int_{\Gamma^c} \varphi_h I_h^{-1} \psi_h ds = \int_{\Gamma^c} I_h^{-1} \varphi_h \psi_h ds, \qquad \varphi_h, \psi_h \in W_h \tag{42}$$

which, by using (22) and (23), allows to write

$$\begin{aligned}
\int_{\Gamma^c} \left|[u](I_h^{-1}\pi_h[v_h] - \pi_h[v_h])\right| ds &= \int_{\Gamma^c} \left|\pi_h[u] I_h^{-1}\pi_h[v_h] - [u]\pi_h[v_h]\right| ds \\
&= \int_{\Gamma^c} \left|(I_h^{-1}\pi_h[u] - [u])\pi_h[v_h]\right| ds \\
&\lesssim \|I_h^{-1}\pi_h[u] - [u]\|_{-\frac{1}{2},\Gamma^c} \|v_h\|_X.
\end{aligned} \tag{43}$$

Using (41) and (43) in (40), we finally get

$$\sup_{v_h \in X_h \backslash \{0\}} \frac{|\tilde{f}_h(v_h) - \tilde{a}_h(u, v_h)|}{\|v_h\|_X} \lesssim \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_{-\frac{1}{2},\Gamma^c} + \|I_h^{-1}\pi_h[u] - [u]\|_{-\frac{1}{2},\Gamma^c},$$

which together with (39) gives (35). In order to estimate the term $\|\lambda - \tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma^c}$, we first remark that, by subtracting (32) from (20), we get

$$a(u_h - \tilde{u}_h, w_h) + b_1(\lambda_h - \tilde{\lambda}_h, w_h) = 0, \qquad w_h \in X_h,$$

which, together with the inf-sup condition (24) and the continuity of the bilinear form $a(\cdot,\cdot)$, yields $\|\lambda_h - \tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma^c} \lesssim \|u_h - \tilde{u}_h\|_X$.

Corollary 2. *Assume that the exact solution u satisfies the same hypothesis of Corollary 1. The following error estimates hold: if $r_1 \geq 0$ and $r_2 > \frac{1}{2}$*

$$\begin{aligned}\|u - \tilde{u}_h\|_X &\lesssim (h_1^{2s_1}\|u\|^2_{s_1+1,\Omega_1} + h_2^{2s_2}\|u\|^2_{s_2+1,\Omega_2})^{\frac{1}{2}}\\ \|\lambda - \tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma^c} &\lesssim (h_1^{2s_1}\|u\|^2_{s_1+1,\Omega_1} + h_2^{2s_2}\|u\|^2_{s_2+1,\Omega_2})^{\frac{1}{2}}.\end{aligned}$$

If $0 \leq r_2 \leq \frac{1}{2}$ and $\lambda \in H^{r_2-\frac{1}{2}}(\Gamma^c)$

$$\begin{aligned}\|u - \tilde{u}_h\|_X &\lesssim (h_1^{2s_1}\|u\|^2_{s_1+1,\Omega_1} + h_2^{2s_2}\|u\|^2_{s_2+1,\Omega_2} + h_2^{2r_2}\|\lambda\|^2_{r_2-\frac{1}{2},\Gamma^c})^{\frac{1}{2}}\\ \|\lambda - \tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma^c} &\lesssim (h_1^{2s_1}\|u\|^2_{s_1+1,\Omega_1} + h_2^{2s_2}\|u\|^2_{s_2+1,\Omega_2} + h_2^{2r_2}\|\lambda\|^2_{r_2-\frac{1}{2},\Gamma^c})^{\frac{1}{2}}.\end{aligned}$$

Proof. In terms of Theorem 3 and Corollary 1, it is sufficient to consider the quantity $\|I_h^{-1}\pi_h[u] - [u]\|_{-\frac{1}{2},\Gamma^c}$, to which we apply the triangular inequality to get the bound

$$\begin{aligned}\|I_h^{-1}\pi_h[u] - [u]\|^2_{-\frac{1}{2},\Gamma^c} &\lesssim \sum_{e\in\mathcal{T}_{\Gamma^c}} \|I_h^{-1}\pi_h[u] - [u]\|^2_{-\frac{1}{2},e}\\ &\lesssim \sum_{e\in\mathcal{T}_{\Gamma^c}} \Big(\|I_h^{-1}\pi_h[u] - c_e\|^2_{-\frac{1}{2},e}\\ &\qquad + \|[u] - c_e\|^2_{-\frac{1}{2},e}\Big), \end{aligned} \tag{44}$$

where c_e is an arbitrary constant depending on the edge or face $e \in \mathcal{T}_{\Gamma^c}$ such that $c_e = I_h^{-1}\pi_h(c_e)$ (see [21] for details on the property of the operator I_h^{-1} of preserving the constant functions). We now aim at estimating the quantity $\|I_h^{-1}\pi_h([u] - c_e)\|^2_{-\frac{1}{2},e}$. Denoting by $Q_h := I_h^{-1}\pi_h$, we write

$$\|Q_h f\|_{-\frac{1}{2},e} := \sup_{\varphi\in H^{\frac{1}{2}}(e)} \frac{\int_e (Q_h f)\varphi\, ds}{\|\varphi\|_{\frac{1}{2},e}} = \sup_{\varphi\in H^{\frac{1}{2}}(e)} \frac{\int_{\Gamma^c} (Q_h f)\tilde{\varphi}\, ds}{\|\varphi\|_{\frac{1}{2},e}},$$

where $\tilde{\varphi}$ is the extension by zero of φ to the whole Γ^c. According to the definition of π_h and to (42), it is easy to prove that Q_h is self adjoint, so that

$$\|Q_h f\|_{-\frac{1}{2},e} = \sup_{\varphi \in H^{\frac{1}{2}}(e)} \frac{\int_{\tilde{e}} f(Q_h \tilde{\varphi})\, ds}{\|\varphi\|_{\frac{1}{2},e}},$$

where $\tilde{e}$ denotes the support of $Q_h\tilde{\varphi}$ (which coincides with a patch of elements near e). Therefore, for arbitrary $0 < \epsilon < \frac{1}{2}$, by using an inverse estimate, the property of Clement's operators [5,8], as well as the well known injection $H^s(e) \hookrightarrow H^s_0(e)$, $u \in H^s(e), s \in (0, \frac{1}{2})$, where the injection bound is given by ([4])

$$\|u\|_{H^s_0(e)} \lesssim \frac{1}{\frac{1}{2} - s} \|u\|_{H^s(e)},$$

we can bound

$$\begin{aligned}
\|Q_h f\|_{-\frac{1}{2},e} &\leq \sup_{\varphi \in H^{\frac{1}{2}}(e)} \frac{\|f\|_{-\frac{1}{2},\tilde{e}} \|Q_h \tilde{\varphi}\|_{\frac{1}{2},\tilde{e}}}{\|\varphi\|_{\frac{1}{2},e}} \\
&\lesssim \sup_{\varphi \in H^{\frac{1}{2}}(e)} \frac{h_e^{-\epsilon} \|f\|_{-\frac{1}{2},\tilde{e}} \|Q_h \tilde{\varphi}\|_{\frac{1}{2}-\epsilon,\tilde{e}}}{\|\varphi\|_{\frac{1}{2},e}} \\
&\lesssim \sup_{\varphi \in H^{\frac{1}{2}}(e)} \frac{h_e^{-\epsilon} \|f\|_{-\frac{1}{2},\tilde{e}} \|\tilde{\varphi}\|_{\frac{1}{2}-\epsilon,\tilde{\tilde{e}}}}{\|\varphi\|_{\frac{1}{2},e}} \\
&\lesssim \sup_{\varphi \in H^{\frac{1}{2}}(e)} \frac{h_e^{-\epsilon} \|f\|_{-\frac{1}{2},\tilde{e}} \|\varphi\|_{H_0^{\frac{1}{2}-\epsilon}(e)}}{\|\varphi\|_{\frac{1}{2},e}} \\
&\lesssim \sup_{\varphi \in H^{\frac{1}{2}}(e)} \frac{\frac{h_e^{-\epsilon}}{\epsilon} \|f\|_{-\frac{1}{2},\tilde{e}} \|\varphi\|_{H^{\frac{1}{2}-\epsilon}(e)}}{\|\varphi\|_{\frac{1}{2},e}} \lesssim \frac{h_e^{-\epsilon}}{\epsilon} \|f\|_{-\frac{1}{2},\tilde{e}},
\end{aligned}$$

where we have denoted by $\tilde{\tilde{e}}$ a suitable patch of elements of $\mathcal{T}_{\Gamma^c}$ with $\tilde{e} \subset \tilde{\tilde{e}}$ containing only a finite number of elements of $\mathcal{T}_{\Gamma^c}$. Choosing $\epsilon = 1/|\log h_e|$, we get $\|Q_h f\|_{-\frac{1}{2},e} \lesssim |\log h_e| \|f\|_{-\frac{1}{2},\tilde{e}}$ which, in (44) yields

$$\begin{aligned}
\|I_h^{-1}\pi_h[u] - \pi_h[u]\|^2_{-\frac{1}{2},\Gamma^c} \lesssim \sum_{e \in \mathcal{T}_{\Gamma^c}} \Big(&(\log h_e)^2 \|[u] - c_e\|^2_{-\frac{1}{2},\tilde{e}} \\
&+ \|[u] - c_e\|^2_{-\frac{1}{2},e} \Big).
\end{aligned}$$

Since $\tilde{e}$ contains only a finite number of elements of Γ^c, the estimate is obtained by choosing c_e as the orthogonal projection of $[u]$ onto the space of constant functions in e. Therefore we finally get

$$\begin{aligned}\|I_h^{-1}\pi_h[u] - \pi_h[u]\|^2_{-\frac{1}{2},\Gamma^c} &\lesssim \sum_{e\in\mathcal{T}_{\Gamma^c}} \Big((\log h_e)^2 h_e \|[u]-c_e\|^2_{0,\tilde{e}} \\ &\qquad + h_e\|[u]-c_e\|^2_{0,e}\Big) \\ &\lesssim (\log h_2)^2 h_2 \Big(h_1^{2s_1+1}\|u\|^2_{s_1+\frac{1}{2},\Gamma^c} \\ &\qquad + h_2^{2s_2+1}\|u\|^2_{s_2+\frac{1}{2},\Gamma^c}\Big) \\ &\lesssim (\log h_1)^2 h_1^{2s_1+2}\|u\|^2_{s_1+1,\Omega_1} \\ &\qquad + (\log h_2)^2 h_2^{2s_2+2}\|u\|^2_{s_2+1,\Omega_2},\end{aligned}$$

which yields the thesis observing that the quantities $|\log h_1| h_1$ and $|\log h_2| h_2$ are bounded.

Remark 2. Equation (38) is the discrete form of (16), which has been introduced for theoretical purpose. It gives rise to a third discrete approach that could in principle be considered. We remark that, in analogy with the situation that occurs in the mortar approach for the Laplace operator (where one can consider either the saddle point formulation on the unconstrained space or the plain formulation in the constrained one), the mortar projections that have to be computed in this case are inside the definition of the bilinear form $\tilde{a}(\cdot,\cdot)$ of equation (16). Such projections are computed by using Lagrange multiplier spaces and give rise to an equivalent saddle point approach. Moreover, error estimates for the Lagrange multiplier cannot be obtained from (16).

Remark 3. Alternative techniques for handling interface conditions and dealing with non-matching grids can be found in the penalty formulations such as the Nitsche method [18] and interior penalty method [1,2]. Originally proposed for enforcing Dirichlet boundary conditions, they have been applied to interface problems. They basically consists of adding the penalty term $h^{-1}\int_{\Gamma^c}[u][v]ds$ (and additional consistency terms involving normal derivatives for the Nitsche technique) that controls the jump of the solution across the interface. We refer to [11] for an application of the Nitsche method to a stationary heat conduction problem with a discontinuity in the conductivity across Γ^c and an inhomogeneous conormal derivative condition on the interface, and to [15] for an interior penalty method (strictly related to the discontinuous Galerkin method) leading to a symmetric and positive definite problem with almost optimal order estimates.

4. Numerical tests

In this section, we show some numerical examples using the approach discussed in the previous section. We assume that the heat transfer parameter $\hat{\alpha}$ is directly proportional to the contact pressure p on the contact interface so that $\hat{\alpha} = \bar{\gamma}_c p$, where $\bar{\gamma}_c$ is the heat transfer coefficient, see [13,14]. The heat transfer parameter has the unit W/m^2°C, whereas the unit of $\bar{\gamma}_c$ is W/N°C. A different model in which the heat transfer parameter $\hat{\alpha}$ depends on the contact pressure p with the relation $\hat{\alpha} = h_{s_0}(\frac{p}{H_c})^{\epsilon}$ can be found in [23], where h_{s_0} is the contact resistance coefficient, H_c the Vickers hardness and ϵ a fixed exponent.

Example 1: In the first example, we test our numerical scheme for a heat transfer problem through the contact interface introduced in [13,14,16]. Here, we solve the steady state heat transfer problem between two bodies $\Omega_1 := (0,1.5)^2$, and $\Omega_2 := (0,1.5)\times(1.5,3)$, which are in contact along the line $y = 1.5$, see the left picture of Figure 1. The material parameters for this problem are given by $\alpha = \alpha_1 = \alpha_2 = 55W/m$°C, and $\bar{\gamma}_c = 1W/N$°C. The temperature at the lower boundary of Ω_1 and the upper boundary of Ω_2 is fixed to be $\theta_1 = 100$°C and $\theta_2 = 200$°C, respectively, and the other boundaries are thermally isolated. This problem can be solved analytically, and the exact solutions at the contact interface are given by $u_1 = \frac{(1+\eta)\theta_1+\eta\theta_2}{1+2\eta}$, and $u_2 = \frac{(1+\eta)\theta_2+\eta\theta_1}{1+2\eta}$ for Ω_1 and Ω_2, respectively, where $\eta = \frac{\bar{\gamma}_c p}{\alpha}$ with p being the pressure per unit length. Since the exact solution is piecewise linear, the numerical solution of the problem is exactly the same as the analytical solution. The temperature at the contact interface for both bodies versus the total contact pressure can be found in the right picture of Figure 1. In this picture, the upper part shows the temperature at the contact interface from Ω_2, whereas the lower part shows the temperature at the contact interface from Ω_1. In case of the perfect conductance, we obtain the temperature 150°C from both sides.

Example 2: In this example, we compare the performance of both approaches using discretization errors in the L^2- and H^1-norms. Here, the domain $\Omega := (-1,1)\times(-1,1)$ is decomposed into two subdomains Ω_1 and Ω_2, where Ω_2 is a circle with radius 0.5 centered at the origin, and $\Omega_1 := \Omega\backslash\bar{\Omega}_2$, see the left picture of Figure 2. Here, the interface Γ is curved, and the analysis should also take into account the polygonal approximation of the interface Γ. For the analysis of mortar methods with a curved interface, we refer to [9]. Setting $\hat{\alpha}(p) = 1$, we compute the right hand side for the second variational equation as $g := \lambda - [u]$, where λ, and $[u]$ are computed by using the exact solution. The exact solutions in Ω_1 and Ω_2 are given, respectively as

$$u_1 := \left(x^2 + y^2 + 1\right) \cos\left(x + 2\,y\right) + 2, \quad u_2 := \left(x^2 + y^2\right) \sin\left(2\,x + y\right).$$

The right-hand side and the Dirichlet boundary condition on $\partial\Omega$ are computed by using the given exact solution. The discretization errors in the L^2 and H^1-norms for both approaches are given in the right picture of Figure 2. Both approaches yield the same qualitative and almost the same quantitative results.

Example 3: In our third example, we consider the case of three bodies, where an iron cube is sliding between two copper bodies $\Omega_1 := (0,5) \times (0,1)^2$ and $\Omega_3 := (0,5) \times (0,1) \times (2,3)$. The length unit is set in centimeters. The iron cube Ω_2 initially occupies the region $(2,3) \times (0,1) \times (1,2)$, and begins to slide with the velocity $2cm/s$ to the right, and when it reaches the right end returns back to the left with the same velocity. In this way, the cube oscillates between the left and right end points of Ω_1 and Ω_3. The material parameters for copper are $\rho_c = 8960kg/m^3$, $c_c = 385J/kg$°C, $\alpha_c = 386W/m$°C; and for iron are $\rho_i = 7860kg/m^3$, $c_i = 444J/kg$°C, $\alpha_i = 80.2W/m$°C. We apply $1MPa$ pressure on the interface Γ at each time step and Robin boundary condition is applied on the upper and lower boundaries of Ω_1 and Ω_3 with Robin parameter $\sigma_1 = \sigma_3 = 25000W/m^2$°C, whereas the other boundaries are thermally isolated. The time discretization is done by using the implicit Euler scheme, and the initial temperature is set to be zero. The domain $\Omega := \cup_{k=1}^{3} \Omega_k$ at time $t = 0$ with the triangulation is shown in the left picture of Figure 3, and the solutions at two different times computed using $\delta t = 0.01$ are shown in the right. In the right picture of Figure 3, we can see the effect of convective boundary condition on the lower surface of Ω_1 and the upper surface of Ω_3, where the cooling process does not allow this area to be heated quickly. In the context of sliding meshes, the advantages of non-matching grids become very obvious. Working with a conforming approach, a complicated re-meshing process is necessary from one time step to the next, whereas we can use the original grid for all time steps. The temperature along the line $y = 0, z = 1$ from the side of copper and iron at time steps $t = 10$ and $t = 100$ is shown in Figure 4. We can see that although in the beginning there is a jump in the temperature across the interface, as time proceeds, the jump decreases.

Example 4: As a last numerical example, we consider the situation illustrated in the left picture of Figure 5. A cylinder Ω_{m} of radius 0.25 cm and height 0.8 cm made of copper rotates at the rate of $10/2\pi$ radians per second inside a cylinder ring Ω_{s} of thickness 0.15 cm and height 0.4 made of iron, where the material parameters are set as in the last example. Zero Dirchlet boundary condition is applied on a small patch of the outer boundary of Ω_{s}, and the rest of the boundary is thermally isolated. As in our third example, we compute the solution by using the implicit Euler scheme using zero initial solution. In the two right pictures of Figure 5, heat distributions at the times $t = 0.01$ and $t = 2$ are visualized, respectively.

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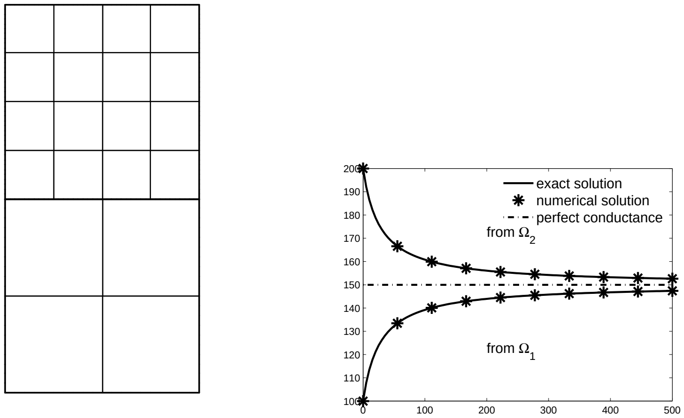

Fig. 1. Two bodies with triangulation (left) and contact temperature in °C versus total contact pressure in N (right), Example 1

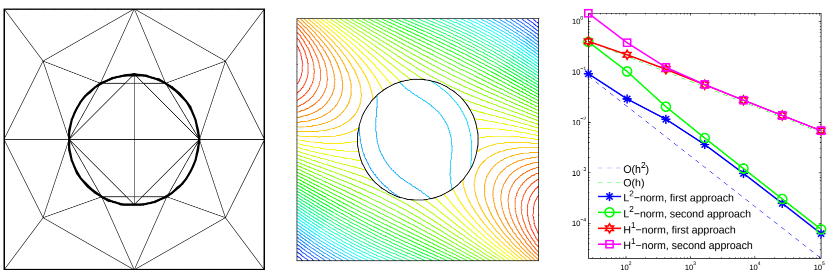

Fig. 2. Decomposition into two subdomains and initial triangulation (left), isolines of the solution (middle) and error plot versus number of elements (right), Example 2

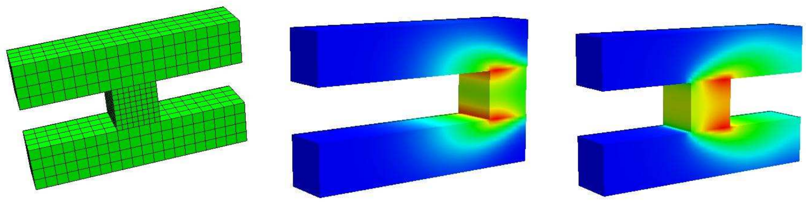

Fig. 3. Decomposition of the domain and initial triangulation (left), and two snapshots of the heat distribution at times $t = 1$ and $t = 2$ (right), Example 3

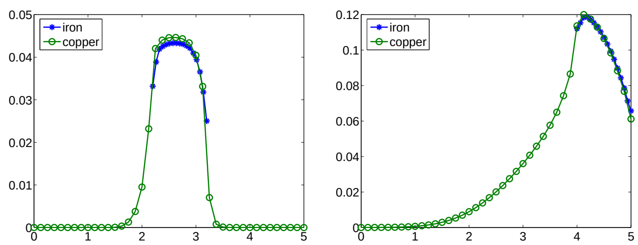

Fig. 4. Temperature along the line $y = 0, z = 1$, after 10 time steps (left) and after 100 time steps (right)

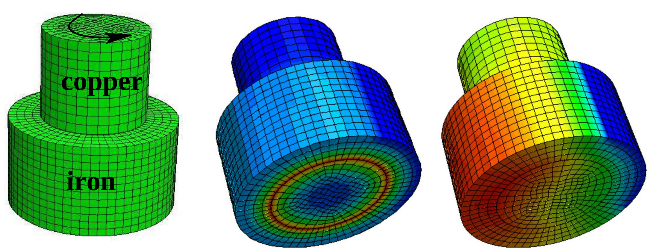

Fig. 5. Decomposition of the domain and triangulation (left), and two snapshots of the heat distribution at times $t = 0.01$ and $t = 2$ (right), Example 4

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